POLITECNICO DI MILANO

Estimation of reliability parameters from experimental data

(Part 1)

In this lecture

- T = failure time is a random variable described by a life distribution $F_T(t)$
- $F_T(t)$ depends from some parameters: $F_T(t;\boldsymbol{\vartheta})$

How to estimate $\boldsymbol{\vartheta}$?

Example: pump with $F_T(t)=1-e^{-\lambda t}$

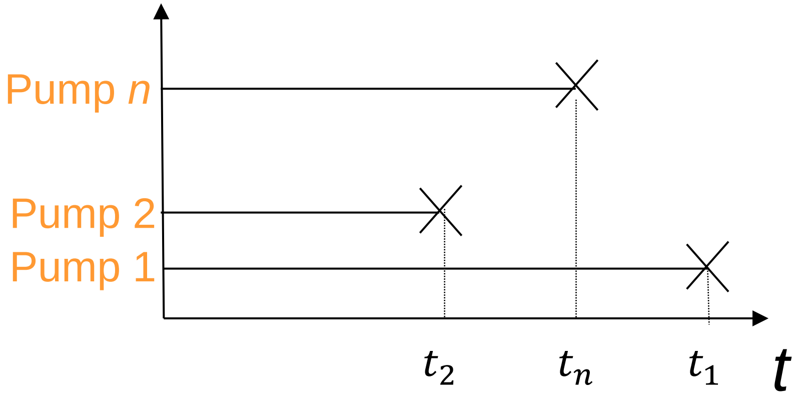
How to estimate λ?

Life tests

- Parameter estimation: $\boldsymbol{\vartheta}$ of $F_T(t; \boldsymbol{\vartheta})$

- Life tests:
  - *n* identical units are activated and their lifetimes are recorded

Example:



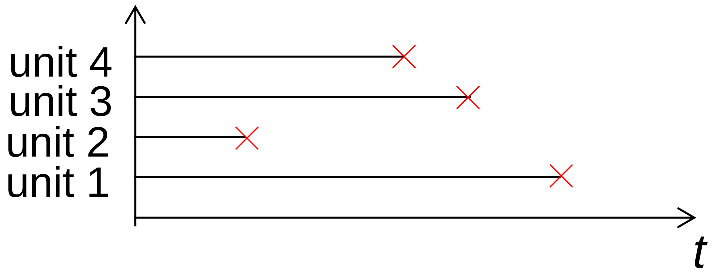
- From $t_1, t_2, \ldots, t_n$ to $\boldsymbol{\vartheta}$

Life tests: basic assumptions

- T_i (failure time of *i*-th unit) are identically distributed
  - Units of the same type
  - same environmental and operational stresses
- T_i are independent
  - components are not affected by the operation or failure of any other component in the set

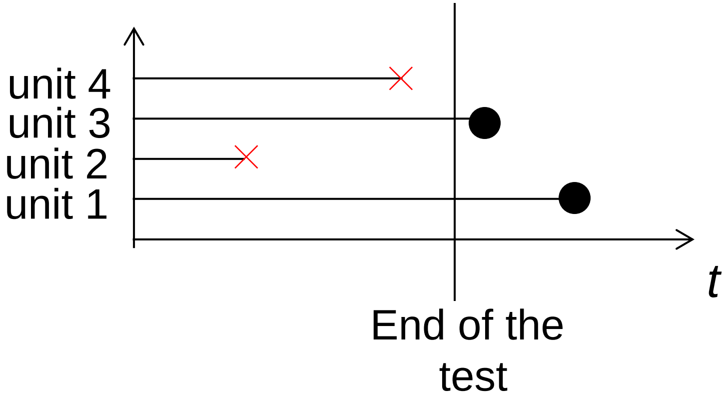
Type of life tests (1)

- Complete: the test runs until all the n components have failed



It may be impractical or too expensive to wait until all components have failed!

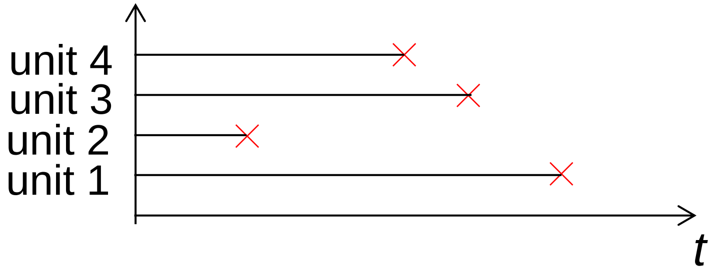
- Censored*:
  - Right censored: composed also by units that did not fail in the test



*new assumption = censoring mechanism is independent of any information gained from previously failed components in the set

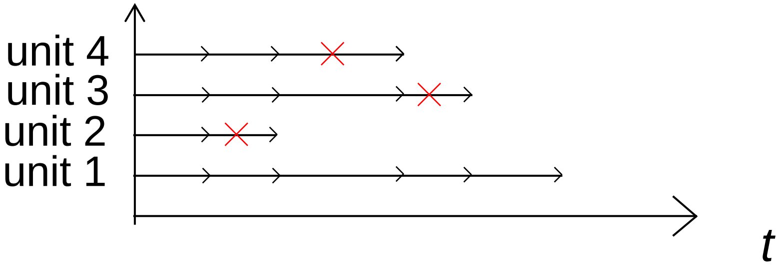
Type of life tests (1)

- Complete: the test runs until all the n components have failed



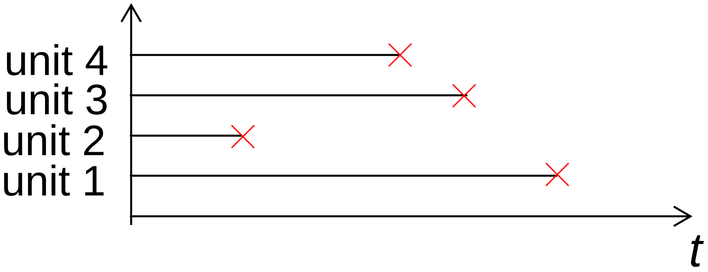
It may be impractical or too expensive
to wait until all components have failed!

- Censored:
  - Right censored
  - Interval censored: units are inspected at fixed times (It is not known the exact time of the failure but only that it occurs between two inspections)



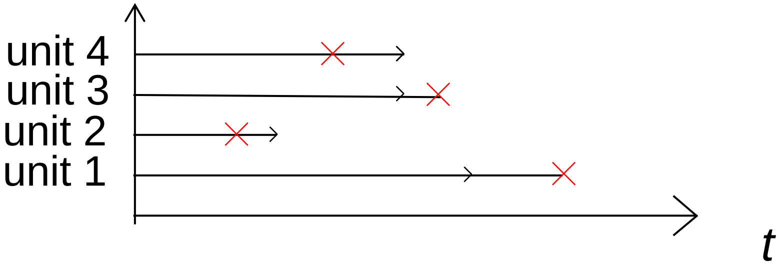
Type of life tests (1)

- Complete: the test runs until all the n components have failed



It may be impractical or too expensive to wait until all components have failed!

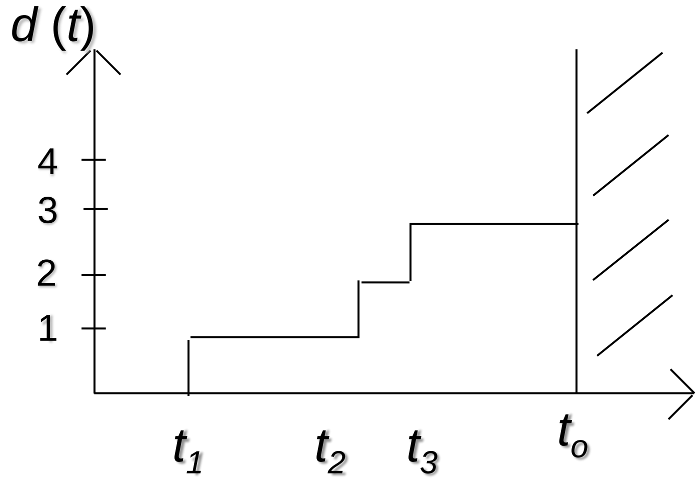
- Censored:
  - Right censored
  - Interval censored
  - Left censored: only one inspection for each unit (failure between time 0 and the inspection or after the inspection)



