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Decision analysis and risk-informed optimization for CI resilience

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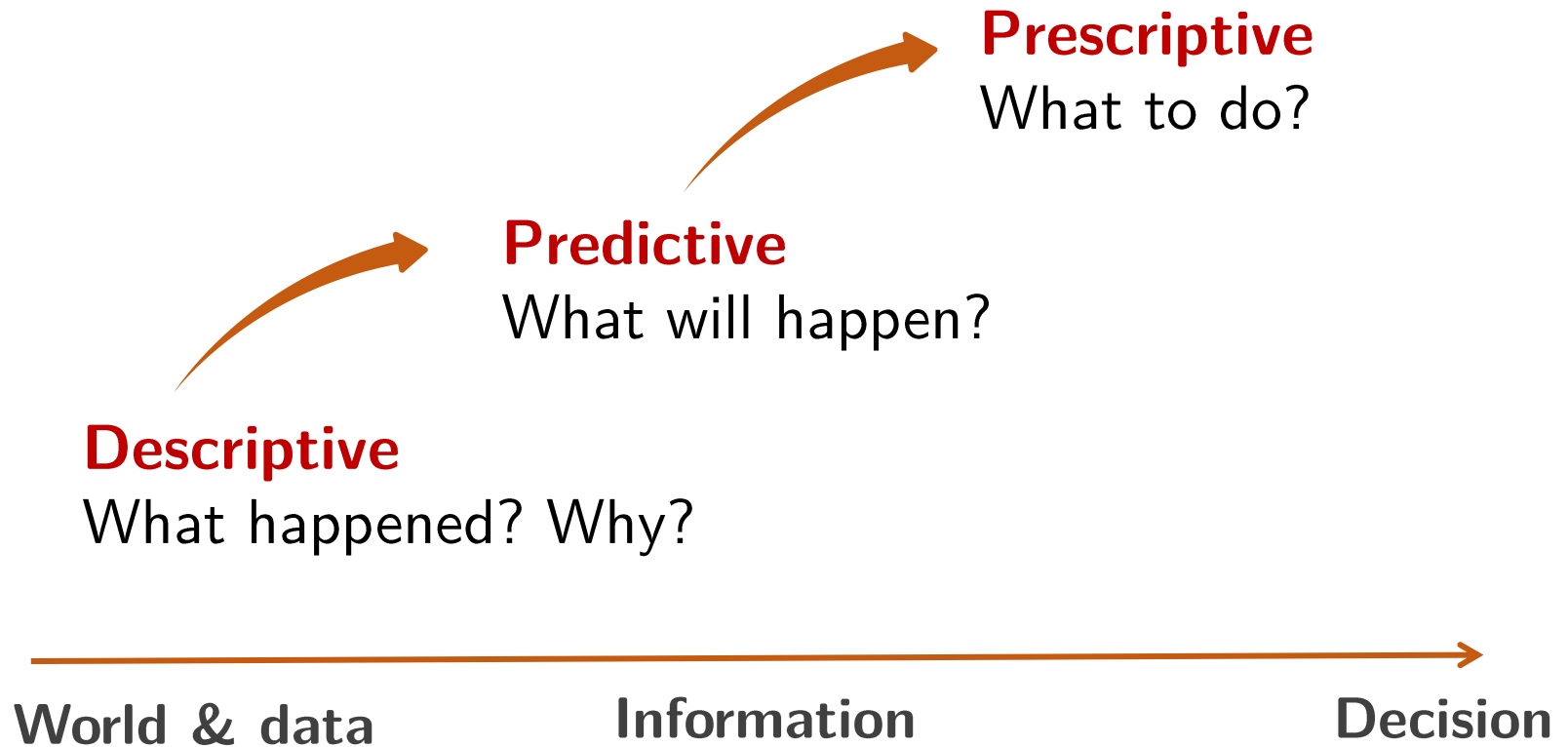
Risk Resilience Reliability (R3) Research Group

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- What to do is often not straightforward, instead very complicated, in real industrial systems.
- Maintenance planning of a wind farm
 - Many turbines
 - Limited repair resources
 - Decide maintenance time
 - Balance maintenance cost v.s. profit
 - Difficult even with **RUL estimated perfectly**...
- Decision theory + Optimization



- Risk measures
- Stochastic decision-making and optimization
 - Optimization model introduction
 - Stochastic optimization formalization
- Sampled average approximation
- Risk-averse stochastic optimization

St. Petersburg paradox

- From Nicolas Bernoulli's letter 1713
- **St. Petersburg game:** flip a fair coin until it comes up heads the 1st time. At that point the player wins € 2^n , where n is the number of times the coin was flipped. How much should one pay for playing this game?

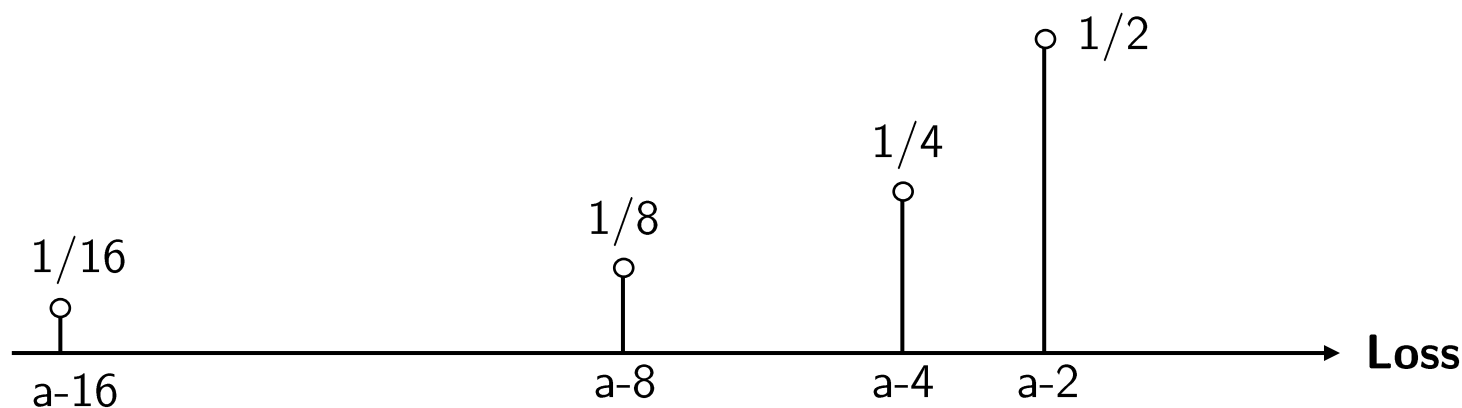
Trial where first tails appears	Probability	Payout
1	$1/2$	2
2	$1/4$	4
3	$1/8$	8
4	$1/16$	16
$\vdots$	$\vdots$	$\vdots$

