



Estimation of reliability parameters from experimental data (Part 1)

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In this lecture

- T = failure time is a random variable described by a life distribution $F_T(t)$
- $F_T(t)$ depends on some parameters: $F_T(t; \boldsymbol{\vartheta})$



How to estimate $\boldsymbol{\vartheta}$?

Example: pump with $F_T(t) = 1 - e^{-\lambda t}$



How to estimate λ ?

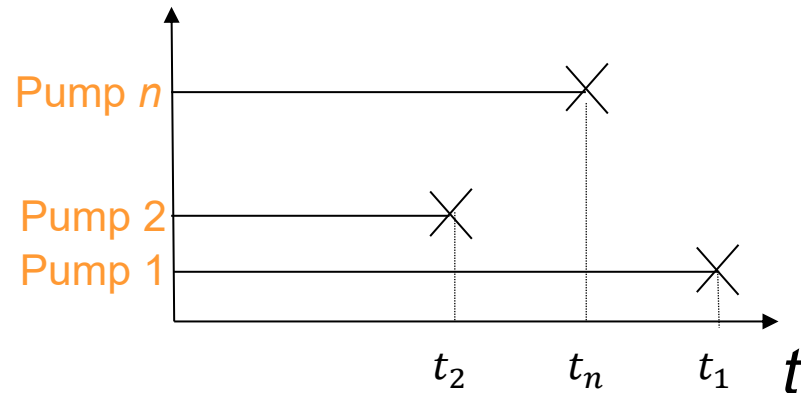
Life tests

- Parameter estimation: ϑ of $F_T(t; \vartheta)$



- Life tests:
 - n identical units are activated and their lifetimes are recorded

Example:



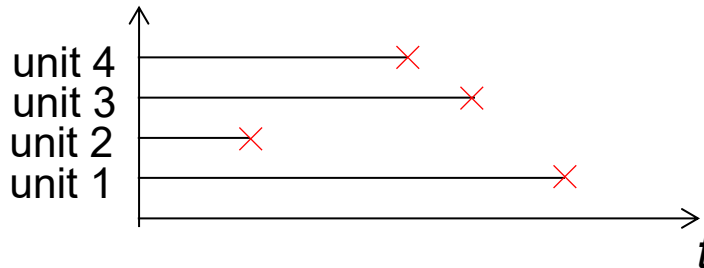
- From $t_1, t_2, \dots, t_n$ to ϑ

Life tests: basic assumptions

- T_i (failure time of i -th unit) are identically distributed
 - Units of the same type
 - same environmental and operational stresses
- T_i are independent
 - components are not affected by the operation or failure of any other component in the set

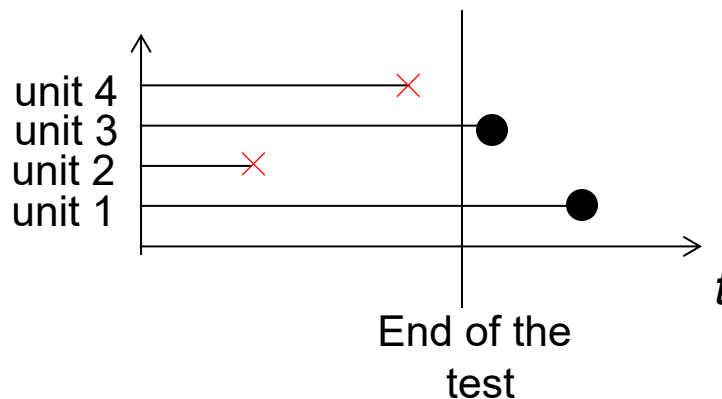
Type of life tests (1)

- **Complete:** the test runs until all the n components have failed



It may be impractical or too expensive to wait until all components have failed!

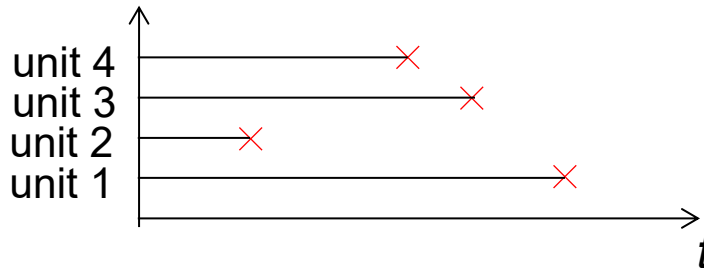
- **Censored*:**
 - **Right censored:** composed also by units that did not fail in the test



*new assumption = censoring mechanism is independent of any information gained from previously failed components in the set

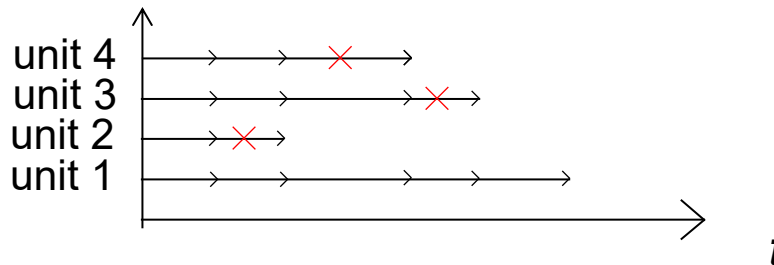
Type of life tests (1)

- **Complete:** the test runs until all the n components have failed



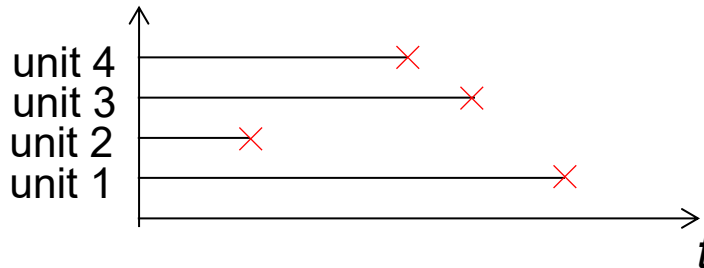
It may be impractical or too expensive to wait until all components have failed!

- **Censored:**
 - **Right censored**
 - **Interval censored:** units are inspected at fixed times (It is not known the exact time of the failure but only that it occurs between two inspections)



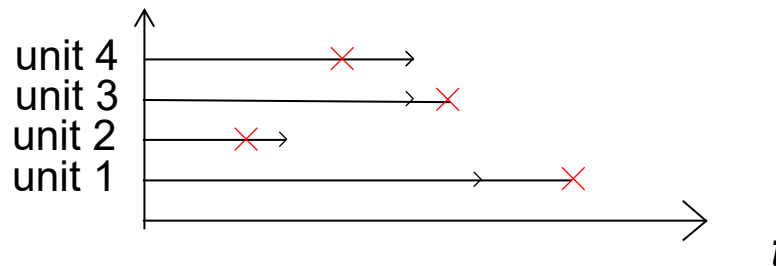
Type of life tests (1)

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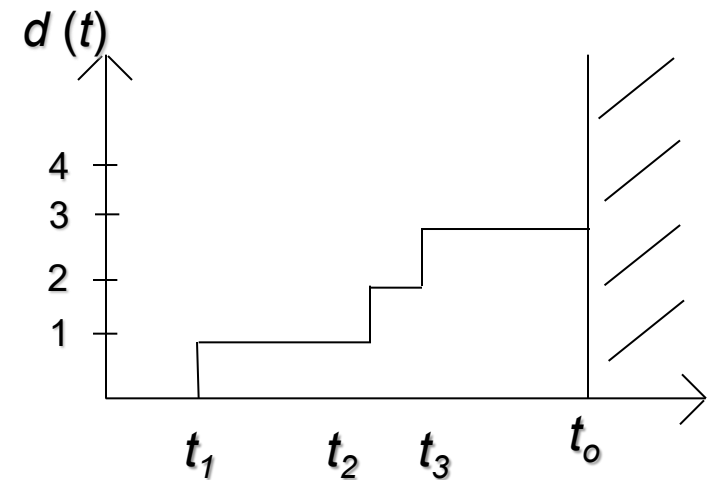
- **Censored:**
 - Right censored
 - Interval censored
 - **Left censored:** only one inspection for each unit (failure between time 0 and the inspection or after the inspection)



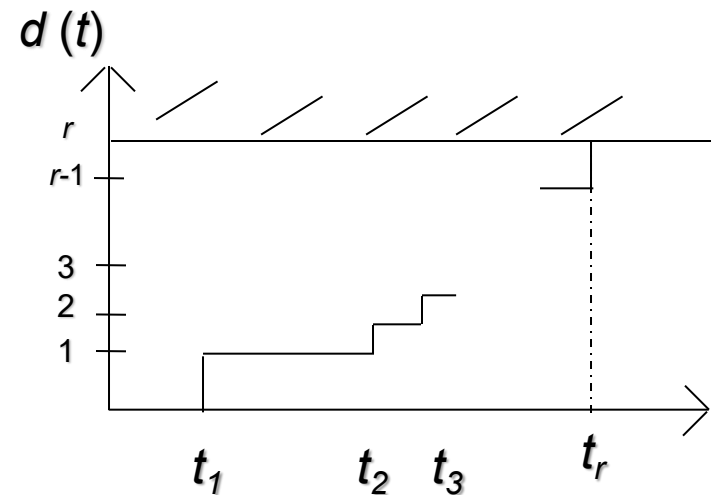
'Terminated' test

- Moment of termination of testing:
 - at fixed t_o (Type 1)

$d(t)$ = number of failures that occur before t



- at the r -th failure (Type 2)