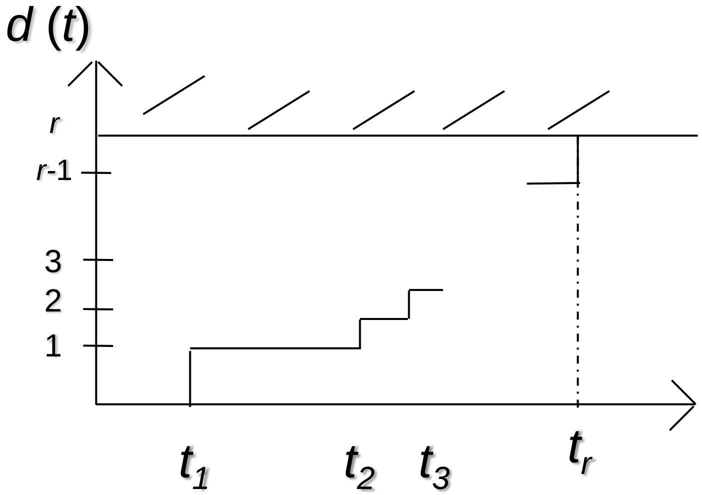
‘Terminated’ test

- Moment of termination of testing:
  - at fixed t_o (Type 1)

$d\,(t)$= number of failures that occur before t

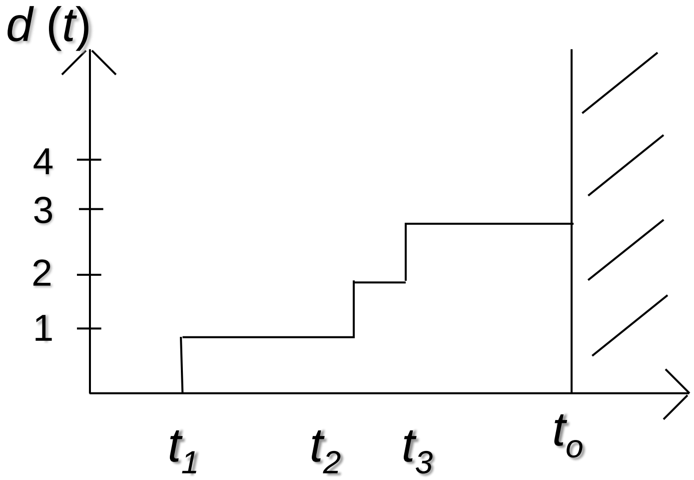


  - at the r-th failure (Type 2)



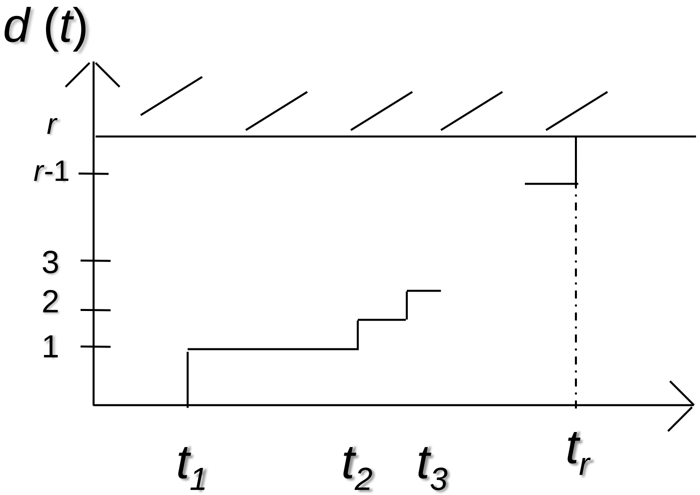
Type of right censored tests

- Moment of termination of testing:
  - at fixed t_o (Type 1)



$d\ (t)$= number of failures that occur before t

  - at the r-th failure (Type 2)



- With Replacement (R) or Without replacement (W) upon unit failure

Estimate Parameters from Observational Data

Life test: $(t_1,t_2,...,t_n) \rightarrow$ Estimate ϑ of $f_T(t;\vartheta)$

For example: λ of $f_T(t)=\lambda e^{-\lambda t}$

- ○ Classical approach:
  - ϑ is a fixed unknown parameter
  - From $(t_1,t_2,...,t_n)$ find an estimator $\hat{\vartheta}$ of ϑ
    - Point Estimation
      - Method of moments
      - Method of maximum likelihood
    - Interval Estimation
      - Confidence interval
- ○ Bayesian Approach [NEXT LECTURE]

Classical approach: Point Estimation

Model:

- ϑ is a fixed unknown quantity
- The estimator $\hat{\vartheta}$ of ϑ:

$$\hat{\vartheta} = g(T_1, \dots, T_n)$$

is a random variable (being a function *g* of the random variables $T_1, \dots, T_n$)

Desiderata:

- Desirable properties of the point estimator $\hat{\vartheta} = g(T_1, \dots, T_n)$ *:*
  - **Unbiased**: $E[\hat{\vartheta}] = \vartheta$ (*Accuracy*)
  - **Consistent**: $\lim_{n \to \infty} Var[\hat{\vartheta}] = 0$ (*Asymptotic precision*)

Point Estimation methods:

- The method of moments
- The method of maximum likelihood

The method of moments

1. Moments of random variables are related with the parameter of the distributions
    - Example - exponential distribution $f_T(t) = \lambda e^{-\lambda t} \rightarrow E[T] = \frac{1}{\lambda}$

2. Sample moments are used to estimate the corresponding moments of the random variables:

$$M_k = \frac{\sum_{i=1}^{n} (t_i)^k}{n} \quad \text{estimates} \quad E[T^k] = \int_{-\infty}^{+\infty} t^k f(t)\,\mathrm{d}t$$

    - Example - exponential distribution

$$M_1 = \frac{\sum_{i=1}^{n} t_i}{n} \quad \text{estimates} \quad \mathrm{E}[T]$$

- Parameters of the distributions can be estimated by using the sample moments of the random variable
    - Example - exponential distribution: $t_1, t_2, \dots, t_n$ are failure times of a component with exponential failure rate

$$\hat{\lambda} = \frac{1}{M_1} = \frac{n}{\sum_{i=1}^{n} t_i}$$

The method of moments

Method of Moments for normal distribution:

Derive the formulas for the parameter of a normal distribution using the method of moments.

Method for the estimation of the parameters:

- The method of moments
- The method of maximum likelihood

The likelihood function

- Sample $\boldsymbol{T} = (T_1, T_2, \dots, T_{n-1}, T_n)$
- Joint pdf (or pmf) of $\boldsymbol{T}$: $f_{\boldsymbol{T}}(t_1, \dots, t_n | \boldsymbol{\theta}), \boldsymbol{\theta} \in \boldsymbol{\Theta}$
- Parameter space $\boldsymbol{\Theta}$= set of all possible values that $\boldsymbol{\theta}$ can assume
- Observations $\boldsymbol{t} = (t_1, t_2, \dots, t_{n-1}, t_n)$

Given that $\boldsymbol{t}$ is observed the function of $\boldsymbol{\theta}$ defined by

$$L(\theta | t_1, \dots, t_n) = f_{\boldsymbol{T}}(t_1, \dots, t_n | \boldsymbol{\theta}), \boldsymbol{\theta} \in \boldsymbol{\Theta}$$

is called **likelihood function**

Let $t = (t_1, \dots, t_n)$ be i.i.d samples from probability mass function $f_T(t \mid \theta)$, where θ is a parameter (or vector of parameters). We define the likelihood of t given θ to be the “probability” of observing t if the true parameter is θ.