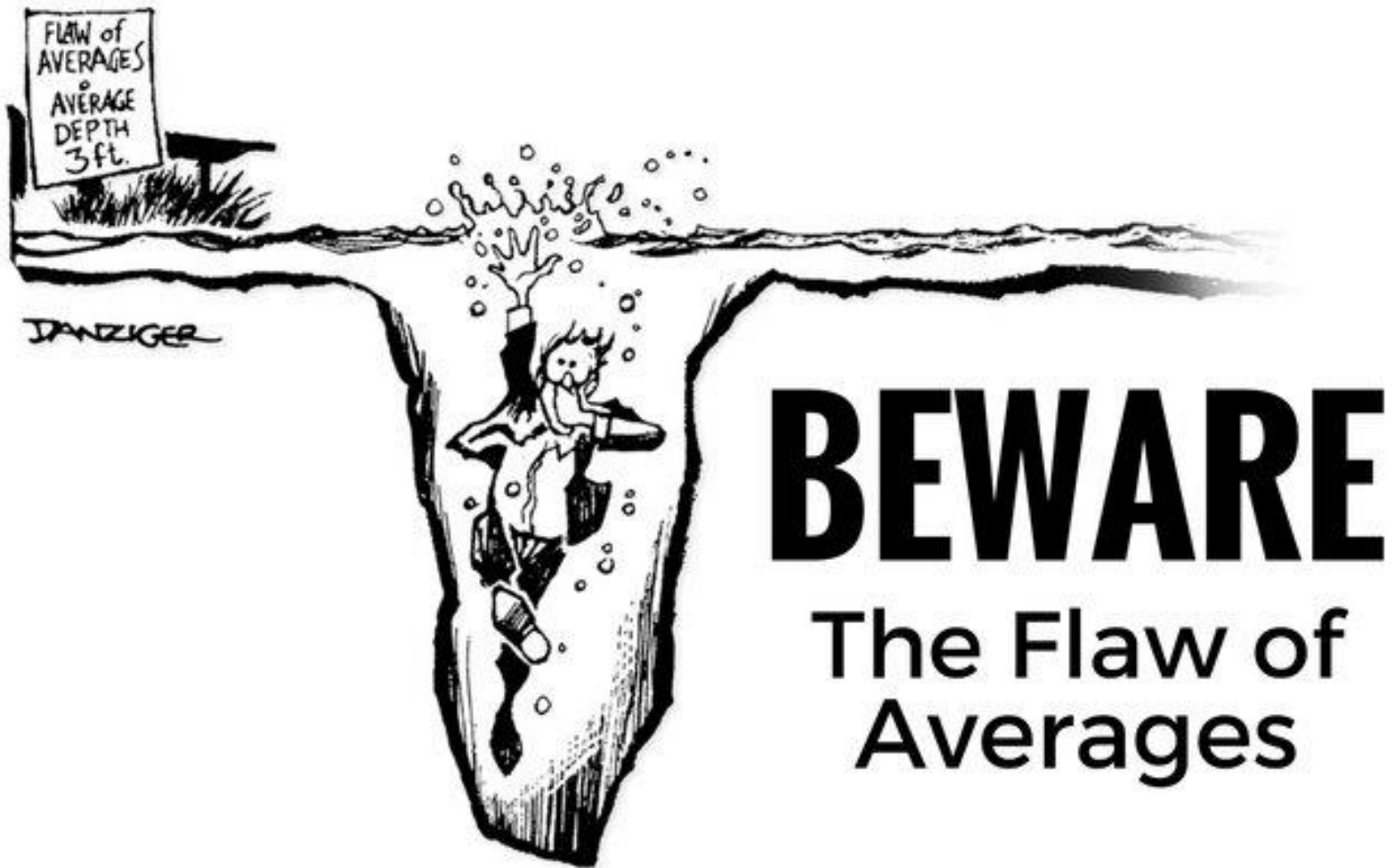
- Decision theorists' advice: the **expected value**

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} P(X = n) \cdot \text{Payout}(n) = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \times 2^n \right) = \sum_{n=1}^{\infty} 1 = \infty$$

St. Petersburg paradox

- The “paradox”: it would be rational to pay any **finite** fee $a = \infty$ to play the game?
- Proposed resolutions, e.g., Daniel Bernoulli’s **expected utility**
 - utility = $\ln(2^n)$ (diminishing marginal utility)
 - expected utility ≈ 1.4 €
 - satisfactory?
- Why we have the paradox?
 - **Expectation neglects the risk of the (bad) outcomes**





- **Different attitudes** towards the risk of the outcomes

Game A:

20% +1000€, 80% -100 €

Game B:

100% +120€

- **Classifications**

- **Risk averse** fear loss and seek sureness
- **Risk neutral** are indifferent to the degree of risky outcome
- **Risk seeking** hope to “win big” and don’t mind losing as much

Risk:

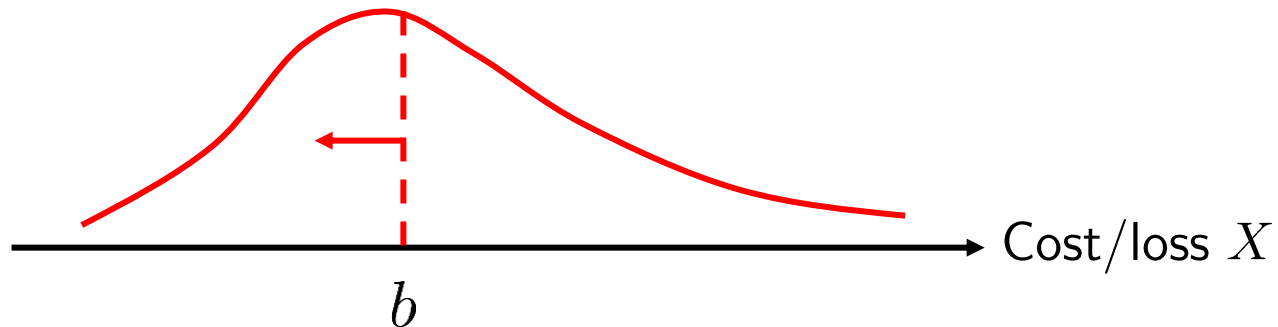
The combination of the **uncertainty of occurrence** of “bad outcome”, and **the severity of that outcome**

- **Random outcomes:** r.v. $X : \Omega \rightarrow \mathbb{R}$
→ “**Cost/loss**” oriented X : high outcomes bad, low outcomes good
- **Risk measure:** a quantification be applied to X that elicits the level of “cost/loss” in X

$\mathcal{R}(X)$ maps X into $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$

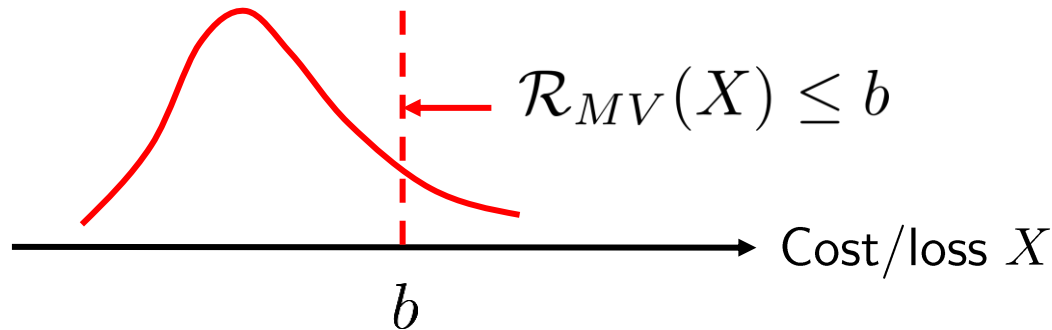
- Risk $\neq$ Uncertainty (statistical dispersion)

- **Expectation:** $\mathcal{R}(X) = \mathbb{E}[X]$
 - $\mathcal{R}(X) \leq b \Leftrightarrow X \leq b$ on average
 - but not risk averse (risk measure?), perhaps too feeble



- **Focusing on worst cases:** $\mathcal{R}(X) = \sup X$
 - $\mathcal{R}(X) \leq b \Leftrightarrow X \leq b$ almost surely
 - Averse, perhaps overly conservative, infeasible

- **Mean variance:** $\mathcal{R}_{MV}(X) = \mathbb{E}[X] + \lambda \mathbb{V}[X]$



- Widely used in finance (portfolio optimization, Harry Markowitz 1952)
- Possible drawback: variance is **symmetric**! penalizes high cost as well as low cost (profit?)