- With Replacement (R) or Without replacement (W) upon unit failure

Estimate Parameters from Observational Data

Life test: $(t_1, t_2, \dots, t_n) \rightarrow$ Estimate ϑ of $f_T(t; \vartheta)$

For example: λ of $f_T(t) = \lambda e^{-\lambda t}$

○ Classical approach:

- ϑ is a fixed unknown parameter
- From $(t_1, t_2, \dots, t_n)$ find an estimator $\hat{\vartheta}$ of ϑ
 - **Point Estimation**
 - Method of moments
 - Method of maximum likelihood
 - **Interval Estimation**
 - Confidence interval

○ Bayesian Approach [NEXT LECTURE]

Classical approach: Point Estimation

Model:

- ϑ is a fixed unknown quantity
- The estimator $\hat{\vartheta}$ of ϑ :

$$\hat{\vartheta} = g(T_1, \dots, T_n)$$

is a random variable (being a function g of the random variables $T_1, \dots, T_n$)

Desiderata:

- Desirable properties of the point estimator $\hat{\vartheta} = g(T_1, \dots, T_n)$:
 - **Unbiased:** $E[\hat{\vartheta}] = \vartheta$ (*Accuracy*)
 - **Consistent:** $\lim_{n \rightarrow \infty} \text{Var}[\hat{\vartheta}] = 0$ (*Asymptotic precision*)

Point Estimation methods:

- The method of moments
- The method of maximum likelihood

The method of moments

1 – True Moment

Moment-Generating Function (mgf): The moment-generating function of a r.v. T with pdf $f(t; \theta)$ is defined for every real number x by

$$\psi_T(x) = E[e^{xT}] = \int_{-\infty}^{\infty} e^{xt} f(t; \theta) dt$$

If the mgf exists for some $|x| > X$, then,

$$\psi_T^{(k)}(0) = \frac{d^k}{dx^k} \psi_T(x) \Big|_{x=0} = \frac{d^k}{dx^k} \left(\int_{-\infty}^{\infty} e^{xt} f(t; \theta) dt \right) \Big|_{x=0} = \int_{-\infty}^{\infty} t^k f(t; \theta) dt = E[T^k]$$

2. Sample moment: Let T be a random variable. Let $t_1, \dots, t_n$ be i.i.d realizations (samples) from T .

The **k th sample moment** of T is

$$M_k = \frac{1}{n} \sum_{i=1}^n t_i^k$$

The method of moments

Method of Moments:

The idea behind Method of Moments (MoM) estimation is that: to find a good estimator, we should have the **true or population** and **(noncentral) sample moments** match as best we can.

$$M_k = \frac{\sum_{i=1}^n (t_i)^k}{n} \quad \text{estimates} \quad E[T^k] = \int_{-\infty}^{+\infty} t^k f(t) dt$$

That is, we should choose the parameters θ (e.g., μ, σ) such that

- the first true moment $E[T] = \int_{-\infty}^{\infty} t f(t; \theta) dt$ is equal to the first sample moment $M_1 = \frac{1}{n} \sum_{i=1}^n t_i$.
- the second true moment $E[T^2] = \int_{-\infty}^{\infty} t^2 f(t; \theta) dt$ is equal to the second sample moment $M_2 = \frac{1}{n} \sum_{i=1}^n t_i^2$.
- ...

The method of moments

1. Moments of random variables are related with the parameter of the distributions

- Example - exponential distribution $f_T(t) = \lambda e^{-\lambda t} \rightarrow E[T] = \frac{1}{\lambda}$

2. Sample moments are used to estimate the corresponding moments of the random variables:

$$M_k = \frac{\sum_{i=1}^n (t_i)^k}{n} \quad \text{estimates} \quad E[T^k] = \int_{-\infty}^{+\infty} t^k f(t) dt$$

- Example - exponential distribution

$$M_1 = \frac{\sum_{i=1}^n t_i}{n} \quad \text{estimates} \quad E[T]$$



- Parameters of the distributions can be estimated by using the sample moments of the random variable

- Example - exponential distribution: $t_1, t_2, \dots, t_n$ are failure times of a component with exponential failure rate

$$\hat{\lambda} = \frac{1}{M_1} = \frac{n}{\sum_{i=1}^n t_i}$$

The method of moments

Method of Moments for normal distribution:

Derive the formulas for the parameter of a normal distribution using the method of moments.

The method of moments - Solution

Method of Moments for normal distribution:

Derive the formulas for the parameter of a normal distribution using the method of moments.

Choose the parameters θ (e.g., μ, σ) such that

• $E[T] = M_1 = \frac{1}{n} \sum_{i=1}^n t_i$ $\Rightarrow$ We know $E[T] = \mu$ $\Rightarrow$ $\mu = \frac{1}{n} \sum_{i=1}^n t_i = \bar{t}$

• $E[T^2] = M_2 = \frac{1}{n} \sum_{i=1}^n t_i^2$ $\Rightarrow$ We know $\sigma^2 = E[(T - \mu)^2] = E[T^2] - (E[T])^2$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n t_i^2 - \bar{t}^2 = \frac{1}{n} \sum_{i=1}^n t_i^2 + \bar{t}^2 - 2\bar{t}^2 = \frac{1}{n} \sum_{i=1}^n t_i^2 + \frac{1}{n} \sum_{i=1}^n \bar{t}^2 - 2\bar{t} \left(\frac{1}{n} \sum_{i=1}^n t_i \right)$$

$$\Rightarrow \sigma^2 = \frac{1}{n} \sum_{i=1}^n (t_i^2 + \bar{t}^2 - 2\bar{t}t_i)$$

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (t_i - \bar{t})^2$$

Method for the estimation of the parameters:

- The method of moments
- The method of maximum likelihood

The likelihood function

- Sample $\mathbf{T} = (T_1, T_2, \dots, T_{n-1}, T_n)$
- Joint pdf (or pmf) of $\mathbf{T}$: $f_{\mathbf{T}}(t_1, \dots, t_n | \theta), \theta \in \Theta$
- Parameter space Θ = set of all possible values that θ can assume
- Observations $\mathbf{t} = (t_1, t_2, \dots, t_{n-1}, t_n)$

Given that $\mathbf{t}$ is **observed**, the **function** of θ defined by

$$L(\theta | \mathbf{t}) = f_{\mathbf{T}}(t_1, \dots, t_n | \theta), \theta \in \Theta$$

is called **likelihood function**

Let $\mathbf{t} = (t_1, \dots, t_n)$ be i.i.d samples from probability mass function $f_{\mathbf{T}}(t | \theta)$, where θ is a parameter (or vector of parameters). We define the likelihood of $\mathbf{t}$ given θ to be the “probability” of observing $\mathbf{t}$ if the true parameter is θ

The likelihood function

Example - exponential distribution

$$L(\theta|t_1, \dots, t_n) = f_T(t_1, \dots, t_n|\theta), \theta \in \Theta$$

i.i.d.
 $T_1, T_2, \dots, T_n \sim \text{Exp}(\lambda)$

$$L(\lambda|t_1, \dots, t_n) = \prod_{i=1}^n \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^n t_i} \quad \lambda > 0$$

Example – Weibull distribution

i.i.d.
 $T_1, T_2, \dots, T_n \sim \text{Weibull}(\alpha, \beta)$

$$L(\lambda|t_1, \dots, t_n) = \prod_{i=1}^n \frac{\beta}{\alpha^\beta} t_i e^{-\left(\frac{t_i}{\alpha}\right)^\beta} = \frac{\beta^n}{\alpha^{\beta n}} \left(\prod_{i=1}^n t_i \right) e^{-\sum_{i=1}^n \left(\frac{t_i}{\alpha}\right)^\beta} \quad \alpha, \beta > 0$$

Example – Poisson distribution

i.i.d.
 $N_1, N_2, \dots, N_n \sim \text{Poisson}(\lambda)$

$$L(\lambda|n_1, \dots, n_n) = \prod_{i=1}^n e^{-\lambda} \frac{\lambda^{n_i}}{n_i!} = e^{-\lambda n} \lambda^{\sum_{i=1}^n n_i} \left(\frac{1}{\prod_{i=1}^n n_i!} \right) \quad \lambda > 0$$

Likelihood principle

Observer 1: $\mathbf{t}^1 = (t_1^1, \dots, t_{n^1}^1) \Rightarrow L(\boldsymbol{\theta}|\mathbf{t}^1)$

Observer 2: $\mathbf{t}^2 = (t_1^2, \dots, t_{n^2}^2) \Rightarrow L(\boldsymbol{\theta}|\mathbf{t}^2)$

Suppose that

$$L(\boldsymbol{\theta}|\mathbf{t}^1) = c(\mathbf{t}^1, \mathbf{t}^2)L(\boldsymbol{\theta}|\mathbf{t}^2)$$

i.e., $L(\boldsymbol{\theta}|\mathbf{t}^1)$ is proportional to $L(\boldsymbol{\theta}|\mathbf{t}^2)$ for less than a constant which does not depend on $\boldsymbol{\theta}$



conclusions drawn from $\mathbf{t}^1$ and $\mathbf{t}^2$ on $\boldsymbol{\theta}$ should be identical

- **Rationale:**

The likelihood is used to compare the **plausibility** of various parameter values