The likelihood function

$$\boxed{L(\theta|t_1,\dots,t_n) = f_{\boldsymbol{T}}(t_1,\dots,t_n|\boldsymbol{\theta}), \boldsymbol{\theta} \in \boldsymbol{\Theta}}$$

Example - exponential distribution

$$T_1, T_2, \dots, T_n \overset{i.i.d.}{\sim} Exp(\lambda)$$

$$L(\lambda|t_1,\dots,t_n) = \prod_{i=1}^{n} \lambda\, e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i} \quad \lambda > 0$$

Example – Weibull distribution

$$T_1, T_2, \dots, T_n \overset{i.i.d.}{\sim} Weibull(\alpha, \beta)$$

$$L(\lambda|t_1,\dots,t_n) = \prod_{i=1}^{n} \frac{\beta}{\alpha^\beta} t_i^{\beta-1}\, e^{-\left(\frac{t_i}{\alpha}\right)^\beta} = \frac{\beta^n}{\alpha^{\beta n}} \left(\prod_{i=1}^{n} t_i^{\beta-1} \right) e^{-\sum_{i=1}^{n}\left(\frac{t_i}{\alpha}\right)^\beta} \quad \alpha, \beta > 0$$

Example – Poisson distribution

$$N_1, N_2, \dots, N_n \overset{i.i.d.}{\sim} Poisson(\lambda)$$

$$L(\lambda|n_1,\dots,n_n) = \prod_{i=1}^{n} e^{-\lambda} \frac{\lambda^{n_i}}{n_i!} = e^{-\lambda n} \lambda^{\sum_{i=1}^{n} n_i} \left(\frac{1}{\prod_{i=1}^{n} n_i!} \right) \quad \lambda > 0$$

Likelihood principle

Observer 1: $\boldsymbol{t}^{\mathbf{1}} = (t_1^1, \dots, \mathrm{t}_{n^1}^1) \Rightarrow \ L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{1}})$

Observer 2: $\boldsymbol{t}^{\mathbf{2}} = (t_1^2, \dots, \mathrm{t}_{n^2}^2) \Rightarrow \ L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{2}})$

Suppose that

$$L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{1}}) = \boldsymbol{C}(\boldsymbol{t}^{\mathbf{1}}, \boldsymbol{t}^{\mathbf{2}})L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{2}})$$

i.e., $L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{1}})$ is proportional to $L(\boldsymbol{\theta}|\boldsymbol{t}^{\mathbf{2}})$ for less than a constant which does not depend on $\boldsymbol{\theta}$

▼

conclusions drawn from $\boldsymbol{t}^{\mathbf{1}}$ and $\boldsymbol{t}^{\mathbf{2}}$ on $\boldsymbol{\theta}$ should be identical

- **Rationale:**

The likelihood is used to compare the **plausibility** of various parameter values

Example

$$L(\boldsymbol{\theta}_{\mathbf{1}}|\boldsymbol{t}\quad) = 2L(\boldsymbol{\theta}_{\mathbf{2}}|\boldsymbol{t}\quad)$$

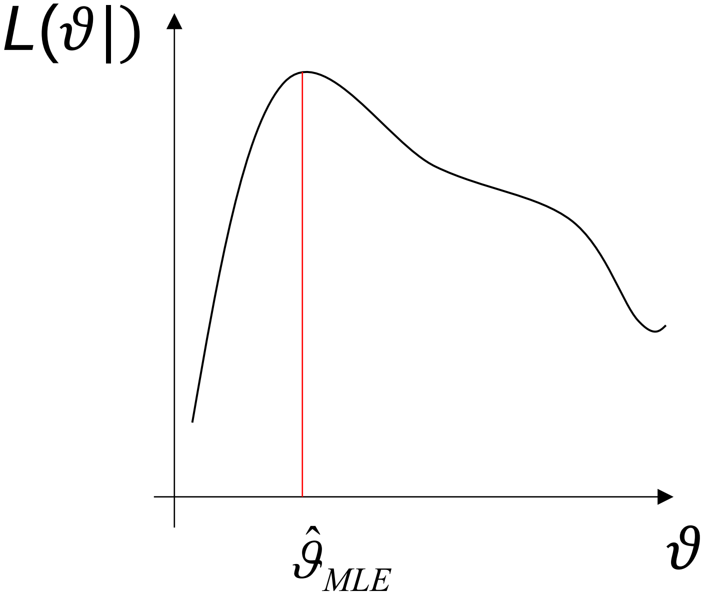
$\boldsymbol{\theta}_{\mathbf{1}}$ is twice as plausible as $\boldsymbol{\theta}_{\mathbf{2}}$

The method of maximum likelihood

- *n* observations $\{t_1, t_2, \ldots, t_n\}$ are available:

$$L(\vartheta|t_1,..,t_n) = \prod_{i=1}^{n} f_T(t_i;\vartheta)$$

- The maximum liklihood estimator $\hat{\vartheta}_{MLE}$ is the value of ϑ which maximizes the likelihood function:



$$\frac{\partial L(\vartheta)}{\partial \vartheta} = 0$$

Example 1: uncensored test

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have observed the failure time of 4 items obtaining:
  - $t_1 = 5d$
  - $t_2 = 7d$
  - $t_3 = 4d$
  - $t_4 = 10d$

Estimate using the method of maximum likelihood the parameter λ

Censoring and likelihood

T time to failure

$T \sim f_T(t|\theta)$: probability density function

$F_T(t|\theta) = \mathbb{P}(T \leq t|\theta)$:cumulative distribution function

$R(t|\theta) = \mathbb{P}(T > t|\theta)$: reliability function

Case 1: Right censoring

The observer only knows that $T > t_1$

Contribute to the likelihood:

$$L(\theta|t_1) = R(t_1|\theta)$$

Case 2: Left censoring

The observer only knows that $T \leq t_1$

Contribute to likelihood:

$$L(\theta|t_1) = F_T(t_1|\theta)$$

Case 3: interval censoring

The observer only knows that $\mathrm{a} \leq T \leq b$

Contribute to likelihood:

$$L(\theta|(a,b)) = F_T(b|\theta) - \ F_T(a|\theta)$$

The method of maximum likelihood applied to test components lifetimes

- Uncensored lifetimes: t_1, t_2, …., t_n: $L(\vartheta|t_1,..,t_n) = \prod_{i=1}^{n} f_T(t_i|\vartheta)$

- Right-censored data: $L(\vartheta\ ) = \prod_i f_T(t_i|\vartheta) \prod_j R(t_j|\vartheta)$

Failures *Right-Censored*

- Generally, one takes $l(\vartheta) = \ln[L(\vartheta)]$

- and the estimator is $\hat{\vartheta}$ which maximizes $l(\vartheta)$:

$$\frac{\partial l}{\partial \vartheta} = 0$$

Method of Maximum likelihood: example exponential distribution

- *r* uncensored observation: failure times $(t_1, t_2, \ldots, t_r)$
- *n-r* observations, each one right-censored at a different time: $(t_{r+1}, t_{r+2}, \ldots, t_n)$

$$L(\lambda) = \underbrace{\prod_i f_T(t_i;\lambda)}_{failures} \cdot \underbrace{\prod_j R(t_j;\lambda)}_{right-censored} = \lambda^r \cdot e^{-\lambda \sum_{i=1}^{r} t_i} \cdot e^{-\lambda \sum_{j=r+1}^{n} t_j}$$

$$= \lambda^r \cdot e^{-\lambda \sum_{k=1}^{n} t_k}$$

$$l(\lambda) = \ln L(\lambda) = r \ln \lambda - \lambda \sum_{k=1}^{n} t_k$$

$$\frac{\partial l}{\partial \lambda} = \frac{r}{\lambda} - \sum_{k=1}^{n} t_k \Rightarrow \hat{\lambda} = \frac{r}{\sum_{k=1}^{n} t_k} = \frac{r}{TTT}$$

TTT=Total Time on Test

Example 2: right censored test – Type 1

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the first type with $t_0 = 8d$ and observed the failures of three items at times:
  - $t_1 = 5d$
  - $t_2 = 7d$
  - $t_3 = 4d$

Estimate using the method of maximum likelihood the parameter λ

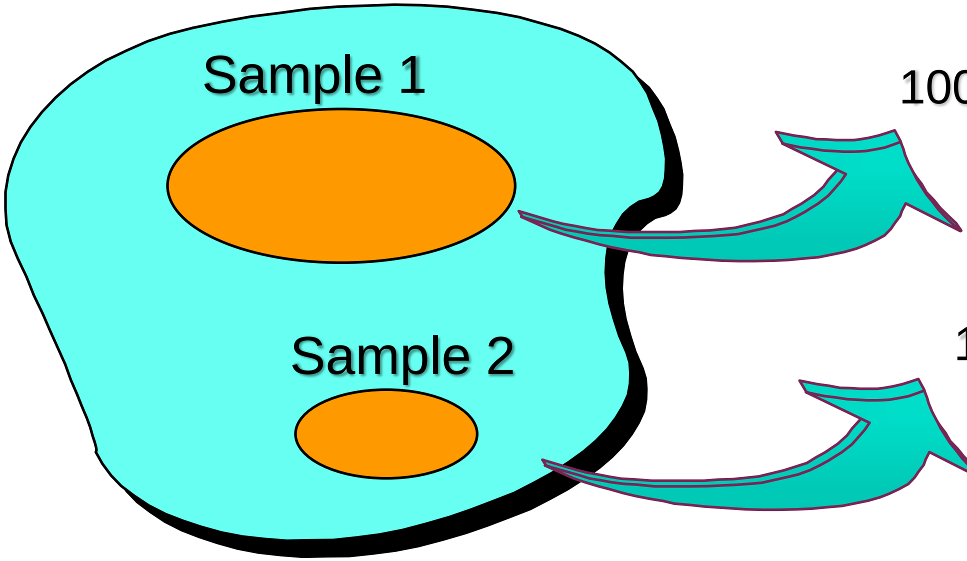
Example 3: right censored test – Type 2

- Assume that the failure time of a certain item is an exponential random variable with failure rate λ
- We have performed a right censored test of the second type with *r*=2 and observed the following failure times:
  - $t_1 = 5d$
  - $t_3 = 4d$