Value-at-Risk (VaR), α -quantile

- Loss/cost r.v. X associated with CDF $F(x)$
- For any $\alpha \in [0, 1]$, the VaR on α is defined by

$$\mathbf{VaR}_\alpha(X) = q_\alpha(X) = \inf \{x \in \mathbb{R} : F(x) \geq \alpha\}$$

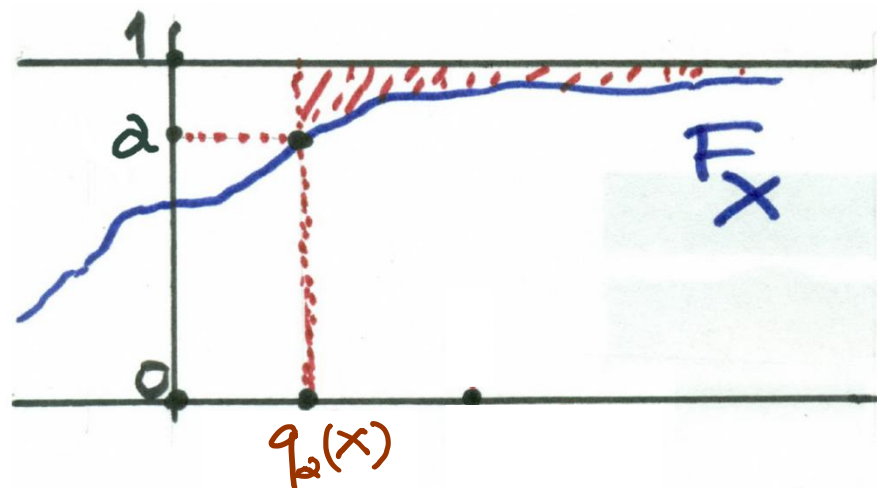
- For **continuous** CDF, $F(\mathbf{VaR}_\alpha) = \alpha$
- For **strictly increasing and continuous** CDF

$$\mathbf{VaR}_\alpha = F^{-1}(\alpha)$$

- Then

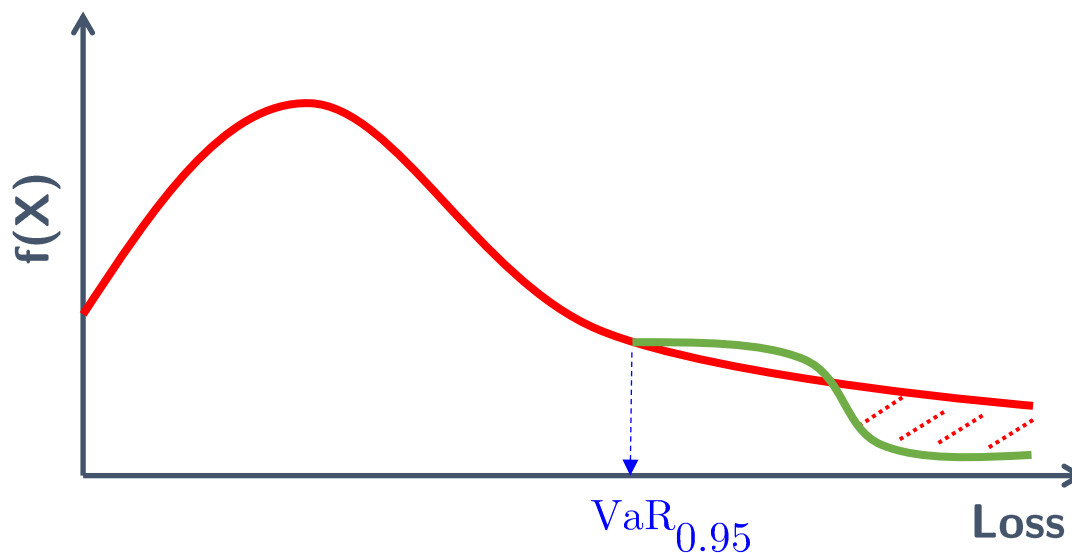
$$\mathcal{R}(X) = \mathbf{VaR}_\alpha(X) \leq b$$

$$\Leftrightarrow \mathbb{P}(X \leq b) \geq \alpha$$



Value-at-Risk (VaR), α -quantile

- Widely used (in Finance)
 - VaR_α pays no attention to the magnitude of losses when the rare event of experiencing a loss above the level VaR_α occurs
 - Bad mathematical & computational behavior: nonconvex, nonlinear
- J.P. Morgan 95%
 - Bank of America 95%
 - Citibank 95.4%
 - Chase Manhattan 97.5%
 - Basel Committee on Bank Supervision 99%



- For $\alpha \in [0, 1]$, the CVaR on α is defined by

$$\mathbf{CVaR}_\alpha(X) = Q_\alpha(X) = \mathbb{E}[X | X \geq \mathbf{VaR}_\alpha(X)]$$

- Or, equivalently

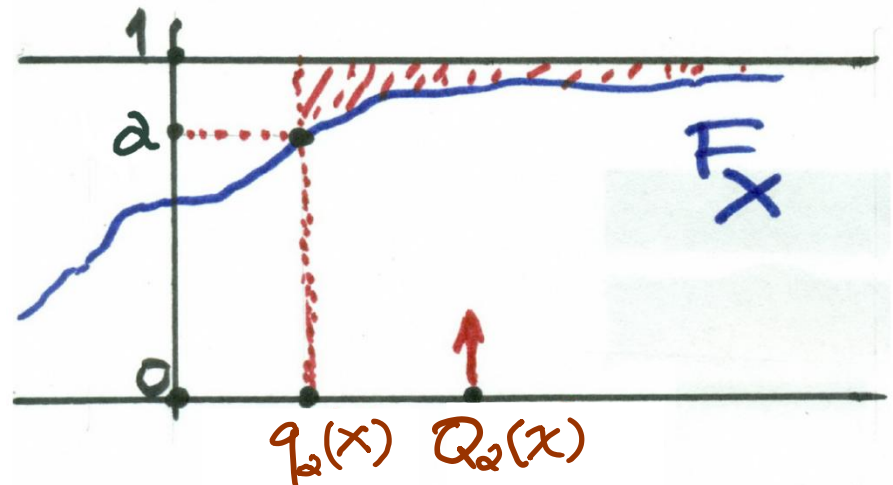
$$\mathbf{CVaR}_\alpha(X) = \frac{1}{1 - \alpha} \int_{x \geq \mathbf{VaR}_\alpha} x dF(x)$$

- Good mathematical behaviors

- Measure the

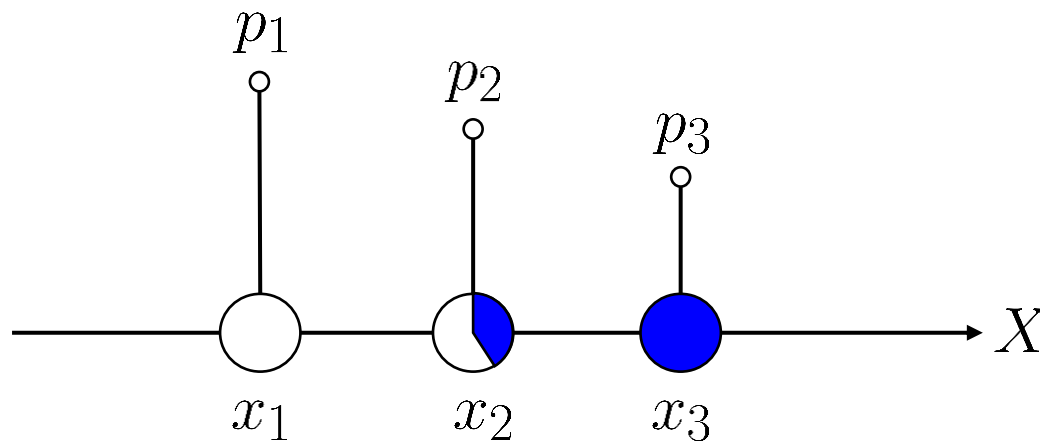
$$\mathcal{R}(X) = Q_\alpha(X) \leq b$$

$\Leftrightarrow X \leq b$ on average in
upper α -tail



- For discrete distributions:

$$\text{CVaR}_\alpha(X) = \frac{1}{1-\alpha} \left[\sum_{x_k > \text{VaR}_\alpha} p_k x_k + \underbrace{\text{VaR}_\alpha \cdot (F(\text{VaR}_\alpha) - \alpha)}_{\text{handles the case where jumps over } \alpha \text{ exactly at VaR}} \right]$$