Example

$$L(\boldsymbol{\theta}_1|\mathbf{t}) = 2L(\boldsymbol{\theta}_2|\mathbf{t})$$

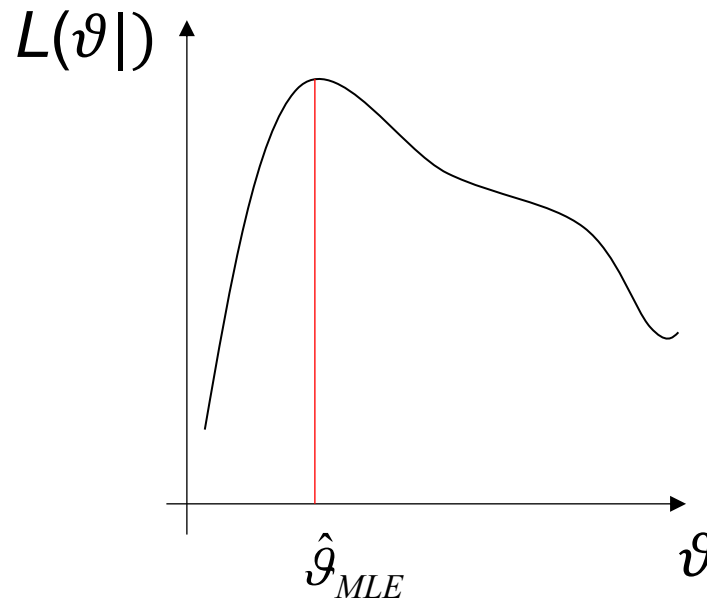
$\boldsymbol{\theta}_1$ is twice as plausible as $\boldsymbol{\theta}_2$

The method of maximum likelihood

- n observations $\{t_1, t_2, \dots, t_n\}$ are available:

$$L(\vartheta|t_1, \dots, t_n) = \prod_{i=1}^n f_T(t_i; \vartheta)$$

- The maximum likelihood estimator $\hat{\vartheta}_{MLE}$ is the value of ϑ which maximizes the likelihood function:



$$\frac{\partial L(\vartheta)}{\partial \vartheta} = 0$$

Example 1: uncensored test

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have observed the failure time of 4 items obtaining:
 - $t_1 = 5d$
 - $t_2 = 7d$
 - $t_3 = 4d$
 - $t_4 = 10d$

Estimate using the method of maximum likelihood the parameter λ

Censoring and likelihood

T time to failure

$T \sim f_T(t|\theta)$: probability density function

$F_T(t|\theta) = \mathbb{P}(T \leq t|\theta)$: cumulative distribution function

$R(t|\theta) = \mathbb{P}(T > t|\theta)$: reliability function

Case 1: Right censoring

The observer only knows that $T > t_1$

Contribute to the likelihood:

$$L(\theta|t_1) = R(t_1|\theta)$$

Case 2: Left censoring

The observer only knows that $T \leq t_1$

Contribute to likelihood:

$$L(\theta|t_1) = F_T(t_1|\theta)$$

Case 3: interval censoring

The observer only knows that $a \leq T \leq b$

Contribute to likelihood:

$$L(\theta|(a, b)) = F_T(b|\theta) - F_T(a|\theta)$$

The method of maximum likelihood applied to test components lifetimes

- Uncensored lifetimes: $t_1, t_2, \dots, t_n$: $L(\vartheta|t_1, \dots, t_n) = \prod_{i=1}^n f_T(t_i|\vartheta)$
- Right-censored data: $L(\vartheta) = \prod_i f_T(t_i|\vartheta) \prod_j R(t_j|\vartheta)$

Failures
↖

Right-Censored
↘
- Generally, one takes $l(\mathcal{G}) = \ln[L(\mathcal{G})]$
- and the estimator is $\hat{\mathcal{G}}$ which maximizes $l(\mathcal{G})$:
$$\frac{\partial l}{\partial \mathcal{G}} = 0$$

- r uncensored observation: failure times $(t_1, t_2, \dots, t_r)$
- $n - r$ observations, each one right-censored at a different time: $(t_{r+1}, t_{r+2}, \dots, t_n)$

$$L(\lambda) = \underbrace{\prod_i f_T(t_i; \lambda)}_{\text{failures}} \cdot \underbrace{\prod_j R(t_j; \lambda)}_{\text{right-censored}} = \lambda^r \cdot e^{-\lambda \sum_{i=1}^r t_i} \cdot e^{-\lambda \sum_{j=r+1}^n t_j}$$

$$= \lambda^r \cdot e^{-\lambda \sum_{k=1}^n t_k}$$

$$l(\lambda) = \ln L(\lambda) = r \ln \lambda - \lambda \sum_{k=1}^n t_k$$

$$\frac{\partial l}{\partial \lambda} = \frac{r}{\lambda} - \sum_{k=1}^n t_k \Rightarrow \hat{\lambda} = \frac{r}{\sum_{k=1}^n t_k} = \frac{r}{TTT}$$

TTT = Total Time
on Test

Example 2: right censored test – Type 1

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the first type with $t_0 = 8d$ and observed the failures of three items at times:
 - $t_1 = 5d$
 - $t_2 = 7d$
 - $t_3 = 4d$

Estimate using the method of maximum likelihood the parameter λ

Example 3: right censored test – Type 1

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the first type with $t_0 = 8d$ **on five items** and observed the failures of three items at times:
 - $t_1 = 5d$
 - $t_2 = 7d$
 - $t_3 = 4d$

Estimate using the method of maximum likelihood the parameter λ

Example 4: right censored test – Type 2

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the second type with $r = 2$ and observed the following failure times:
 - $t_1 = 5d$
 - $t_2 = 4d$

Estimate using the method of maximum likelihood the parameter λ

Example 5: right censored test – Type 2

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the second type on **five** components, with $r = 3$ and observed the following failure times:
 - $t_1 = 5d$
 - $t_2 = 4d$
 - $t_3 = 7d$
- Estimate using the method of maximum likelihood the parameter λ

The method of maximum likelihood – Weibull Distribution

- n observations $\{t_1, t_2, \dots, t_n\}$ are available:

$$L(\theta|t_1, \dots, t_n) = \prod_{i=1}^n f_T(t_i; \theta)$$

where θ is a parameter (or
vector of parameters):

- Find the expressions for parameters

$$f_T(\alpha, \beta) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^\beta}$$

The method of maximum likelihood – Weibull Distribution

- n observations $\{t_1, t_2, \dots, t_n\}$ are available:

$$L(\theta|t_1, \dots, t_n) = \prod_{i=1}^n f_T(t_i; \theta) \quad \text{where } \theta \text{ is a parameter (or vector of parameters):}$$

- The maximum likelihood estimator $\hat{\theta}_{MLE}$ is the value θ of which maximizes the likelihood function:

$$f_T(\alpha, \beta) = \frac{\beta}{\alpha} \left(\frac{t}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t}{\alpha}\right)^\beta}$$
$$L = \prod_{i=1}^n f_T(t_i; \alpha, \beta) = \prod_{i=1}^n \frac{\beta}{\alpha} \left(\frac{t_i}{\alpha}\right)^{\beta-1} e^{-\left(\frac{t_i}{\alpha}\right)^\beta}$$
$$l = \sum_{i=1}^n \left(\ln \beta + (\beta - 1) \ln t_i - \beta \ln \alpha - \left(\frac{t_i}{\alpha}\right)^\beta \right)$$

$$l = n \ln \beta - n\beta \ln \alpha + (\beta - 1) \sum_{i=1}^n \ln t_i - \left(\frac{1}{\alpha}\right)^\beta \sum_{i=1}^n t_i^\beta$$

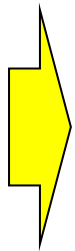
The method of maximum likelihood – Weibull Distribution

$$l = n \ln \beta - n \beta \ln \alpha + (\beta - 1) \sum_{i=1}^n \ln t_i - \frac{1}{\alpha^\beta} \sum_{i=1}^n t_i^\beta$$

$$\frac{\partial l}{\partial \alpha} = -n \frac{\beta}{\alpha} \sum_{i=1}^n \left(\frac{\beta}{\alpha} \frac{1}{\alpha^\beta} t_i^\beta \right) = 0 \rightarrow n \frac{\beta}{\alpha} = \frac{\beta}{\alpha} \frac{1}{\alpha^\beta} \sum_{i=1}^n (t_i^\beta) \quad \Rightarrow \quad \alpha = \left(\frac{\sum_{i=1}^n t_i^\beta}{n} \right)^{\frac{1}{\beta}}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - n \ln \alpha + \sum_{i=1}^n \ln t_i - \frac{1}{\alpha^\beta} \sum_{i=1}^n t_i^\beta \ln \frac{t_i}{\alpha} = 0$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - n \ln \alpha + \sum_{i=1}^n \ln t_i - \frac{n}{\sum_{i=1}^n t_i^\beta} \left(\sum_{i=1}^n t_i^\beta \ln t_i - \ln \alpha \sum_{i=1}^n t_i^\beta \right) = 0$$



$$\frac{n}{\beta} + \sum_{i=1}^n \ln t_i - \frac{n \sum_{i=1}^n t_i^\beta \ln t_i}{\sum_{i=1}^n t_i^\beta} = 0$$

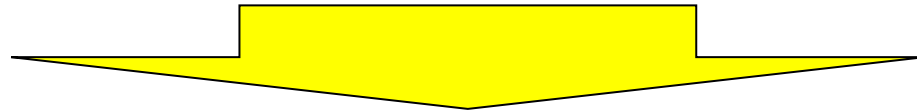
The method of maximum likelihood – Weibull Distribution

$$\alpha = \left(\frac{\sum_{i=1}^n t_i^\beta}{n} \right)^{\frac{1}{\beta}}$$

$$\frac{n}{\beta} + \sum_{i=1}^n \ln t_i - \frac{n \sum_{i=1}^n t_i^\beta \ln t_i}{\sum_{i=1}^n t_i^\beta} = 0$$



$$\beta = \frac{1}{\frac{\sum_{i=1}^n t_i^\beta \ln t_i}{\sum_{i=1}^n t_i^\beta} - \frac{\sum_{i=1}^n \ln t_i}{n}}$$