Estimate using the method of maximum likelihood the parameter λ

Limitation of the point estimates

Coin toss: binomial process $\hat{p}_{MLE} = \frac{\#\ of\ \ successes}{\#\ of\ \ trials}$

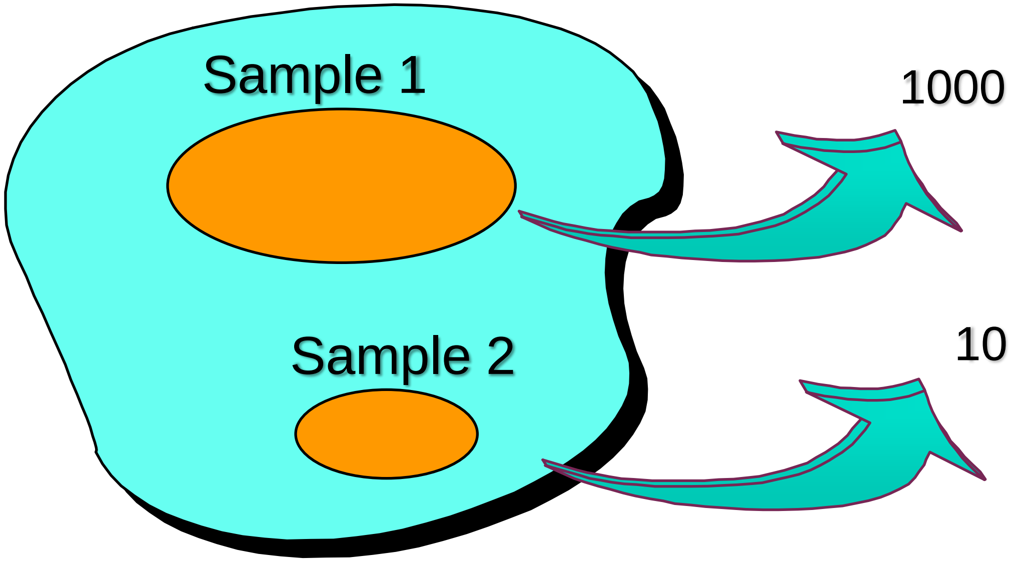


1000 tosses, 400 heads $\rightarrow \hat{p}_{MLE1} = 0.4$

10 tosses, 3 heads $\rightarrow \hat{p}_{MLE2} = 0.3$

Limitation of the point estimates

Coin toss: binomial process $\hat{p}_{MLE} = \frac{\#\ of\ \ successes}{\#\ of\ \ trials}$



1000 tosses, 400 heads → $\hat{p}_{MLE1} = 0.4$

10 tosses, 3 heads → $\hat{p}_{MLE2} = 0.3$

We have more confidence in $\hat{p}_{MLE1} = 0.4$ computed using Sample 1, but the point estimates $\hat{p}_{MLE}$ do not give a measure of the confidence in the result

Confidence limits for the reliability parameters

- Degree of confidence in the estimates ⇒ ~~Point estimates $\hat{\vartheta}$~~

Intervalar estimation

▼

Confidence Interval

▼

Confidence limits $(\vartheta_1, \vartheta_2)$

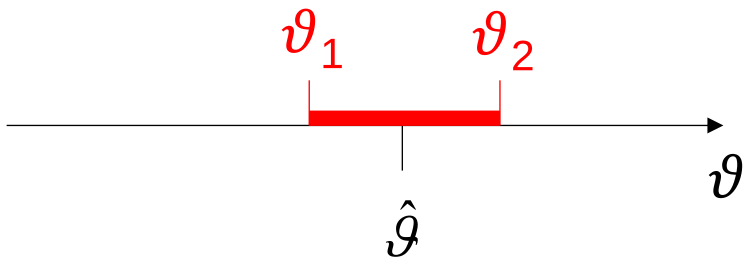
Confidence limits for the reliability parameters

- Degree of confidence in the estimates ⇨ ~~Point estimates $\hat{\vartheta}$~~

Interval estimation

Confidence Interval

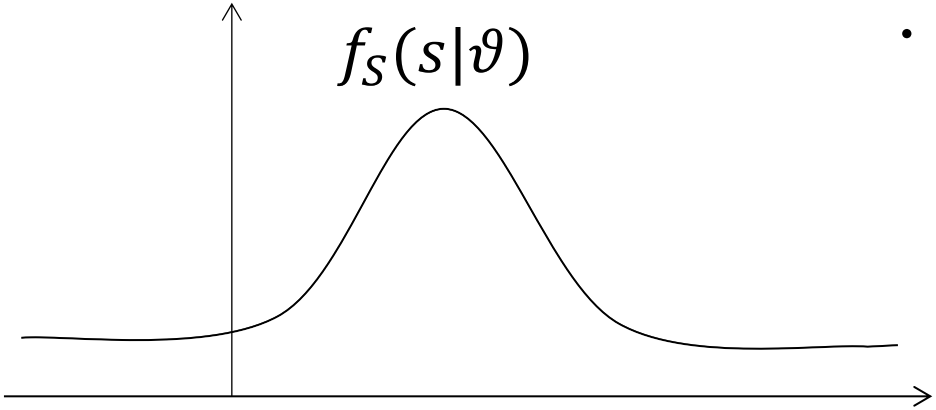
Confidence limits (ϑ_1, ϑ_2)



- We fix a value of confidence α<1, and we consider interval (ϑ_1, ϑ_2) such that we are α confident that the true value of the parameter ϑ is between (ϑ_1, ϑ_2)

Interval estimates of reliability parameters

- $t_1, t_2, \ldots, t_n$ = sample from the population distribution
- ϑ = unknown characteristic of the population,
  - Example: $\vartheta = \mu$ of a normal distribution with known σ: $f_T(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$
- $S = \hat{\vartheta} = g(t_1, t_2, \ldots, t_n)$ = estimated characteristic,
  - Example: $S = \bar{t} = \frac{\sum_{i=1}^{n} T_i}{n}$
- S is a random variable (being a function of random failure times) → S is characterized by a probability density function, $f_S(s|\vartheta)$.



What does this distribution mean?

- I repeat the test over the n components several times:

First test: $\left(t_1^{(1)}, t_2^{(1)}, \ldots, t_n^{(1)}\right) \rightarrow \bar{t}^{(1)} = \frac{\sum_{i=1}^{n} t_i^{(1)}}{n}$

Second test: $\left(t_1^{(2)}, t_2^{(2)}, \ldots, t_n^{(2)}\right) \rightarrow \bar{t}^{(2)} = \frac{\sum_{i=1}^{n} t_i^{(2)}}{n}$

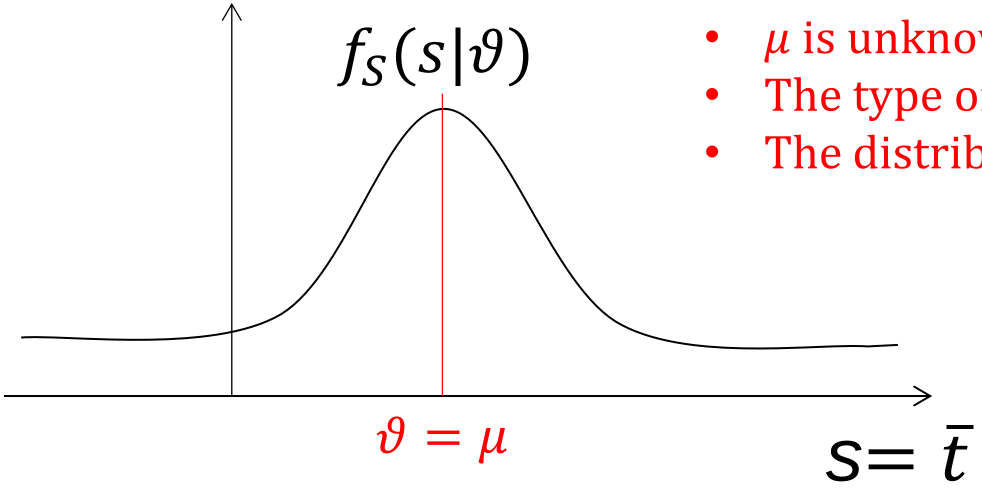
…

M-th test $\left(t_1^{(M)}, t_2^{(M)}, \ldots, t_n^{(M)}\right) \rightarrow \bar{t}^{(M)} = \frac{\sum_{i=1}^{n} t_i^{(M)}}{n}$

$\bar{t}^{(1)}, \bar{t}^{(2)}, \ldots, \bar{t}^{(M)}$ are distributed according to $f_S(s|\vartheta)$

Interval estimates of reliability parameters

- t_1, t_2, ….,t_n = sample from the population distribution
- ϑ = unknown characteristic of the population,
  - Example: $\vartheta = \mu$ of a normal distribution with known σ
- $S = \hat{\vartheta}$ = $g(t_1,t_2,….,t_n)$ = estimated characteristic,
  - Example: $S = \bar{t} = \frac{\sum_{i=1}^{n} T_i}{n}$
- S is a random variable (being a function of random failure times) → S is characterized by a probability density function, $f_S(s|\vartheta)$.
  - Example: Central limit theorem → $\bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$



- μ is unknown!
- The type of the distribution (normal) is known!
- The distribution variance is known!