$\text{VaR}_\alpha = x_2$ but $F(x_2) > \alpha$

A simple numeric example

- Annual maintenance cost X (k€) with 5 discrete outcomes

Cost x_k (k€)	p_k	$F(x_k)$
5	0.50	0.50
15	0.25	0.75
40	0.15	0.90
80	0.08	0.98
150	0.02	1.00

- Expectation: $\mathcal{R}_{\text{mean}} = \mathbb{E}[X] = 21.65$
- Worst case: $\mathcal{R}_{\text{WC}} = \sup X = 150$
- Std. deviation: $\mathbb{V}[X] \approx 802$
 $\Rightarrow \mathcal{R}_{\text{MV}}(\lambda = 0.1) = \mathbb{E}[X] + 0.1 \cdot \mathbb{V}[X] \approx 101.85$

A simple numeric example

- $\text{VaR}_{0.90}(X) = \inf\{x : F(x) \geq 0.90\}$
 $= 40$
- With 90% confidence, cost ≤ 40 , but what happens in the other 10%?
- VaR assigns **the same number** whether the tail contains 80 or 150 outcomes

Cost x_k (k€)	p_k	$F(x_k)$
5	0.50	0.50
15	0.25	0.75
40	0.15	0.90
80	0.08	0.98
150	0.02	1.00

- $\text{VaR}_{0.85}(X) = \inf\{x : F(x) \geq 0.85\} = 40$
 $\text{VaR}_{0.95}(X) = \inf\{x : F(x) \geq 0.95\} = 80$

A simple numeric example

Cost x_k (k€)	p_k	$F(x_k)$
5	0.50	0.50
15	0.25	0.75
40	0.15	0.90
80	0.08	0.98
150	0.02	1.00

$$\text{CVaR}_{0.90} = \frac{1}{0.1} [0.08 \times 80 + 0.02 \times 150 + 40 \times (0.9 - 0.9)] = \mathbf{94}$$

$$\text{CVaR}_{0.85} = \frac{1}{0.15} [0.08 \times 80 + 0.02 \times 150 + 40 \times (0.9 - 0.85)] = \mathbf{76}$$

$$\text{CVaR}_{0.95} = \frac{1}{0.05} [0.02 \times 150 + 80 \times (0.98 - 0.95)] = \mathbf{108}$$

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Stochastic DM problems for resilience

- Maintenance planning of wind farms
 - Maintenance time, spare parts
 - **Uncertain** lifetimes (RUL) of items
- Mitigate flooding risk of an NPP
 - Dike, what height?
 - **Uncertain** flooding level & outcomes
- Design resilient supply chain under unexpected perturbations (e.g., Covid)
 - Backup/secondary suppliers? inventory?
 - **Uncertain** perturbation type, frequency, outcomes



A simple example

- Use **optimization to plan maintenance actions** of a wind farm composed of 10 turbines based on their **RUL forecasts**



- Key question: “**When** should we schedule maintenance actions across time while minimizing cost?” Why:
 - Some turbines are close to failure → act early
 - Resources (technicians/equipment) are limited
 - Other considerations in practice: opportunistic cost, overall system performance requirement, etc.

A super simple example



- **Problem setup:** we manage 10 turbine over 8 time periods (e.g., weeks)
- **Each turbine has**
 - a predicted RUL_i
 - a baseline maintenance cost c_i^p
 - a penalty cost if it fails c_i^f
- **Constraints:**
 - max **2** maintenance slots per period
 - Each is maintained **at most once** in the planning horizon
- **Objective:** schedule maintenance time of each turbine to minimize total maintenance + failure costs

A simple example

- **Decision variables:** the quantities you can control or choose to achieve the best possible outcome

	$t = 1$	2	3	4	5	6	7	$T = 8$
M1	?	?	?	?	?	x	?	?
M2	?	?	?	x	?	?	?	?
...								
M10	?	?	?	?	x	?	?	?

$$x_{it} = \begin{cases} 1 & \text{if turbine } i \text{ is maintained in time } t \\ 0 & \text{otherwise} \end{cases}$$

- Binary decision variables

A simple example



- **Constraints:** on the decision variables
- Maintenance recourses limit per time

$$\sum_{i=1}^{10} x_{it} \leq 2, \forall t = 1, \dots, 8$$

- Maintenance window: each turbine must be maintained at most once in the horizon

$$\sum_{t=1}^8 x_{it} \leq 1, \forall i = 1, \dots, 10$$

A simple example



- **Objective function:** the quantities you can control or choose to achieve the best possible outcome

$$\min \sum_{i=1}^{10} \sum_{t=1}^8 c_i^p x_{it} + \sum_{i=1}^{10} c_i^f \left(1 - \sum_{t=1}^{\text{RUL}_i} x_{it} \right)$$

- First term = total maintenance cost
- Second term = total penalty cost for failure

- The whole model:

$$\begin{aligned} \min_{x_{it}} \quad & \sum_{i=1}^{10} \sum_{t=1}^8 c_i^p x_{it} + \sum_{i=1}^{10} c_i^f \left(1 - \sum_{t=1}^{\text{RUL}_i} x_{it} \right) \\ \text{s.t.} \quad & \sum_{i=1}^{10} x_{it} \leq 2, \quad \forall t = 1, \dots, 8 \\ & \sum_{t=1}^8 x_{it} \leq 1, \quad \forall i = 1, \dots, 10 \\ & x_{it} \in \{0, 1\}, \quad \forall i = 1, \dots, 10, t = 1, \dots, 8 \end{aligned}$$

- **Key elements:**

- **Decision variables:** the quantities you can control or choose (e.g., maintenance time, height of a dike, supplier selection, etc.)
- **Constraints:** on decision variables (physical, operational, economical, etc.)
- **Objective function:** rewards (economic gain, system performance) or costs/loss
- **Uncertainty:** in the objective and constraint function parameters

- Often classified according to the types of the **decision variables**, **constraints**, and the **objective function**

$$\begin{aligned} \min_{x_{it}} \quad & \sum_{i=1}^{10} \sum_{t=1}^8 c_i^p x_{it} + \sum_{i=1}^{10} c_i^f \left(1 - \sum_{t=1}^{\text{RUL}_i} x_{it} \right) \\ \text{s.t.} \quad & \sum_{i=1}^{10} x_{it} \leq 2, \quad \forall t = 1, \dots, 8 \\ & \sum_{t=1}^8 x_{it} \leq 1, \quad \forall i = 1, \dots, 10 \\ & x_{it} \in \{0, 1\}, \quad \forall i = 1, \dots, 10, t = 1, \dots, 8 \end{aligned}$$

- **Integer (binary) linear program**: decision variables are **integers**, constraint and objective functions are **linear**

Model classification



Decision variable	Continuous LP, NLP, QP, SDP	Discrete (integer) IP, ILP, BIP	Mixed-integer MILP, MINLP
Constraint structure	Unconstrained gradient descent, L-BFGS, Newton		Constrained equality / inequality / bounds
Function form	Linear (LP) all functions linear	Quadratic (QP) quad. obj. + lin. cst.	Nonlinear (NLP) SOCP, SDP, general NL
Convexity	Convex global optimum guaranteed	Nonconvex local optima, NP-hard in general	
Uncertainty in parameters	Deterministic fixed parameters	Stochastic (SP) random scenarios	Robust / DRO worst-case / ambiguity set