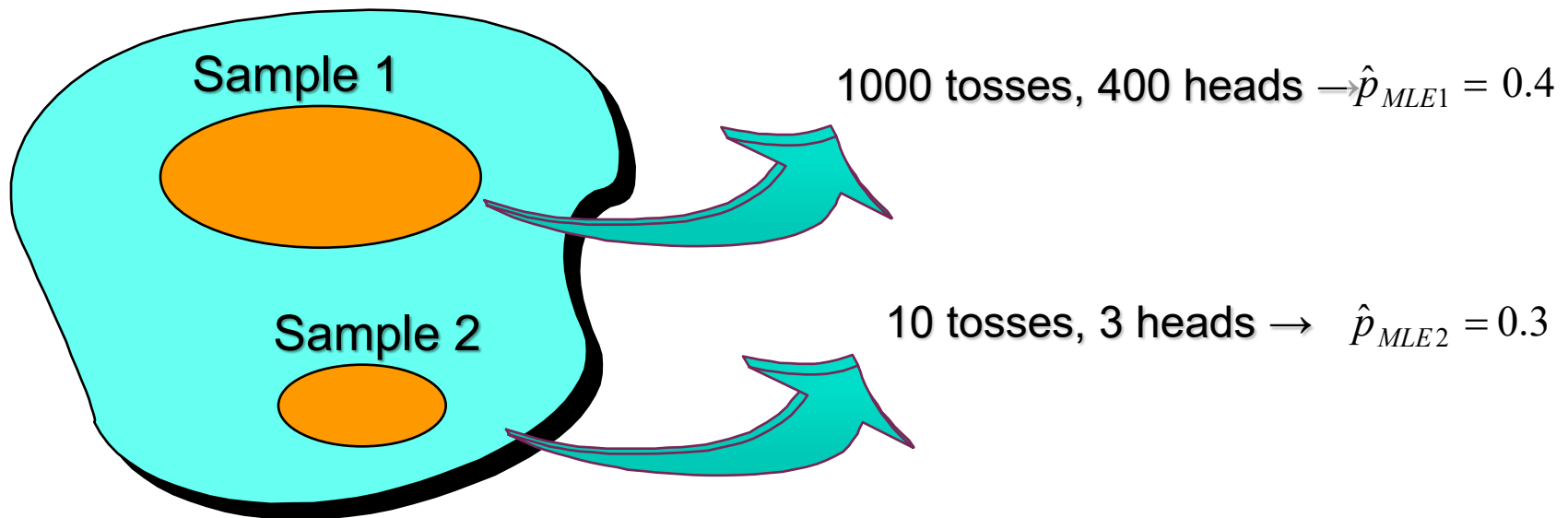
$$\beta_{k+1} = \frac{1}{\frac{\sum_{i=1}^n t_i^{\beta_k} \ln t_i}{\sum_{i=1}^n t_i^{\beta_k}} - \frac{\sum_{i=1}^n \ln t_i}{n}} \quad |\beta_{k+1} - \beta_k| < \varepsilon$$

$$\alpha = \left(\frac{\sum_{i=1}^n t_i^\beta}{n} \right)^{\frac{1}{\beta}}$$

Limitation of the point estimates

Coin toss: binomial process

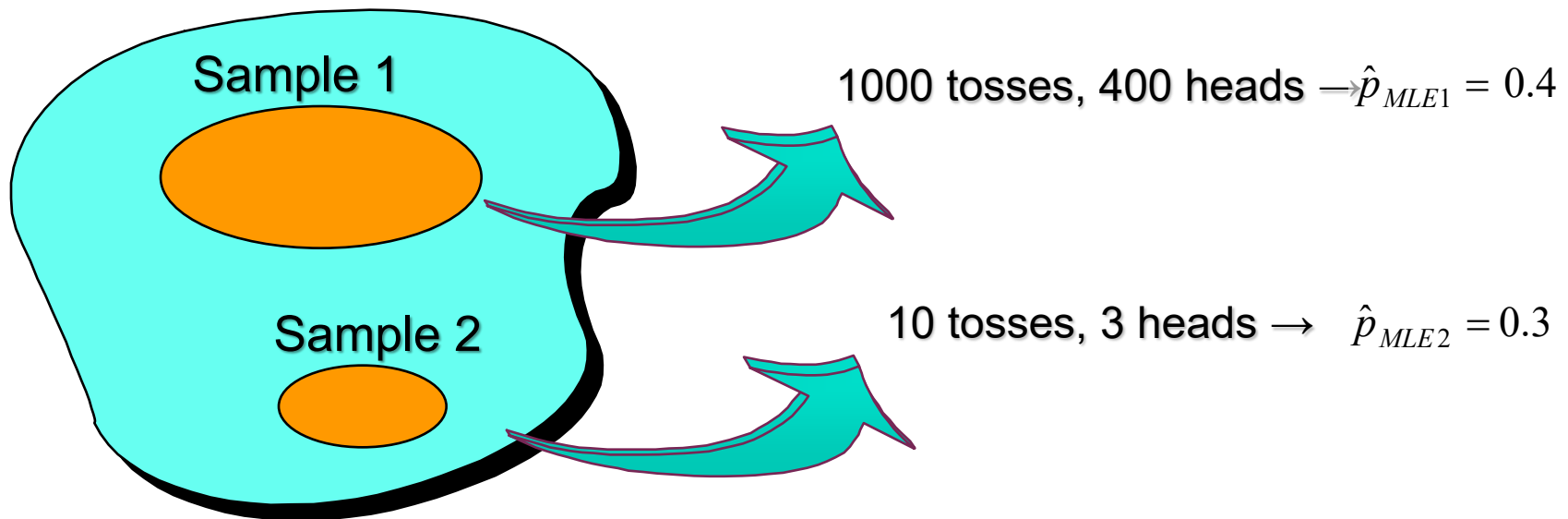
$$\hat{p}_{MLE} = \frac{\# \text{ of successes}}{\# \text{ of trials}}$$



Limitation of the point estimates

Coin toss: binomial process

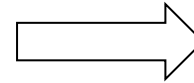
$$\hat{p}_{MLE} = \frac{\# \text{ of successes}}{\# \text{ of trials}}$$



We have more confidence in $\hat{p}_{MLE1} = 0.4$ computed using Sample 1, but **the point estimates $\hat{p}_{MLE}$ do not give a measure of the confidence in the result**

Confidence limits for the reliability parameters

- Degree of confidence in the estimates



~~Point estimates $\hat{g}$~~

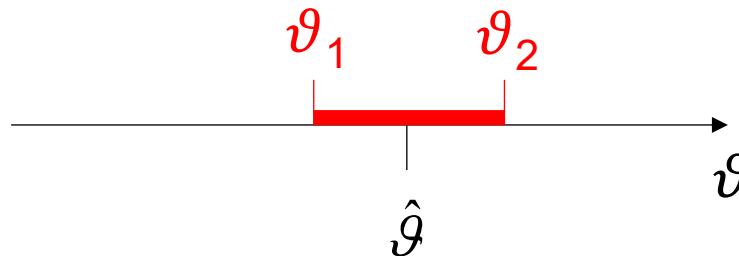
Interval estimation



Confidence Interval



Confidence limits $(\vartheta_1, \vartheta_2)$

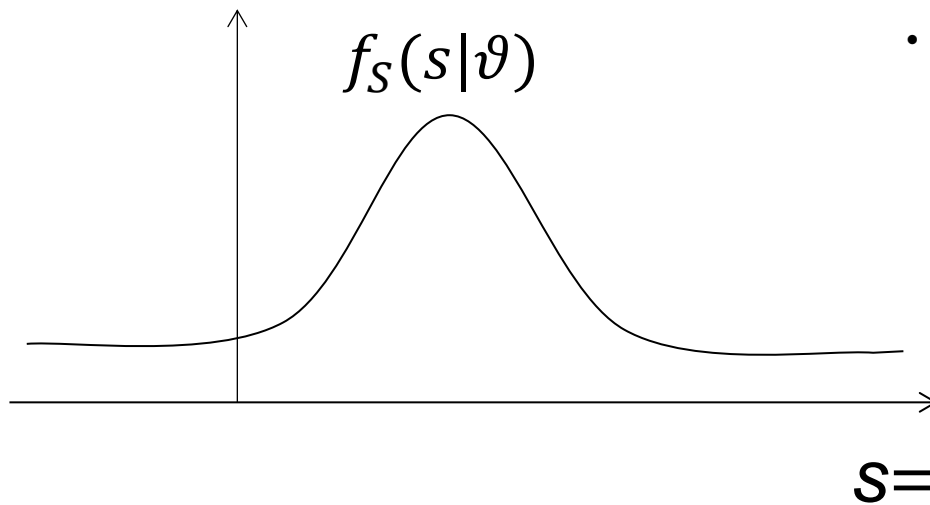


- We fix a value of confidence $\alpha < 1$, and we consider interval $(\vartheta_1, \vartheta_2)$ such that we are α confident that the true value of the parameter ϑ is between $(\vartheta_1, \vartheta_2)$

•

Interval estimates of reliability parameters

- $t_1, t_2, \dots, t_n$ = sample from the population distribution
- ϑ = unknown characteristic of the population,
 - Example: $\vartheta = \mu$ of a normal distribution with known σ : $f_T(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t - \mu)^2}{2\sigma^2}}$
- $S = \hat{\vartheta} = g(t_1, t_2, \dots, t_n)$ = estimated characteristic,
 - Example: $S = \bar{t} = \frac{\sum_{i=1}^n T_i}{n}$
- S is a random variable (being a function of random failure times) $\rightarrow S$ is characterized by a probability density function, $f_S(s|\vartheta)$.