The central limit theorem: For a population with mean μ and standard deviation σ, if we take sufficiently large random samples from the population with replacement, then the *distribution of the sample means* will be approximately normally distributed.

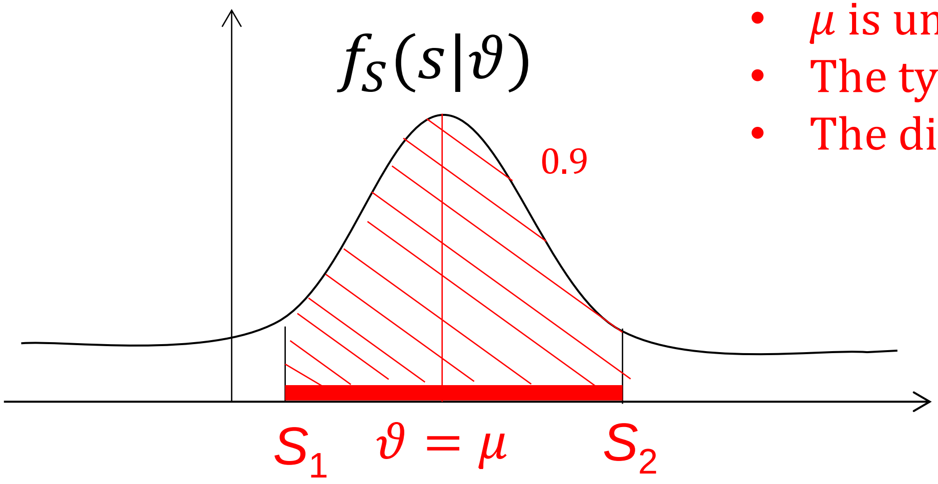
$$\bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Example: confidence limits for the μ of a normal distribution

- Estimate of the confidence interval (S_1,S_2)
  - 90% confidence interval → I need the 5 and 95 percentile of the distribution

unknown

$$\bar{t} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$



- μ is unknown!
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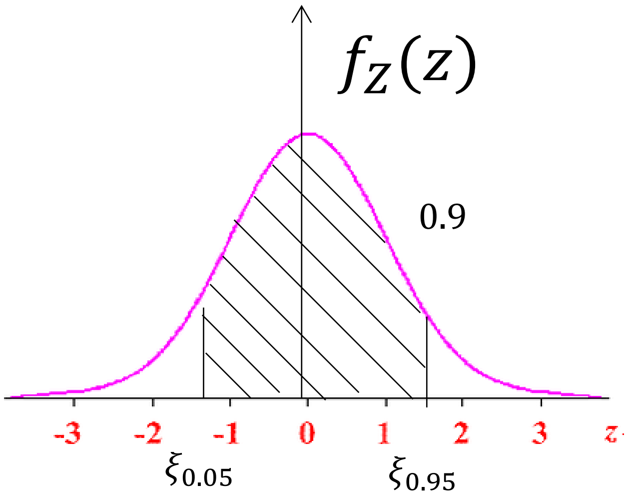
$s = \bar{t}$

Example: confidence limits for the μ of a normal distribution

- Estimate of the confidence interval (S_1, S_2)
  - 90% confidence interval → I need the 5 and 95 percentile of the distribution

$$\bar{t} \sim N\left(\overset{\text{unknown}}{\mu}, \frac{\sigma^2}{n}\right)$$



Standard normal

$$z = \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

$$P\left(\xi_{0.05} < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < \xi_{0.95}\right) = P\left(-1.645 < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645\right) = 0.9$$

Table of Standard Normal Probability

$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\xi^2}\,d\xi$$

x	Φ(x)
1.50	0.933193
1.51	0.934478
1.52	0.935744
1.53	0.936992
1.54	0.938220
1.55	0.939429
1.56	0.940620
1.57	0.941792
1.58	0.942947
1.59	0.944083
1.60	0.945201
1.61	0.946301
1.62	0.947384
1.63	0.948449
1.64	0.949497
1.65	0.950529
1.66	0.951543
1.67	0.952540
1.68	0.953521
1.69	0.954486
1.70	0.955435
1.71	0.956367
1.72	0.957284
1.73	0.958185
1.74	0.959071
1.75	0.959941
1.76	0.960796
1.77	0.961636
1.78	0.962426
1.79	0.963273
1.80	0.964070
1.81	0.964852
1.82	0.965621
1.83	0.966375
1.84	0.967116
1.85	0.967843
1.86	0.968557
1.87	0.969258
1.88	0.969946
1.89	0.970621
1.90	0.971284
1.91	0.971933
1.92	0.972571
1.93	0.973197
1.94	0.973810
1.95	0.974412
1.96	0.975002
1.97	0.975581
1.98	0.976148
1.99	0.976705
2.00	0.977250
2.01	0.977784
2.02	0.978308
2.03	0.978822
2.04	0.979325
2.05	0.979818
2.06	0.980301
2.07	0.980774
2.08	0.981237
2.09	0.981691
2.10	0.982136
2.11	0.982571
2.12	0.982997
2.13	0.983414
2.14	0.983823
2.15	0.984223
2.16	0.984614
2.17	0.984997
2.18	0.985371
2.19	0.985738
2.20	0.986097
2.21	0.986447
2.22	0.986791
2.23	0.987126
2.24	0.987455
2.25	0.987776
2.26	0.988089
2.27	0.988396
2.28	0.988696
2.29	0.988989
2.30	0.989276
2.31	0.989556
2.32	0.989830
2.33	0.990097
2.34	0.990358
2.35	0.990613
2.36	0.990863
2.37	0.991106
2.38	0.991344
2.39	0.991576
2.40	0.991802
2.41	0.992024
2.42	0.992240
2.43	0.992451
2.44	0.992656
2.45	0.992857
2.46	0.993053
2.47	0.993244
2.48	0.993431
2.49	0.993613
2.50	0.993790
2.51	0.993963
2.52	0.994132
2.53	0.994267
2.54	0.994457
2.55	0.994614
2.56	0.994766
2.57	0.994915
2.58	0.995060
2.59	0.995201
2.60	0.995339
2.61	0.995473
2.62	0.995604
2.63	0.995731
2.64	0.995855
2.65	0.995975
2.66	0.996093
2.67	0.996207
2.68	0.996319
2.69	0.996427
2.70	0.996533
2.71	0.996636
2.72	0.996736
2.73	0.996833
2.74	0.996928
2.75	0.997020
2.76	0.997110
2.77	0.997197
2.78	0.997282
2.79	0.997365
2.80	0.997445
2.81	0.997523
2.82	0.997599
2.83	0.997673
2.84	0.997744
2.85	0.997814
2.86	0.997882
2.87	0.997948
2.88	0.998012
2.89	0.998074
2.90	0.998134
2.91	0.998193
2.92	0.998250
2.93	0.998305
2.94	0.998359
2.95	0.998411
2.96	0.998462
2.97	0.998511
2.98	0.998559
2.99	0.998605