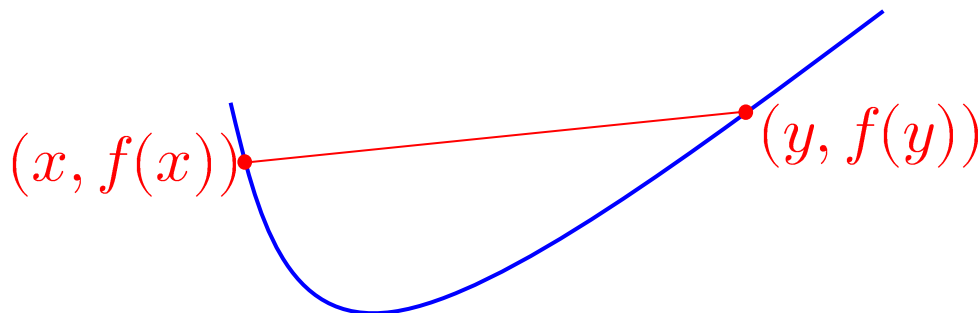
- **General optimization problem**
 - Very **difficult** to solve
 - Methods involve **compromise**, e.g., very long computation time, or not always finding the optimal solution
- **Convex optimization problems:** the **clearest dividing line** between efficiently solvable problems and numerical problems for which we have no hope to solve it easily, e.g.
 - Least square (LS) problems
 - Linear programming (LP) problems

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

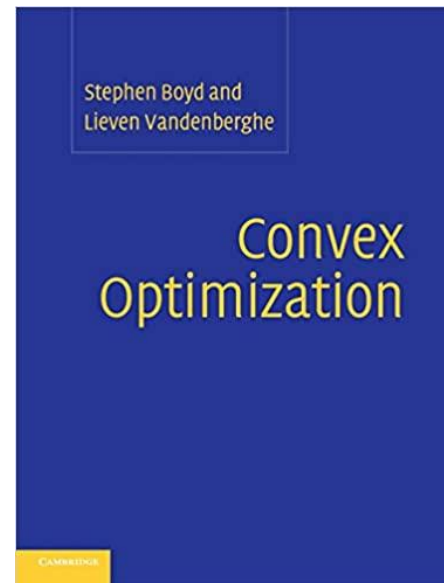
- **Objective and constraint functions** are convex:

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

For all $x, y \in \text{dom } f$, $0 \leq \theta \leq 1$

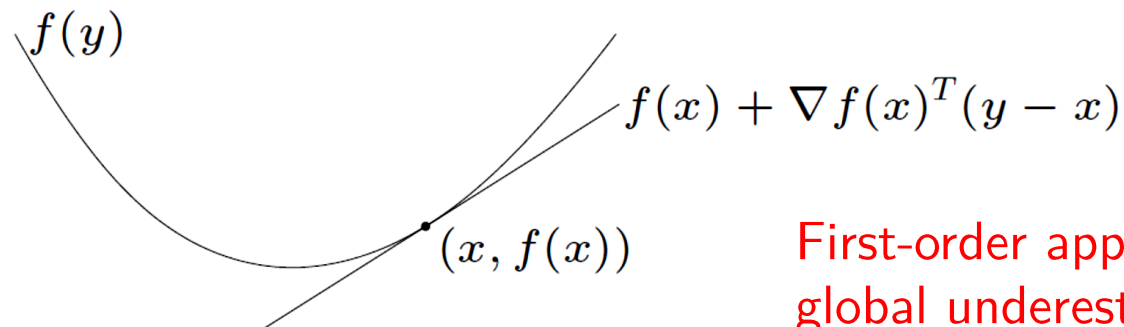


- **Solving convex optimization**
 - No analytical solution
 - **Reliable and efficient** algorithms
 - Almost a technology
- **Using convex optimization**
 - Often **difficult** to recognize
 - Many tricks for transforming problems into convex form
 - Surprisingly, many problems can be solved via convex optimization



- **First-order condition:** differentiable f with convex domain is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom} f$$



First-order approximation of f is global underestimator

- **Theorem:** any locally optimal point of a convex problem is (globally) optimal

$$\min_x \|Ax - b\|_2^2 = \sum_{i=1}^k (a_i^\top x - b_i)^2$$

- **Solving LS problems:**

- Analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- Reliable and efficient algorithms and software
- A **mature technology**

- **Using LS:**

- LS problems are easy to recognize

$$\begin{aligned} \min_x \quad & f_0(x) = c^\top x \\ \text{s.t.} \quad & f_i(x) = a_i^\top x - b_i \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- **Solving LP:**

- No analytical formula for solution
- Reliable and efficient algorithms and software
- A **mature technology** a mature technology, especially for problems of reasonable size

- **Using LP:**

- Not as easy to recognize as LS problems, e.g., problems involving piecewise-linear functions

- Risk measures
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- A stand optimization problem with uncertainty

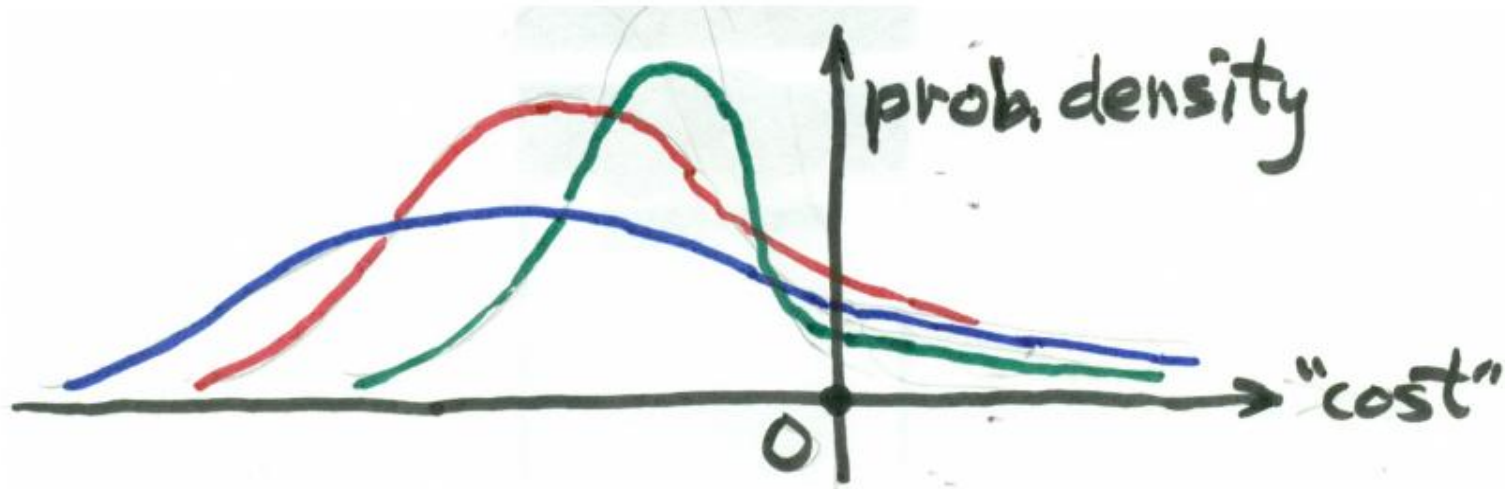
$$\begin{aligned} \min_x \quad & f_0(x, \xi) \\ \text{s.t.} \quad & f_i(x, \xi) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- Decision x must be made with **uncertain** parameter ξ
- Objective $f_0(x, \xi)$ cost/loss function (to be minimized), depending on our decision x , and the realization of uncertain nature ξ
- Same to the constraint functions $f_i(x, \xi)$

- Example:
 - x the height of a dike to be constructed
 - ξ = heights of the possible flooding
 - $f_0(x, \xi)$ the economic loss associated with a given (x, ξ)
- How to model the uncertainty ξ ?
- The Stochastic Programming (SP) approach
 - $\xi \in \Xi$ = random variable with known probability law
 - Then, $f_i(x, \xi)$ are **random variables** for any given x

Key issue in problem formulation

- The distribution of $f_i(x, \xi)$ can only be shaped by the choice of x
- How then can **constraints and minimization** be understood?