What does this distribution mean?

- I repeat the test over the n components several times:

$$\text{First test: } (t_1^{(1)}, t_2^{(1)}, \dots, t_n^{(1)}) \rightarrow \bar{t}^{(1)} = \frac{\sum_{i=1}^n t_i^{(1)}}{n}$$

$$\text{Second test: } (t_1^{(2)}, t_2^{(2)}, \dots, t_n^{(2)}) \rightarrow \bar{t}^{(2)} = \frac{\sum_{i=1}^n t_i^{(2)}}{n}$$

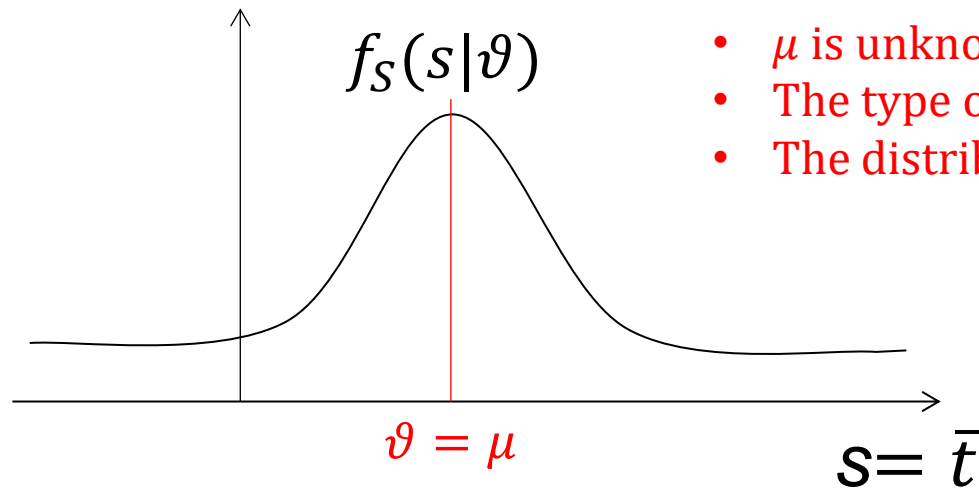
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$$\text{M-th test } (t_1^{(M)}, t_2^{(M)}, \dots, t_n^{(M)}) \rightarrow \bar{t}^{(M)} = \frac{\sum_{i=1}^n t_i^{(M)}}{n}$$

$\bar{t}^{(1)}, \bar{t}^{(2)}, \dots, \bar{t}^{(M)}$ are distributed according to $f_S(s|\vartheta)$

Interval estimates of reliability parameters

- $t_1, t_2, \dots, t_n$ = sample from the population distribution
- ϑ = unknown characteristic of the population,
 - Example: $\vartheta = \mu$ of a normal distribution with known σ
- $S = \hat{\vartheta} = g(t_1, t_2, \dots, t_n)$ = estimated characteristic,
 - Example: $S = \bar{t} = \frac{\sum_{i=1}^n T_i}{n}$
- S is a random variable (being a function of random failure times) $\rightarrow S$ is characterized by a probability density function, $f_S(s|\vartheta)$.
 - Example: Central limit theorem $\rightarrow \bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$



- μ is unknown!
- The type of the distribution (normal) is known!
- The distribution variance is known!

The central limit theorem: For a population with mean μ and standard deviation σ , if we take sufficiently large random samples from the population with replacement, then the *distribution of the sample means* will be approximately normally distributed.

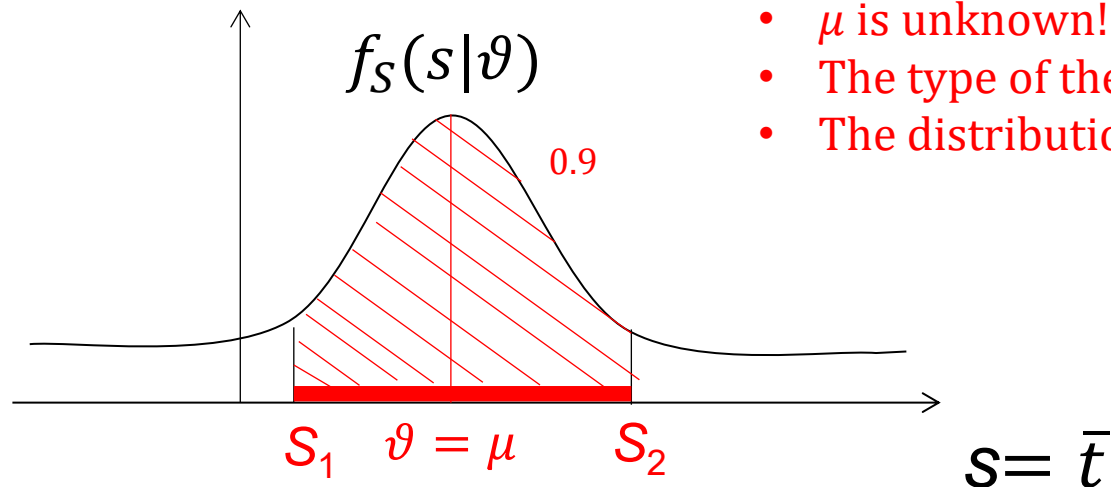
$\bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

Example: confidence limits for the μ of a normal distribution

- Estimate of the confidence interval (S_1, S_2)
 - 90% confidence interval $\rightarrow$ I need the 5 and 95 percentile of the distribution

$$\bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

unknown known



- μ is unknown!
- The type of the distribution (normal) is known!
- The distribution variance is known!

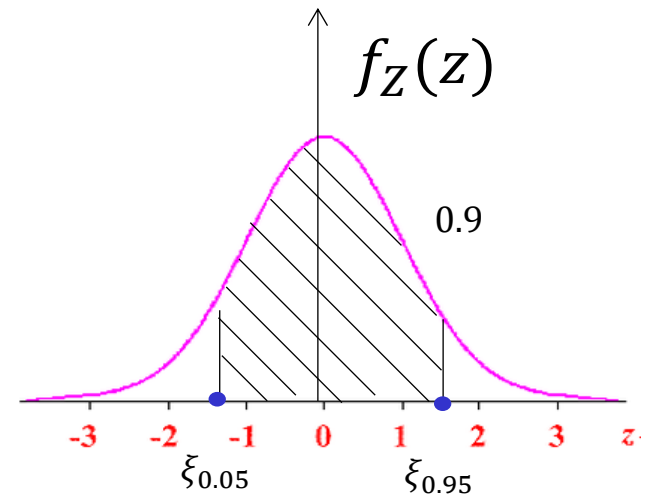
Example: confidence limits for the μ of a normal distribution

- Estimate of the confidence interval (S_1, S_2)
 - 90% confidence interval $\rightarrow$ I need the 5 and 95 percentile of the distribution

$$\bar{t} \sim N\left(\overset{\text{unknown}}{\mu}, \frac{\sigma^2}{n}\right)$$

Standard normal

$$z = \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$



$$P\left(\xi_{0.05} < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < \xi_{0.95}\right) = P\left(-1.645 < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645\right) = 0.9$$

Table of Standard Normal Probability

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\xi^2} d\xi$$

x	$\Phi(x)$
1.50	0.933193
1.51	0.934478
1.52	0.935744
1.53	0.936992
1.54	0.938220
1.55	0.939429
1.56	0.940620
1.57	0.941792
1.58	0.942947
1.59	0.944083
1.60	0.945201
1.61	0.946301
1.62	0.947384
1.63	0.948449
1.64	0.949497
1.65	0.950529
1.66	0.951543
1.67	0.952540
1.68	0.953521
1.69	0.954486
1.70	0.955435
1.71	0.956367
1.72	0.957284
1.73	0.958185
1.74	0.959071
1.75	0.959941
1.76	0.960796
1.77	0.961636
1.78	0.962426
1.79	0.963273
1.80	0.964070
1.81	0.964852
1.82	0.965621
1.83	0.966375
1.84	0.967116
1.85	0.967843
1.86	0.968557
1.87	0.969258
1.88	0.969946
1.89	0.970621
1.90	0.971284
1.91	0.971933
1.92	0.972571
1.93	0.973197
1.94	0.973810
1.95	0.974412
1.96	0.975002
1.97	0.975581
1.98	0.976148
1.99	0.976705

x	$\Phi(x)$
2.00	0.977250
2.01	0.977784
2.02	0.978308
2.03	0.978822
2.04	0.979325
2.05	0.979818
2.06	0.980301
2.07	0.980774
2.08	0.981237
2.09	0.981691
2.10	0.982136
2.11	0.982571
2.12	0.982997
2.13	0.983414
2.14	0.983823
2.15	0.984223
2.16	0.984614
2.17	0.984997
2.18	0.985371
2.19	0.985738
2.20	0.986097
2.21	0.986447
2.22	0.986791
2.23	0.987126
2.24	0.987455
2.25	0.987776
2.26	0.988089
2.27	0.988396
2.28	0.988696
2.29	0.988989
2.30	0.989276
2.31	0.989556
2.32	0.989830
2.33	0.990097
2.34	0.990358
2.35	0.990613
2.36	0.990863
2.37	0.991106
2.38	0.991344
2.39	0.991576
2.40	0.991802
2.41	0.992024
2.42	0.992240
2.43	0.992451
2.44	0.992656
2.45	0.992857
2.46	0.993053
2.47	0.993244
2.48	0.993431
2.49	0.993613