Table of Standard Normal Probability

x	Φ(x)
1.50	0.933193
1.51	0.934478
1.52	0.935744
1.53	0.936992
1.54	0.938220
1.55	0.939429
1.56	0.940620
1.57	0.941792
1.58	0.942947
1.59	0.944083
1.60	0.945201
1.61	0.946301
1.62	0.947384
1.63	0.948449
1.64	0.949497
1.65	0.950529
1.66	0.951543
1.67	0.952540
1.68	0.953521
1.69	0.954486
1.70	0.955435
1.71	0.956367
1.72	0.957284
1.73	0.958185
1.74	0.959071
1.75	0.959941
1.76	0.960796
1.77	0.961636
1.78	0.962426
1.79	0.963273
1.80	0.964070
1.81	0.964852
1.82	0.965621
1.83	0.966375
1.84	0.967116
1.85	0.967843
1.86	0.968557
1.87	0.969258
1.88	0.969946
1.89	0.970621
1.90	0.971284
1.91	0.971933
1.92	0.972571
1.93	0.973197
1.94	0.973810
1.95	0.974412
1.96	0.975002
1.97	0.975581
1.98	0.976148
1.99	0.976705
2.00	0.977250
2.01	0.977784
2.02	0.978308
2.03	0.978822
2.04	0.979325
2.05	0.979818
2.06	0.980301
2.07	0.980774
2.08	0.981237
2.09	0.981691
2.10	0.982136
2.11	0.982571
2.12	0.982997
2.13	0.983414
2.14	0.983823
2.15	0.984223
2.16	0.984614
2.17	0.984997
2.18	0.985371
2.19	0.985738
2.20	0.986097
2.21	0.986447
2.22	0.986791
2.23	0.987126
2.24	0.987455
2.25	0.987776
2.26	0.988089
2.27	0.988396
2.28	0.988696
2.29	0.988989
2.30	0.989276
2.31	0.989556
2.32	0.989830
2.33	0.990097
2.34	0.990358
2.35	0.990613
2.36	0.990863
2.37	0.991106
2.38	0.991344
2.39	0.991576
2.40	0.991802
2.41	0.992024
2.42	0.992240
2.43	0.992451
2.44	0.992656
2.45	0.992857
2.46	0.993053
2.47	0.993244
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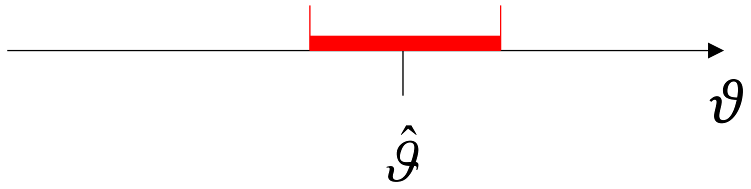
2.49	0.993613
2.50	0.993790
2.51	0.993963
2.52	0.994132
2.53	0.994267
2.54	0.994457
2.55	0.994614
2.56	0.994766
2.57	0.994915
2.58	0.995060
2.59	0.995201
2.60	0.995339
2.61	0.995473
2.62	0.995604
2.63	0.995731
2.64	0.995855
2.65	0.995975
2.66	0.996093
2.67	0.996207
2.68	0.996319
2.69	0.996427
2.70	0.996533
2.71	0.996636
2.72	0.996736
2.73	0.996833
2.74	0.996928
2.75	0.997020
2.76	0.997110
2.77	0.997197
2.78	0.997282
2.79	0.997365
2.80	0.997445
2.81	0.997523
2.82	0.997599
2.83	0.997673
2.84	0.997744
2.85	0.997814
2.86	0.997882
2.87	0.997948
2.88	0.998012
2.89	0.998074
2.90	0.998134
2.91	0.998193
2.92	0.998250
2.93	0.998305
2.94	0.998359
2.95	0.998411
2.96	0.998462
2.97	0.998511
2.98	0.998559
2.99	0.998605

Example: confidence limits for the μ of a normal distribution

- Solving for μ:

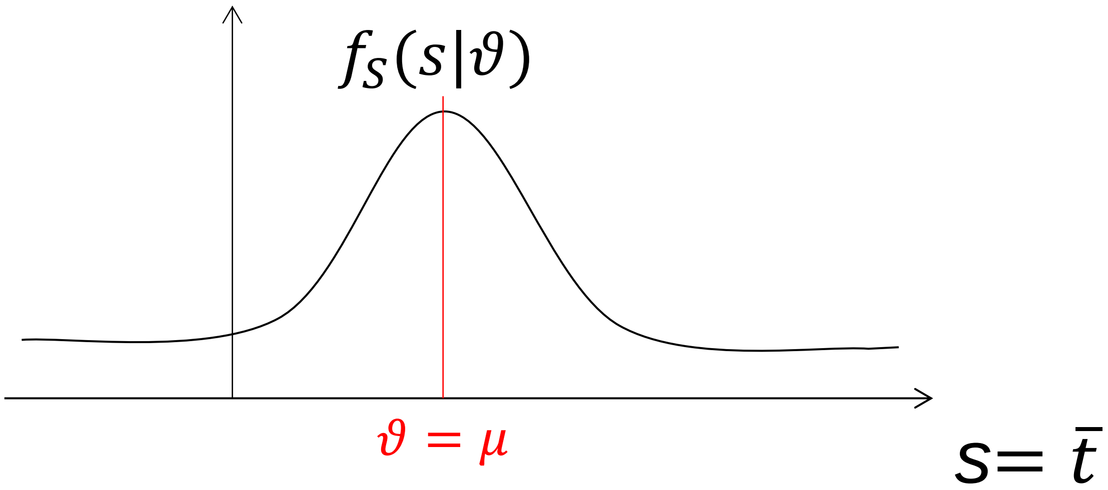
$$P\left(-1.645 < \frac{\bar{t} - \mu}{\frac{\sigma}{\sqrt{n}}} < 1.645\right) = 0.9$$

$$P(\underbrace{\bar{t} - 1.645\frac{\sigma}{\sqrt{n}}}_{\vartheta_1} < \mu < \underbrace{\bar{t} - 1.645\frac{\sigma}{\sqrt{n}}}_{\vartheta_2}) = 0.9$$



Interval estimates of reliability parameters

- t_1, t_2, ….,t_n = samples from the population distribution
- ϑ = unknown characteristic of the population,
- $S = \hat{\vartheta} = g(t_1,t_2,\ldots,t_n)$ = estimated characteristic,
- S is a random variable (being a function of random failure times) → S is characterized by a probability density function, $f_S(s|\vartheta)$.



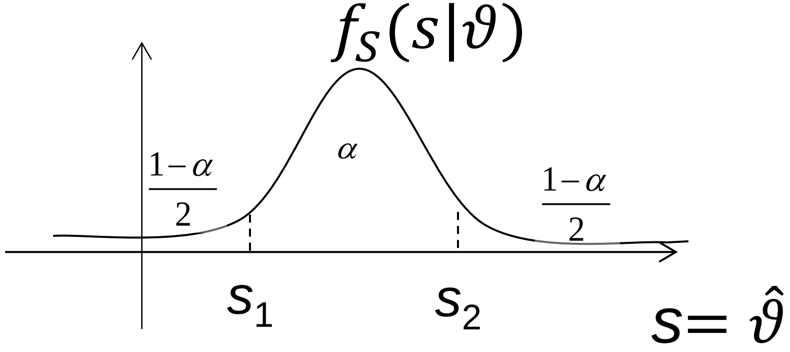
- We fix a value of confidence α<1, and we want to find the interval (S_1,S_2) such that the probability that the true value of the parameter be in the interval is α

Two-sided confidence interval

- Two-sided confidence interval of $\hat{\vartheta}$ at a level of confidence α:
- We want to find $s_1(\vartheta)$ and $s_2(\vartheta)$ such that:

$$P[s_1(\vartheta) \le \hat{\vartheta} = s \le s_2(\vartheta)] = \alpha$$

$$\int_{s_1(\vartheta)}^{s_2(\vartheta)} f_S(s \mid \vartheta) ds = \alpha$$



- The above expression can be rewritten to express the inequality in terms of the unknown characteristic ϑ:
  - μ of a normal distribution with known σ

$$s_1(\mu) = \mu - A$$
$$s_2(\mu) = \mu + A$$

$\xi_{0.95} = 1.645$

A is known! ($1.645\ \sigma/\sqrt{n}$)
- it can computed from the variance of the distribution

$$P[s_1(\vartheta) < s = \hat{\vartheta} < s_2(\vartheta)] = P[\mu - A < \hat{\vartheta} < \mu + A] = \alpha$$