- Outcome < 0 , if any, correspond to “reward/profit”

Systematic prescription

- Articulate r.v. $f(x, \xi)$ numerically as $\mathcal{R}[f(x, \xi)]$ for a chosen risk measure $\mathcal{R}(\cdot)$
- **Constraints:** keeping $f_i(x, \xi)$ “acceptably” $\leq b_i$
modeled as: constraint $\mathcal{R}[f_i(x, \xi)] \leq b_i$
- **Objective:** keeping $l_0(x, \xi)$ “as reasonably low as possible”
modeled as: minimizing $\mathcal{R}[f_0(x, \xi)]$
- **Key assumption:** the probability distribution $\mathbf{P}$ of ξ is known
(e.g., estimated from historical data)

- **Objective** is small **on average**

$$\min_x F_0(x) = \mathbb{E}[f_0(x, \xi)]$$

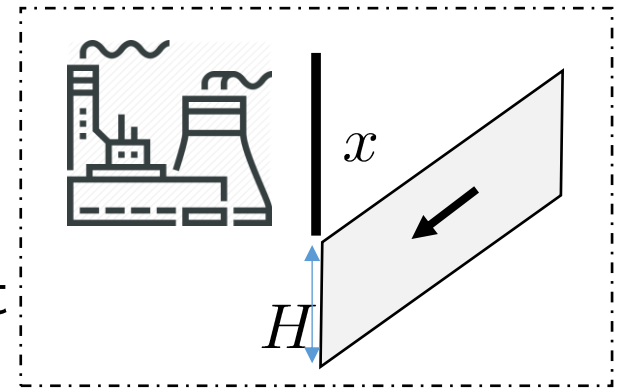
- **Constraints** are satisfied **on average**

$$F_i(x) = \mathbb{E}[f_i(x, \xi)] \leq 0, \quad i = 1, \dots, m$$

- The expectation is suitable for “soft” constraint, e.g., quality of service, whose violation is tolerable
- F_i have analytical expressions only in very few cases; mostly, need to solve the problem approximately, e.g., **sample average approximation (SAA)** (Monte Carlo sampling)

Problem: protect a power plant from potential river flooding by building a dike with a height x meter

- Linear investment cost $c \cdot x$ (annualized)
- **Uncertain** annual maximum flooding water height H
- Annual loss is linear related to the height of overflow $H - x$ when it's positive with **uncertain** coefficient b
- **Objective:** minimize investment cost + expected economic loss



$$\begin{aligned} \min_x \quad & cx + \mathbb{E}[b(H - x)_+] \\ \text{s.t.} \quad & x \geq 0 \end{aligned}$$

- Risk measures
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- **Sampled average approximation**
- Risk-averse stochastic optimization

$$\begin{aligned} \min_x \quad & L(x) = \mathbb{E}[l(x, \xi)] \\ \text{s.t.} \quad & F_i(x) = \mathbb{E}[f_i(x, \xi)] \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- **Idea:** replace the expectation with a **sampled model** when the support Ξ is continuous
- Generate N i.i.d samples $\xi_1, \xi_2, \dots, \xi_N$, each with $p_j = 1/N$
- Now solve the finite event **deterministic SAA problem**

$$\begin{aligned} \min_x \quad & L_N(x) = \sum_{j=1}^N p_j l(x, \xi_j) \\ \text{s.t.} \quad & F_{Ni}(x) = \sum_{j=1}^N p_j f_i(x, \xi_j) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

- Now, a deterministic optimization model

- Denote by L_N^* the optimal value of the SAA problem and $x_N^* \in O_N^*$ an optimal solution (set)
- How good is the quality of the sampled SAA solution
- Does it approach the true optimal value L^* and the true set of optimal solution O^* (as N increases)?
- How fast?
- How to validate the approximation in practice?

- Annual cost function of the dike design problem

$$\begin{aligned} \min_x \quad & 0.9x + \mathbb{E}[b(H - x)_+] \\ \text{s.t.} \quad & 0 \leq x \leq 3 \end{aligned}$$

$$H \sim \text{Gamma}(8, 4.5) \quad b \sim \text{Lognorm}(0.7, 0.16^2)$$

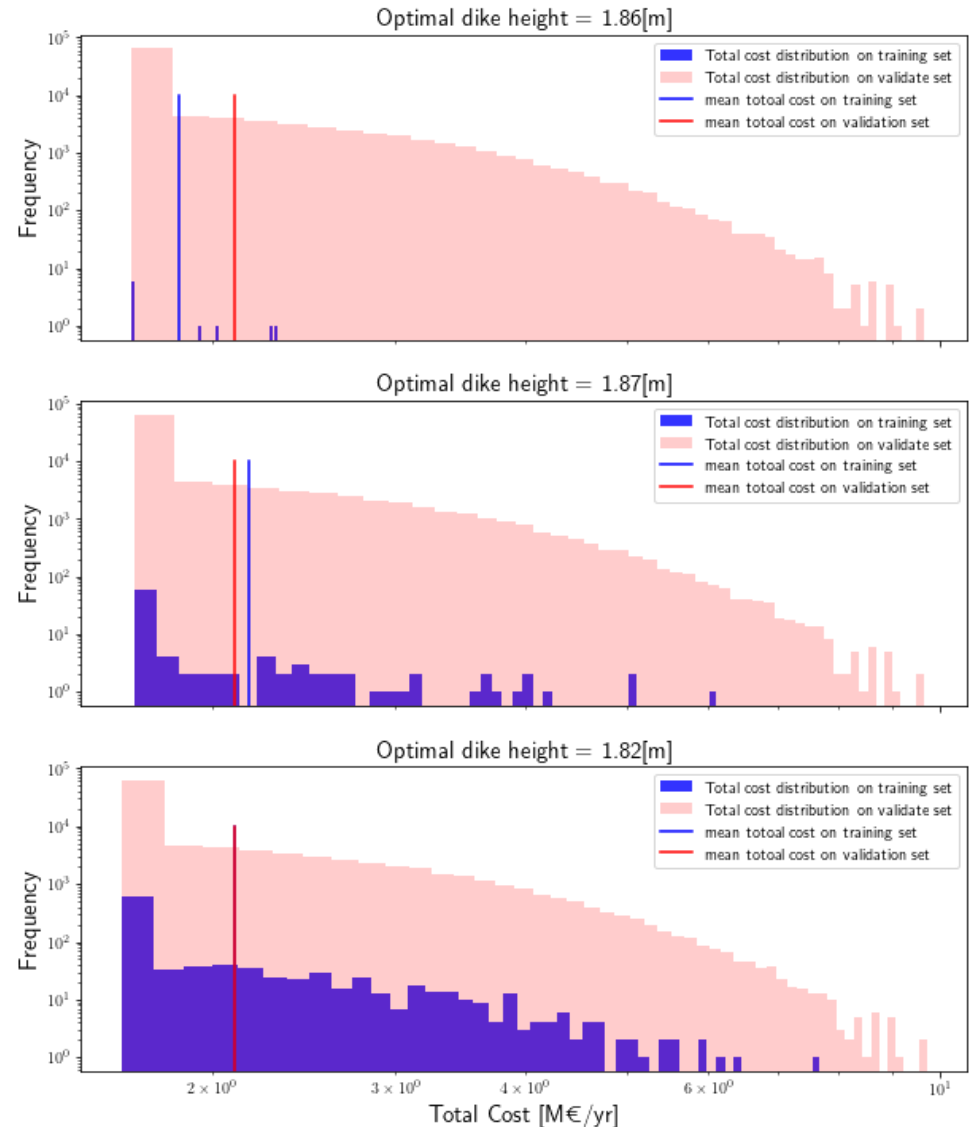
$$\begin{aligned} \min_x \quad & 0.9x + \frac{1}{N} \sum_{i=1}^N [b_i(H_i - x)_+] \\ \text{s.t.} \quad & 0 \leq x \leq 3 \end{aligned}$$