$\Phi(x)$	$\Phi(x)$
2.50	0.993790
2.51	0.993963
2.52	0.994132
2.53	0.994267
2.54	0.994457
2.55	0.994614
2.56	0.994766
2.57	0.994915
2.58	0.995060
2.59	0.995201
2.60	0.995339
2.61	0.995473
2.62	0.995604
2.63	0.995731
2.64	0.995855
2.65	0.995975
2.66	0.996093
2.67	0.996207
2.68	0.996319
2.69	0.996427
2.70	0.996533
2.71	0.996636
2.72	0.996736
2.73	0.996833
2.74	0.996928
2.75	0.997020
2.76	0.997110
2.77	0.997197
2.78	0.997282
2.79	0.997365
2.80	0.997445
2.81	0.997523
2.82	0.997599
2.83	0.997673
2.84	0.997744
2.85	0.997814
2.86	0.997882
2.87	0.997948
2.88	0.998012
2.89	0.998074
2.90	0.998134
2.91	0.998193
2.92	0.998250
2.93	0.998305
2.94	0.998359
2.95	0.998411
2.96	0.998462
2.97	0.998511
2.98	0.998559
2.99	0.998605

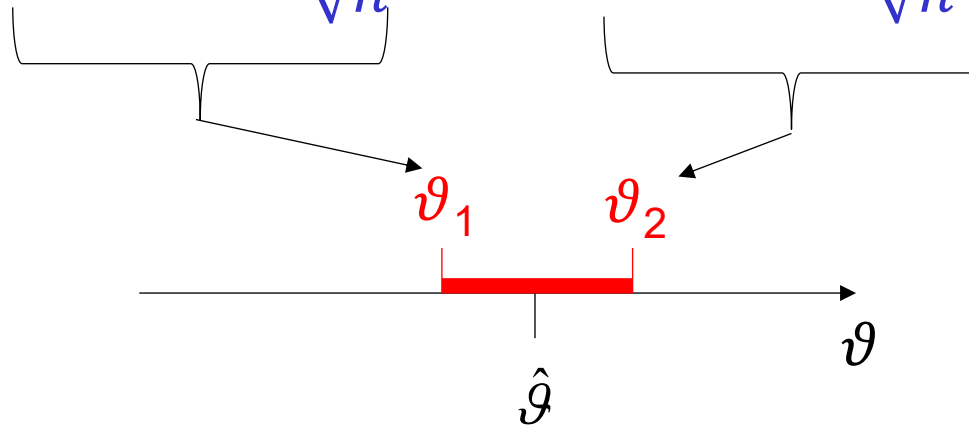
Example: confidence limits for the μ of a normal distribution

- Solving for μ :

$$P\left(-1.645 < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645\right) = 0.9$$

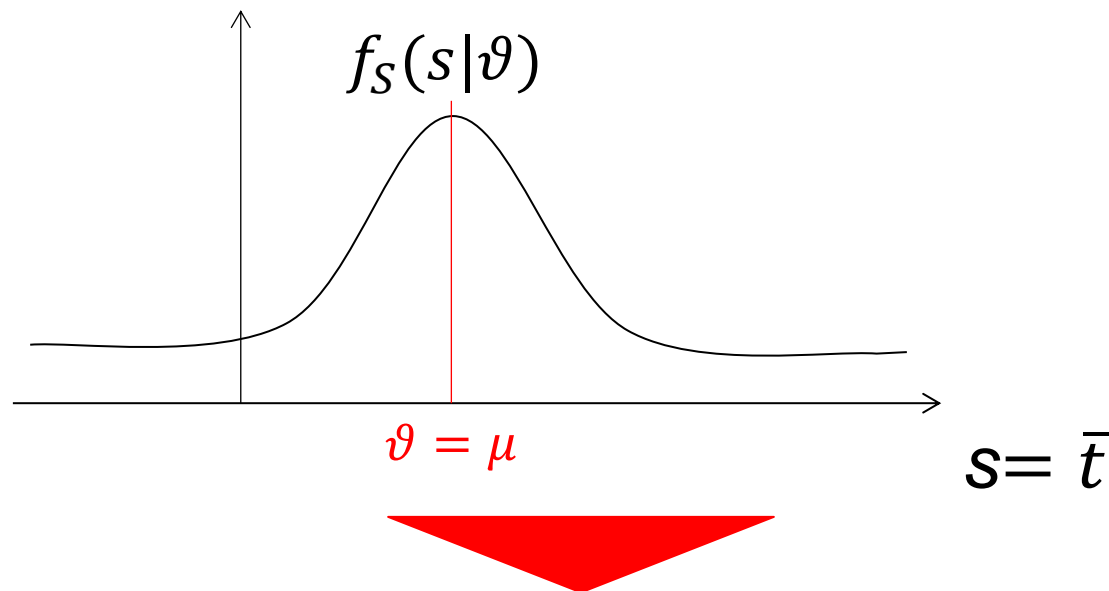


$$P\left(\bar{t} - 1.645 \frac{\sigma}{\sqrt{n}} < \mu < \bar{t} + 1.645 \frac{\sigma}{\sqrt{n}}\right) = 0.9$$



Interval estimates of reliability parameters

- $t_1, t_2, \dots, t_n$ = samples from the population distribution
- ϑ = unknown characteristic of the population,
- $S = \hat{\vartheta} = g(t_1, t_2, \dots, t_n)$ = estimated characteristic,
- S is a random variable (being a function of random failure times) $\rightarrow S$ is characterized by a probability density function, $f_S(s|\vartheta)$.



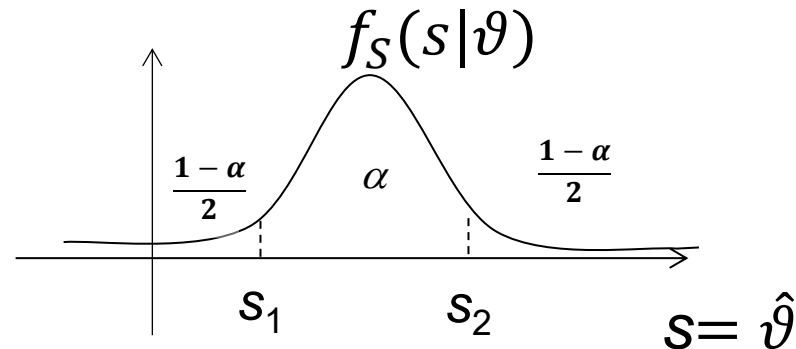
- We fix a value of confidence $\alpha < 1$, and we want to find the interval (S_1, S_2) such that the probability that the true value of the parameter be in the interval is α

Two-sided confidence interval

- Two-sided confidence interval of $\hat{\vartheta}$ at a level of confidence α :
- We want to find $s_1(\vartheta)$ and $s_2(\vartheta)$ such that:

$$P[s_1(\vartheta) \leq \hat{\vartheta} = s \leq s_2(\vartheta)] = \alpha$$

$$\int_{s_1(\vartheta)}^{s_2(\vartheta)} f_S(s|\vartheta) ds = \alpha$$



- The above expression can be rewritten to express the inequality in terms of the unknown characteristic ϑ :

- μ of a normal distribution with known σ

$$s_1(\mu) = \mu - A$$

$$s_2(\mu) = \mu + A$$

A is known! ($1.645 \sigma/\sqrt{n}$)

- it can be computed from the variance of the distribution

$$\xi_{0.95} = 1.645$$

$$P[s_1(\vartheta) < s = \hat{\vartheta} < s_2(\vartheta)] = P[\mu - A < \hat{\vartheta} < \mu + A] = \alpha$$

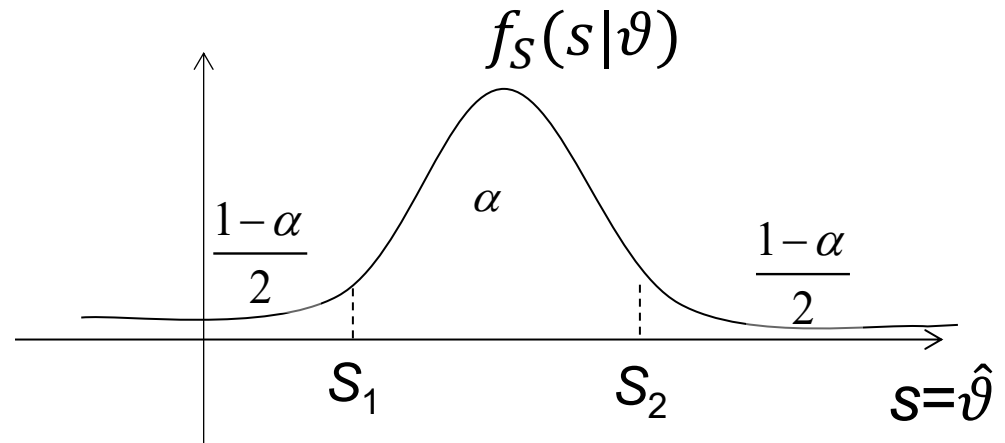
$$P[\hat{\vartheta} - A < \mu < \hat{\vartheta} + A] = \alpha$$

Two-sided confidence interval

- Two-sided confidence interval of $\hat{\vartheta}$ at a level of confidence $1-2\alpha$:
- We want to find $s_1(\vartheta)$ and $s_2(\vartheta)$ such that:

$$P[s_1(\mathcal{G}) \leq \hat{\mathcal{G}} = s \leq s_2(\mathcal{G})] = \alpha$$

$$\int_{s_1(\mathcal{G})}^{s_2(\mathcal{G})} f_S(s | \mathcal{G}) ds = \alpha$$



- The above expression can be rewritten to express inequality in terms of the unknown characteristic ϑ :

$$P(\mathcal{G}_1 \leq \vartheta < \mathcal{G}_2) = \alpha$$

Fixed, but unknown

known

(but it depends from the
samples $t_1, t_2, \dots, t_n$)

known

(but it depends from the
samples $t_1, t_2, \dots, t_n$)

One sided confidence limits

- Lower confidence limit with confidence α

$$P(\mathcal{G} > \vartheta_1) = \alpha$$



- Upper confidence limit with confidence α

$$P(\mathcal{G} < \vartheta_2) = \alpha$$



Confidence limits for exponential distribution

- $TTT = \sum_{j=1}^n T_j$ (Total Time on Test)
- n = number of failures
- Since T_j is exponentially distributed with parameter λ , $Y_j = 2\lambda T_j$ will be chi-squared distributed with 2 degrees of freedom for $j = 1, \dots, n$.