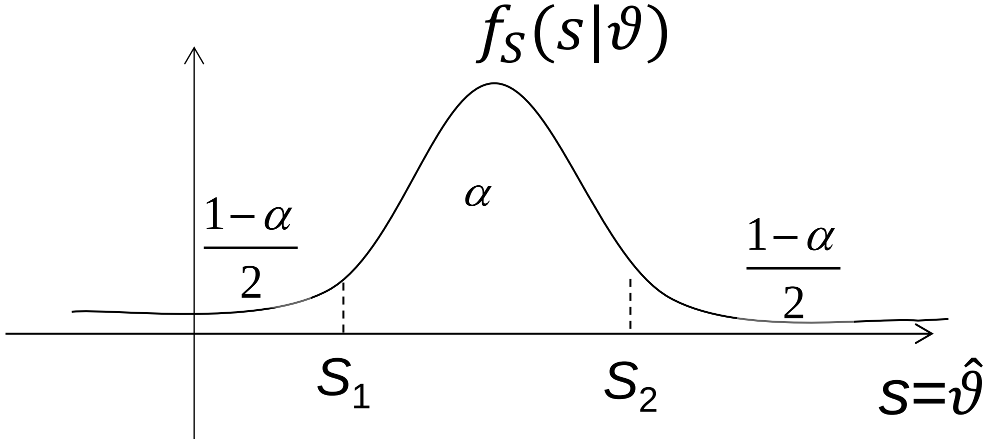
$$P[\hat{\vartheta} - A < \mu < \hat{\vartheta} + A] = \alpha$$

Two-sided confidence interval

- Two-sided confidence interval of $\hat{\vartheta}$ at a level of confidence 1-2α:
- We want to find $s_1(\vartheta)$ and $s_2(\vartheta)$ such that:

$$P[s_1(\vartheta) \le \hat{\vartheta} = s \le s_2(\vartheta)] = \alpha$$

$$\int_{s_1(\vartheta)}^{s_2(\vartheta)} f_S(s \mid \vartheta) ds = \alpha$$



- The above expression can be rewritten to express inequality in terms of the unknown characteristic ϑ:

$$P(\vartheta_1 \le \vartheta < \vartheta_2) = \alpha$$

Fixed, but unknown

known
(but it depends from the samples t_1, t_2, ….,t_n)

known
(but it depends from the samples t_1, t_2, ….,t_n)

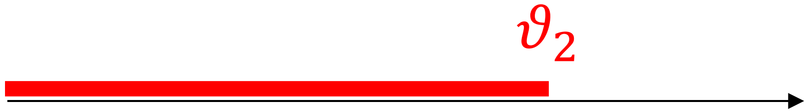
One sided confidence limits

- Lower confidence limit with confidence α

$$P(\vartheta > \vartheta_1) = \alpha$$

- Upper confidence limit with confidence α

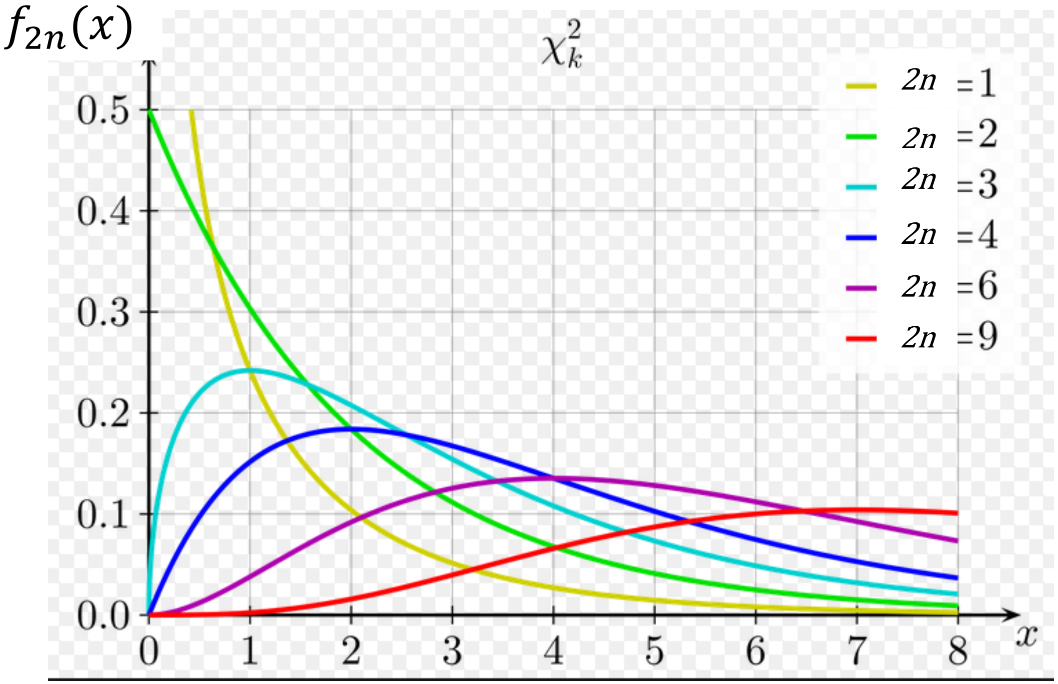
$$P(\vartheta < \vartheta_2) = \alpha$$



Confidence limits for exponential distribution

- T=TTT $= \sum_{j=1}^{n} T_j$ (Total Time on Test)
- n = number of failures

It can be shown that $2\lambda T$ is distributed according to a χ^2 distribution with 2n degree of freedom: $\chi^2(2n)$ (Dudewicz and Misra, 1988, p. 276)

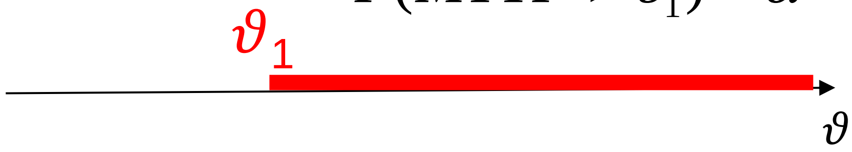


Lower Confidence limits for estimate of MTTF = $\frac{1}{\hat{\lambda}}$

- T = TTT = $\sum_{j=1}^{n} T_j$ (Total Time on Test)
- n = number of failures

It can be shown that $2\lambda T$ is distributed according to a χ^2 distribution with 2*r* degree of freedom: $\chi^2(2\mathrm{n})$ (Dudewicz and Misra, 1988, p. 276)

$$P(MTTF > \vartheta_1) = \alpha$$



Percentile (from tables)

$$P[2\lambda T \leq \chi^2_\alpha(2n)] = \alpha$$

$$P\left[\lambda \leq \frac{\chi^2_\alpha(2n)}{2T}\right] = \alpha$$

Lower confidence limit

$$P\left[MTTF \geq \frac{2T}{\chi^2_\alpha(2n)}\right] = \alpha$$

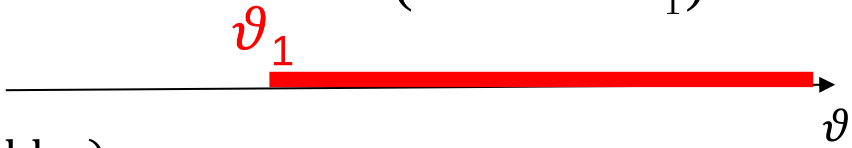
Lower Confidence limits for estimate of MTTF = $\frac{1}{\hat{\lambda}}$

- Type 2 censored test (stop at the r-th failure)
- T=TTT (Total Time on Test)
- n= number of components on test
- r=number of failures
- The last (i.e., r-th) failure has occurred at T_r

$$T = \sum_{j=1}^{r} T_j + (n-r)T_r$$

It can be shown that $2\lambda T$ is distributed according to a χ^2 distribution with 2r degree of freedom: $\chi^2(2r)$ (Dudewicz and Misra, 1988, p. 276)

$$P(MTTF > \vartheta_1) = \alpha$$



Percentile (from tables)