- We solve SAA with $N = 10, 100, 1000$, validation set uses $M=100k$
- Solved by **LP** (piecewise min can be transformed to LP)

SAA and out-of-sample example

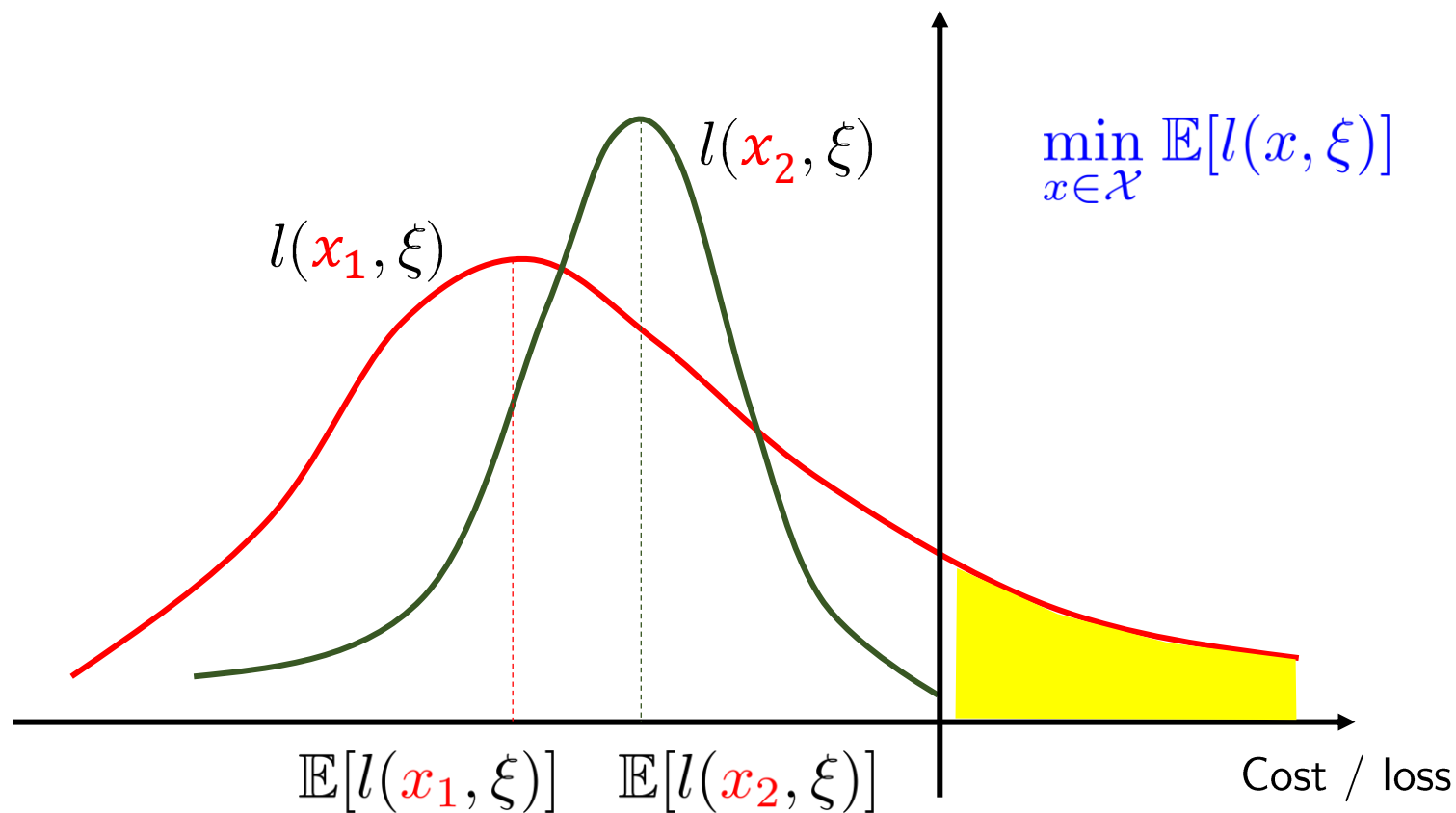
N	10	100	1000
$x_N^\star$	1.86	1.87	1.82
C_N	1.859	2.172	2.101
C^{val}	2.104	2.106	2.102

- $N \geq 1000$ is probably fine
- Computational cost increases very quickly along with N
 - Better sampling
 - Decomposition



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- To control of the **tail risk** in risk averse situations



- Let $l(x, \xi)$ represent a random loss function by definition:

$$\mathbf{VaR}_\alpha[l(x, \xi)] = \inf \{ \gamma | \mathbb{P}[l(x, \xi) \leq \gamma] \geq \alpha \}$$

- **Chance constraint:**

$$\mathbf{VaR}_\alpha[l(x, \xi)] \leq b \Leftrightarrow \mathbb{P}[l(x, \xi) \leq b] \geq \alpha$$

- VaR in the **objective function**

$$\begin{aligned} \min_x \mathbf{VaR}_\alpha[l(x, \xi)] &\Leftrightarrow \min_{x, \gamma} \gamma \\ &\text{s.t. } \mathbb{P}[l(x, \xi) \leq \gamma] \geq \alpha \end{aligned}$$

- **In general**, chance constraints are **nonconvex and non-smooth**,
very difficult to optimize

- For $\alpha \in [0, 1]$, the CVaR on α is defined by

$$\mathbf{CVaR}_\alpha(X) = \mathbb{E}[X | X \geq \mathbf{VaR}_\alpha(X)] = \frac{1}{1 - \alpha} \int_{x \geq \mathbf{VaR}_\alpha} x dF(x)$$

- Computing CVaR generally requires the computation of VaR
- CVaR-minimization might be **even harder** than VaR-minimization problem?

Under quite reasonable modeling assumptions, the opposite is true because that CVaR is coherent!

VaR vs CVaR in risk optimization

CVaR is **superior** to VaR in risk optimization

- VaR is difficult to optimize numerically when losses are not normally distributed
- CVaR_α **preserves convexity** and can be expressed as a minimization formulation
- $\text{CVaR}_\alpha \geq \text{VaR}_\alpha$ always holds, CVaR is a **conservative approximation of VaR**
- Able to control the **expected** tail risk

- Recall $l(x, \xi)$ represents a random loss function
- Consider the auxiliary function

$$\begin{aligned} F_\alpha(x, \gamma) &= \gamma + \frac{1}{1-\alpha} \mathbb{E} \{ [l(x, \xi) - \gamma]_+ \} \\ &= \gamma + \frac{1}{1-\alpha} \int_{\xi \in \Xi} [l(x, \xi) - \gamma]_+ p(\xi) d\xi \end{aligned}$$

Theorem

1. For any fixed x , the function $\gamma \mapsto F_\alpha(x, \gamma)$ is convex.
2. $\mathbf{VaR}_\alpha(x)$ is a minimizer of the problem $\min_\gamma F_\alpha(x, \gamma)$.
3. $F_\alpha(x, \mathbf{VaR}_\alpha(x)) = \mathbf{CVaR}_\alpha(x)$.