Proof

$$Y_j = 2\lambda T_j \rightarrow T_j = \frac{Y_j}{2\lambda}$$

Since T_j is exponentially distributed

$$\int_0^{\infty} f(t) dt = \int_0^{\infty} \lambda e^{-\lambda t} dt = \int_0^{\infty} \lambda e^{-\lambda \left(\frac{y_j}{2\lambda}\right)} d\left(\frac{y_j}{2\lambda}\right) = \int_0^{\infty} \frac{1}{2} e^{-\frac{1}{2}y_j} dy_j = 1$$

$f(y_j) = \frac{1}{2} e^{-\frac{1}{2}y_j}$ is the exponential distribution pdf with parameter 1/2 or the χ^2 distributed with 2 degrees of freedom

$$f(t) = \frac{1}{\frac{m}{2^{\frac{m}{2}}} \Gamma(m/2)} t^{\frac{m}{2}-1} e^{-\frac{t}{2}} \quad \xrightarrow{m=2} \quad f(t) = \frac{1}{2} e^{-\frac{1}{2}t}$$

Confidence limits for exponential distribution

- $TTT = \sum_{j=1}^n T_j$ (Total Time on Test)
- n = number of failures
- Since T_j is exponentially distributed with parameter λ , $Y_j = 2\lambda T_j$ will be chi-squared distributed with 2 degrees of freedom for $j = 1, \dots, n$.
- Since T_j 's are independent, then $W = 2\lambda \sum_{j=1}^n T_j$ will be χ^2 distributed with $2n$ degrees of freedom.

Proof

Moment-generating function of χ_m^2 distribution:

$$\psi_T(x) = E[e^{xT}] = \int_{-\infty}^{\infty} e^{xt} \frac{1}{2^{\frac{m}{2}} \Gamma(m/2)} t^{\frac{m}{2}-1} e^{-\frac{t}{2}} dt = \dots = \frac{1}{(1-2x)^{\frac{m}{2}}}$$

$$\psi_W(x) = E[e^{xW}] = E\left[e^{x \sum_{j=1}^n 2\lambda T_j}\right] = E\left[e^{2x\lambda T_1} \cdot e^{2x\lambda T_2} \dots \cdot e^{2x\lambda T_n}\right]$$

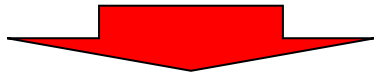
$$= E[e^{2x\lambda T_1}] E[e^{2x\lambda T_2}] \dots E[e^{2x\lambda T_r}] = \psi_{2\lambda T_1}(x) \psi_{2\lambda T_2}(x) \dots \psi_{2\lambda T_n}(x)$$

Confidence limits for exponential distribution

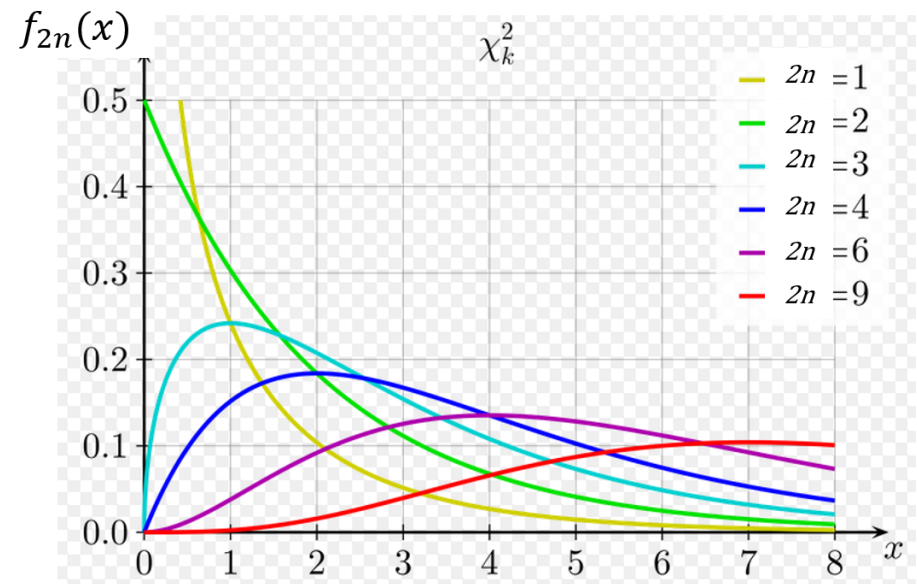
$$\psi_W(x) = \psi_{2\lambda T_1}(x) \psi_{2\lambda T_2}(x) \dots \psi_{2\lambda T_r}(x)$$

where, $\psi_T(x) = \frac{1}{(1-2x)^{\frac{m}{2}}}$

$$\psi_W(x) = \prod_{j=1}^n \frac{1}{(1-2x)^{\frac{2}{2}}} = \frac{1}{(1-2x)^{\frac{2n}{2}}}$$



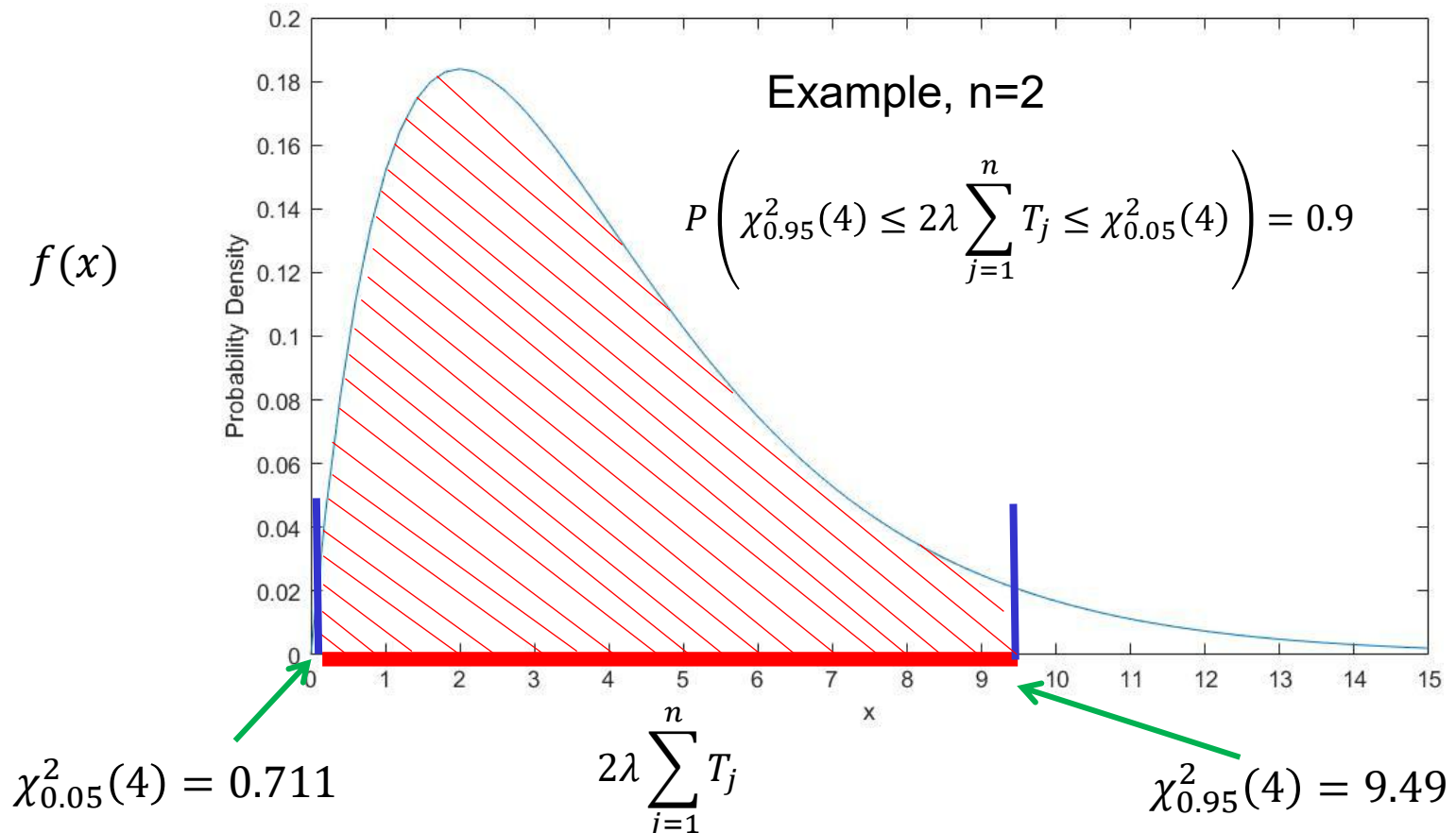
- $W = 2\lambda \sum_{j=1}^n T_j$ is χ^2 distributed with $2n$ degrees of freedom.