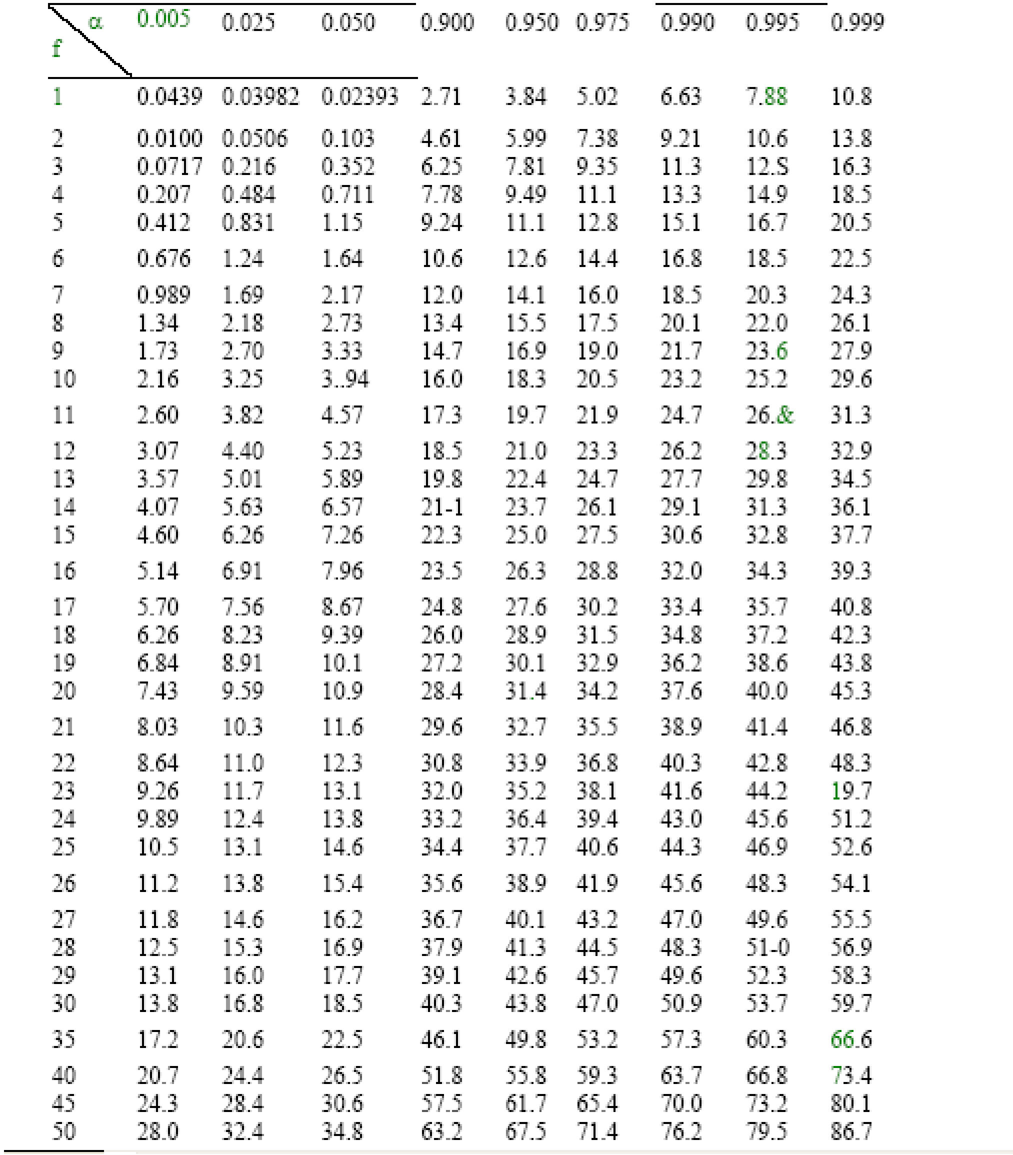
$$P[2\lambda T \leq \chi^2_\alpha(2r)] = \alpha$$

$$P\left[\lambda \leq \frac{\chi^2_\alpha(2r)}{2T}\right] = \alpha$$

Lower confidence limit

$$P\left[MTTF \geq \frac{2T}{\chi^2_\alpha(2r)}\right] = \alpha$$

α Percentile values of the χ^2(f) distribution



α f	0.005	0.025	0.050	0.900	0.950	0.975	0.990	0.995	0.999
1	0.0439	0.03982	0.02393	2.71	3.84	5.02	6.63	7.88	10.8
2	0.0100	0.0506	0.103	4.61	5.99	7.38	9.21	10.6	13.8
3	0.0717	0.216	0.352	6.25	7.81	9.35	11.3	12.S	16.3
4	0.207	0.484	0.711	7.78	9.49	11.1	13.3	14.9	18.5
5	0.412	0.831	1.15	9.24	11.1	12.8	15.1	16.7	20.5
6	0.676	1.24	1.64	10.6	12.6	14.4	16.8	18.5	22.5
7	0.989	1.69	2.17	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	2.18	2.73	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.70	3.33	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	3.25	3..94	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.82	4.57	17.3	19.7	21.9	24.7	26.&	31.3
12	3.07	4.40	5.23	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	5.01	5.89	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	5.63	6.57	21-1	23.7	26.1	29.1	31.3	36.1
15	4.60	6.26	7.26	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	6.91	7.96	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	7.56	8.67	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	8.23	9.39	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	8.91	10.1	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	9.59	10.9	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	10.3	11.6	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	11.0	12.3	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	11.7	13.1	32.0	35.2	38.1	41.6	44.2	19.7
24	9.89	12.4	13.8	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	13.1	14.6	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	13.8	15.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	14.6	16.2	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	15.3	16.9	37.9	41.3	44.5	48.3	51-0	56.9
29	13.1	16.0	17.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	16.8	18.5	40.3	43.8	47.0	50.9	53.7	59.7
35	17.2	20.6	22.5	46.1	49.8	53.2	57.3	60.3	66.6
40	20.7	24.4	26.5	51.8	55.8	59.3	63.7	66.8	73.4
45	24.3	28.4	30.6	57.5	61.7	65.4	70.0	73.2	80.1
50	28.0	32.4	34.8	63.2	67.5	71.4	76.2	79.5	86.7

Confidence limits for estimate of MTTF = $\frac{1}{\hat{\lambda}}$

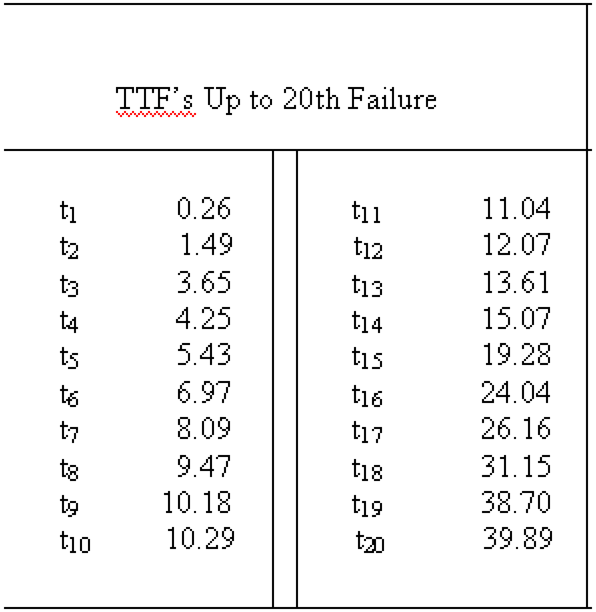
T=TTT (Total Time on Test)
r=number of failures

	I, fixed t_0	II, fixed r
one-sided (lower) $P(MTTF > \vartheta_1) = \alpha$  	$\vartheta_1 = \frac{2T}{\underset{\substack{\downarrow \\ \text{percentile}}}{\chi^2_\alpha} \quad \underbrace{(2r+2)}_{\text{\#of degrees of freedom}}}$	$\vartheta_1 = \frac{2T}{\chi^2_\alpha(2r)}$
two-sided (lower and upper) $P(\vartheta_1 < MTTF < \vartheta_2) = \alpha$  	$(\vartheta_1, \vartheta_2) = \frac{2T}{\chi^2_{\frac{1+\alpha}{2}}(2r+2)}, \frac{2T}{\chi^2_{\frac{1-\alpha}{2}}2r}$	$(\vartheta_1, \vartheta_2) = \frac{2T}{\chi^2_{\frac{1+\alpha}{2}}(2r)}, \frac{2T}{\chi^2_{\frac{1-\alpha}{2}}(2r)}$

Confidence limits for MTTF: Example 1

- 30 identical components,
- Type II censoring with r = 20

Find the 95% two-sided confidence
limits for the MTTF,



TTF's Up to 20th Failure

t_1	0.26	t_{11}	11.04
t_2	1.49	t_{12}	12.07
t_3	3.65	t_{13}	13.61
t_4	4.25	t_{14}	15.07
t_5	5.43	t_{15}	19.28
t_6	6.97	t_{16}	24.04
t_7	8.09	t_{17}	26.16
t_8	9.47	t_{18}	31.15
t_9	10.18	t_{19}	38.70
t_{10}	10.29	t_{20}	39.89

Confidence limits for MTTF: Example 2

- Consider failures in NPP with remarkable release of radiation. Identify the 95% lower confidence limit for the mean time to failure (MTTF=1/λ) of western design reactors, assuming:
  - 10000 reactor year of operation
  - 2 failures with remarkable release of radiations (Three Miles Island + Fukushima)
  - Constant failure rate, equal for all the reactors