The theorems imply

$$\begin{aligned}\min_{x \in \mathcal{X}} \mathbf{CVaR}_\alpha[l(x, \xi)] &\Leftrightarrow \min_{x, \gamma \in \mathcal{X} \times \mathbb{R}} F_\alpha[l(x, \xi), \gamma] \\ &\Leftrightarrow \min_{x, \gamma \in \mathcal{X} \times \mathbb{R}} \gamma + \frac{1}{1 - \alpha} \mathbb{E} \{[l(x, \xi) - \gamma]_+\}\end{aligned}$$

- If the loss function $l(x, \xi)$ is convex in x , then $F_\alpha(x, \gamma)$ is convex in x and γ , the problem is **convex optimization** that can be well solved
- If (x^*, γ^*) minimizes $F_\alpha(x, \gamma)$ over $\mathcal{X} \times \mathbb{R}$, then $F_\alpha(x^*, \gamma^*)$ is the optimal value of the CVaR minimization problem.

- If CVaR is in the constraint

$$\mathbf{CVaR}_\alpha[l(x, \xi)] \leq L \Leftrightarrow \exists \gamma \in \mathbb{R}, F_\alpha[l(x, \xi), \gamma] \leq L$$

- Thus,

$$\begin{cases} \min_{x \in \mathcal{X}} & f_0(x) \\ \text{s.t.} & \mathbf{CVaR}_\alpha[l(x, \xi)] \leq L \end{cases} \Leftrightarrow \begin{cases} \min_{x, \gamma \in \mathcal{X} \times \mathbb{R}} & f_0(x) \\ \text{s.t.} & F_\alpha[l(x, \xi), \gamma] \leq L \end{cases}$$

- Convex optimization when $l(x, \xi)$ is convex in x

- In the **discrete case** (e.g., sampling)

$$F_{\alpha}(x, \gamma) = \gamma + \frac{1}{1 - \alpha} \sum_{k=1}^N p_k [l(x, \xi^k) - \gamma]_+$$

- Replace $[l(x, \xi^k) - \gamma]_+$ by additional variables z^k ,

$$\begin{aligned} \min_{x \in \mathcal{X}} \mathbf{CVaR}_{\alpha}[l(x, \xi)] &\Leftrightarrow \min_{x, \gamma \in \mathcal{X} \times \mathbb{R}} \gamma + \frac{1}{1 - \alpha} \sum_{k=1}^N p_k z^k \\ &\text{s.t. } z^k \geq 0, \forall k \\ &\quad z^k \geq l(x, \xi^k) - \gamma, \forall k \end{aligned}$$

- When $l(x, \xi)$ affine in x , the CVaR minimization is an LP
- When $l(x, \xi)$ convex in x , convex optimization

- Optimal dike construction

$$\begin{aligned} \min_x \quad & 0.9x + \frac{1}{N} \sum_{k=1}^N [b_k (H_k - x)_+] \\ \text{s.t.} \quad & 0 \leq x \leq 3 \end{aligned}$$

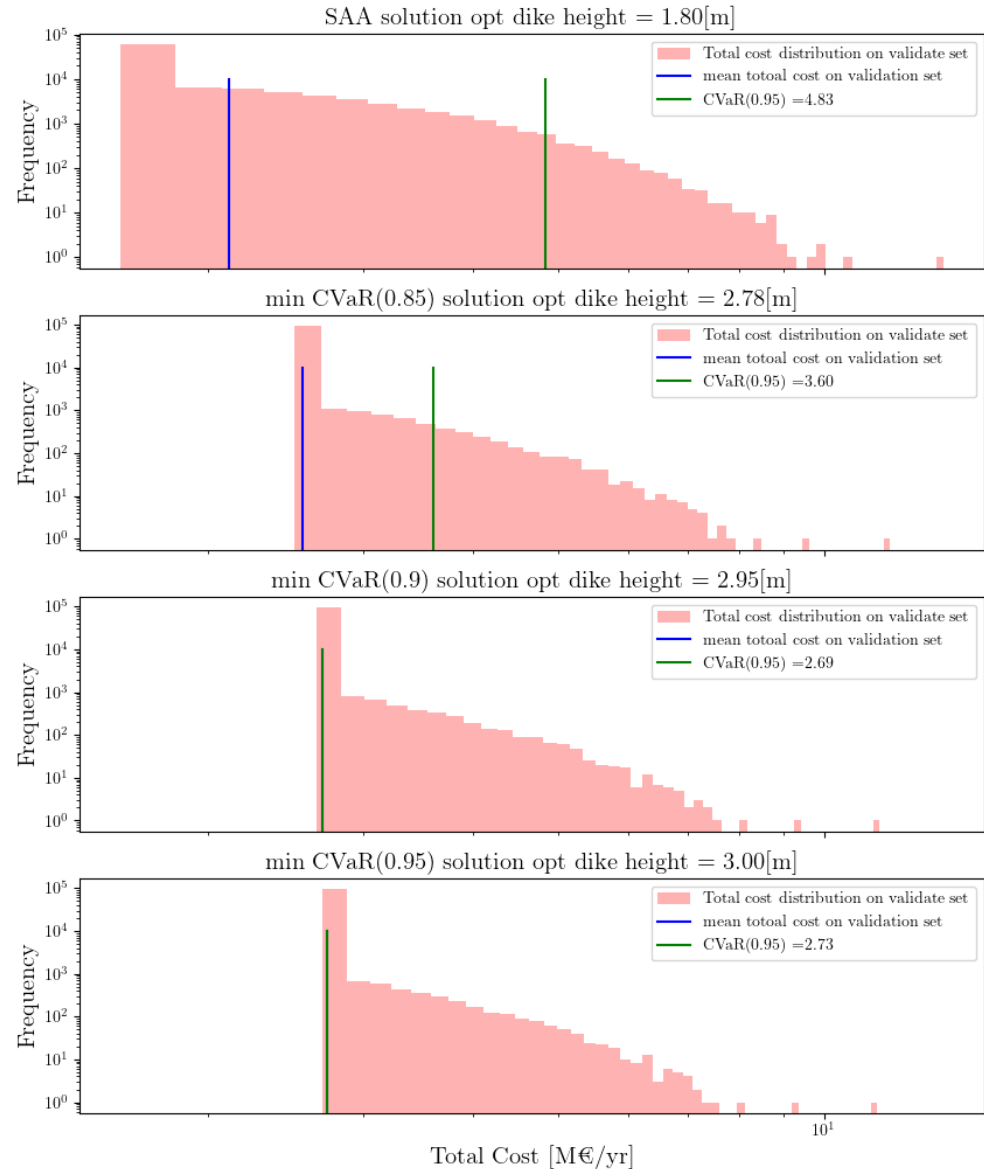
$c = 0.9$, $H \sim \text{Gamma}(8, 4.5)$ and $b \sim \text{Lognorm}(0.7, 0.16^2)$

$$\begin{aligned} \min_{x, \gamma \in [0, 3] \times \mathbb{R}} \quad & \gamma + \frac{1}{(1 - \alpha)N} \sum_{k=1}^N z^k \\ \text{s.t.} \quad & z^k \geq 0, \forall k \\ & h^k \geq H_k - x, h_k \geq 0, \forall k \\ & z^k \geq 0.9x + b_k h_k - \gamma, \forall k \end{aligned}$$

CVaR optimization example



$(N = 2000)$	x [m]
SAA	1.80
CVaR _{0.85}	2.78
CVaR _{0.9}	2.95
CVaR _{0.95}	3.00



- Decision analysis and risk-informed optimization for resilience



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