Confidence limits for exponential distribution

- Since, $W = 2\lambda \sum_{j=1}^n T_j$ is χ^2 distributed with $2n$ degrees of freedom, we can write:

$$P\left(\chi_{\frac{1-\alpha}{2}}^2(2n) \leq 2\lambda \sum_{j=1}^n T_j \leq \chi_{\frac{1+\alpha}{2}}^2(2n)\right) = \alpha$$



α Percentile values of the χ^2 with d.o.f f

$\alpha \backslash f$	0.005	0.025	0.050	0.900	0.950	0.975	0.990	0.995	0.999
1	0.0439	0.03982	0.02393	2.71	3.84	5.02	6.63	7.88	10.8
2	0.0100	0.0506	0.103	4.61	5.99	7.38	9.21	10.6	13.8
3	0.0717	0.216	0.352	6.25	7.81	9.35	11.3	12.8	16.3
4	0.207	0.484	0.711	7.78	9.49	11.1	13.3	14.9	18.5
5	0.412	0.831	1.15	9.24	11.1	12.8	15.1	16.7	20.5
6	0.676	1.24	1.64	10.6	12.6	14.4	16.8	18.5	22.5
7	0.989	1.69	2.17	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	2.18	2.73	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.70	3.33	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	3.25	3.94	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.82	4.57	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	4.40	5.23	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	5.01	5.89	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	5.63	6.57	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	6.26	7.26	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	6.91	7.96	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	7.56	8.67	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	8.23	9.39	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	8.91	10.1	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	9.59	10.9	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	10.3	11.6	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	11.0	12.3	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	11.7	13.1	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	12.4	13.8	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	13.1	14.6	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	13.8	15.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	14.6	16.2	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	15.3	16.9	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	16.0	17.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	16.8	18.5	40.3	43.8	47.0	50.9	53.7	59.7
35	17.2	20.6	22.5	46.1	49.8	53.2	57.3	60.3	66.6
40	20.7	24.4	26.5	51.8	55.8	59.3	63.7	66.8	73.4
45	24.3	28.4	30.6	57.5	61.7	65.4	70.0	73.2	80.1
50	28.0	32.4	34.8	63.2	67.5	71.4	76.2	79.5	86.7

Confidence limits for exponential distribution

- Since, $W = 2\lambda \sum_{j=1}^n T_j$ is χ^2 distributed with $2n$ degrees of freedom, we can write:

$$P\left(\chi_{\frac{1-\alpha}{2}}^2(2n) \leq 2\lambda \sum_{j=1}^n T_j \leq \chi_{\frac{1+\alpha}{2}}^2(2n)\right) = \alpha$$


$$P\left(\frac{\chi_{\frac{1-\alpha}{2}}^2(2n)}{2 \sum_{j=1}^n T_j} \leq \lambda \leq \frac{\chi_{\frac{1+\alpha}{2}}^2(2n)}{2 \sum_{j=1}^n T_j}\right) = \alpha$$

- So the α confidence interval for parameter λ can be given as:

$$\lambda \in \left[\frac{\chi_{\frac{1-\alpha}{2}}^2(2n)}{2 \sum_{j=1}^n T_j}, \frac{\chi_{\frac{1+\alpha}{2}}^2(2n)}{2 \sum_{j=1}^n T_j} \right]$$

- So the α confidence interval for MTTF can be given as:

$$MTTF \in \left[\frac{2 \sum_{j=1}^n T_j}{\chi_{\frac{1+\alpha}{2}}^2(2n)}, \frac{2 \sum_{j=1}^n T_j}{\chi_{\frac{1-\alpha}{2}}^2(2n)} \right]$$

Lower Confidence limits for estimate of MTTF = $\frac{1}{\hat{\lambda}}$

- $TTT = \sum_{j=1}^n T_j$ (Total Time on Test)
- n = number of failures



It can be shown that $2\lambda \sum_{j=1}^n T_j$ is distributed according to a χ^2 distribution with $2n$ degree of freedom: $\chi^2(2n)$



$$P \left[2\lambda \sum_{j=1}^n T_j \leq \chi_{\alpha}^2(2n) \right] = \alpha$$

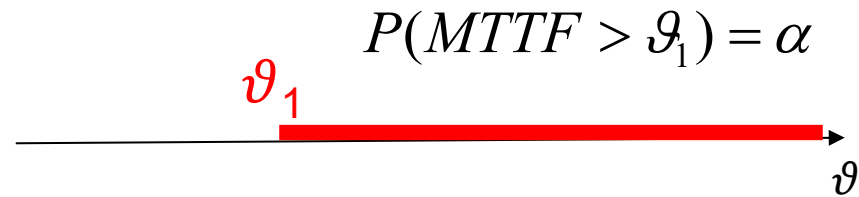
Percentile (from tables) $\nearrow$



$$P \left[\lambda \leq \frac{\chi_{\alpha}^2(2n)}{2 \sum_{j=1}^n T_j} \right] = \alpha$$



$$P \left[MTTF \geq \frac{2 \sum_{j=1}^n T_j}{\chi_{\alpha}^2(2n)} \right] = \alpha$$



$$P(MTTF > \vartheta_1) = \alpha$$

Lower confidence limit

Lower Confidence limits for estimate of MTTF for Type II

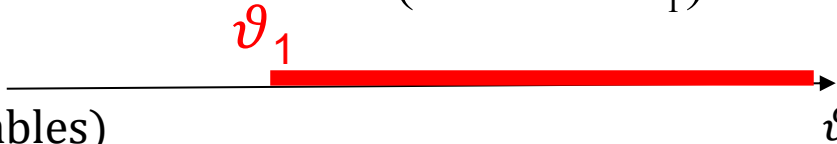
censored test

- Type 2 censored test (stop at the r-th failure)
- $T = TTT$ (Total Time on Test)
- n = Number of components on test
- r = Number of failures
- The last (i.e., r-th) failure has occurred at T_r

$$T = \sum_{j=1}^r T_j + (n - r)T_r$$

It can be shown that $2\lambda T$ is distributed according to a χ^2 distribution with $2r$ degree of freedom: $\chi^2(2r)$

$P(MTTF > \vartheta_1) = \alpha$



Percentile (from tables)

$P[2\lambda T \leq \chi_{\alpha}^2(2r)] = \alpha$

Lower confidence limit

$$P\left[\lambda \leq \frac{\chi_{\alpha}^2(2r)}{2T}\right] = \alpha$$



$$P\left[MTTF \geq \frac{2T}{\chi_{\alpha}^2(2r)}\right] = \alpha$$

Confidence limits for estimate of MTTF = $\frac{1}{\hat{\lambda}}$

- $T = TTT$ (Total Time on Test)
- r = number of failures

	I, fixed t_0	II, fixed r
one-sided (lower) $P(MTTF > \vartheta_1) = \alpha$ 	$\vartheta_1 = \frac{2T}{\chi_{\alpha}^2 \underbrace{(2r+2)}_{\substack{\text{\# of degrees of freedom} \\ \downarrow \\ \text{percentile}}}}$	$\vartheta_1 = \frac{2T}{\chi_{\alpha}^2(2r)}$
two-sided (lower and upper) $P(\vartheta_1 < MTTF < \vartheta_2) = \alpha$ 	$(\vartheta_1, \vartheta_2) = \left(\frac{2T}{\chi_{\frac{1+\alpha}{2}}^2(2r+2)}, \frac{2T}{\chi_{\frac{1-\alpha}{2}}^2(2r)} \right)$	$(\vartheta_1, \vartheta_2) = \left(\frac{2T}{\chi_{\frac{1+\alpha}{2}}^2(2r)}, \frac{2T}{\chi_{\frac{1-\alpha}{2}}^2(2r)} \right)$

Confidence limits for MTTF: Example 1

- 30 identical components,
- Type II censoring with $r = 20$

Find the 95% two-sided
confidence
limits for the MTTF,

TTF's Up to 20th Failure			
t_1	0.26	t_{11}	11.04
t_2	1.49	t_{12}	12.07
t_3	3.65	t_{13}	13.61
t_4	4.25	t_{14}	15.07
t_5	5.43	t_{15}	19.28
t_6	6.97	t_{16}	24.04
t_7	8.09	t_{17}	26.16
t_8	9.47	t_{18}	31.15
t_9	10.18	t_{19}	38.70
t_{10}	10.29	t_{20}	39.89

Confidence limits for MTTF: Example 2

• Consider failures in NPP with remarkable release of radiation. Identify the 95% lower confidence limit for the mean time to failure ($MTTF=1/\lambda$) of western design reactors, assuming:

- 10000 reactor year of operation
- 2 failures with remarkable release of radiations (Three Miles Island + Fukushima)
- Constant failure rate, equal for all the reactors

Confidence limits for MTTF: Example 3

Example: 30 items are placed on test. Items which fail are replaced. The life test is stopped after 100 hours have elapsed. Five failures were observed in the course of the experiment. Assuming that the underlying distribution of life is exponential, find one and two-sided 95% confidence intervals for $MTTF$.