Monte Carlo Simulation

31.03.2025 | Luca Pinciroli

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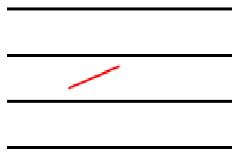
The history of Monte Carlo simulation



Random sampling of numbers

Buffon — 1707-88

Kelvin (kinetic theory of gas - multidimensional integrals) — 1824-1907

Gosset (coefficient of the t-Student) — 1908

Fermi, von Neumann, Ulam — 1930-40's

(Manhattan project, interaction neutron-matter)

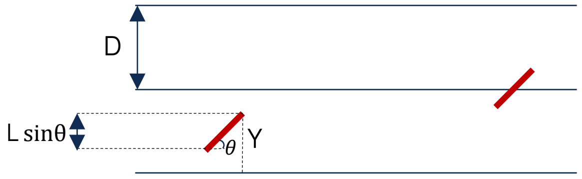
Neutron transport — 1950's

(design of shielding in nuclear devices - 6 dimension-integral)

System Transport (RAMS) — 1990's

Buffon's needle

Buffon considered a set of parallel straight lines a distance *D* apart onto a plane and computed the probability *P* that a needle of length *L* ‹ *D* randomly positioned on the plane would intersect one of these lines.



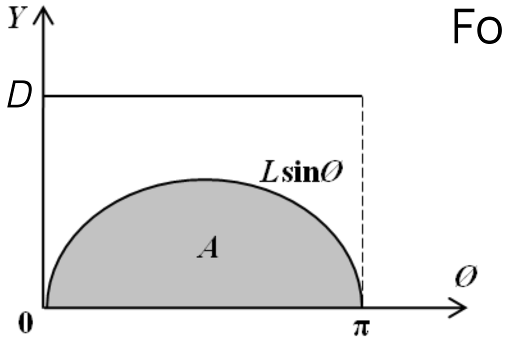
$$P = P\{Y \leq Lsin\theta\}$$

$$Y: f_Y(y) = \frac{1}{D} \qquad y \in [0, D]$$

$$\Theta: f_\Theta(\theta) = \frac{1}{\pi} \qquad \theta \in [0, \pi]$$

For a fixed value of θ:

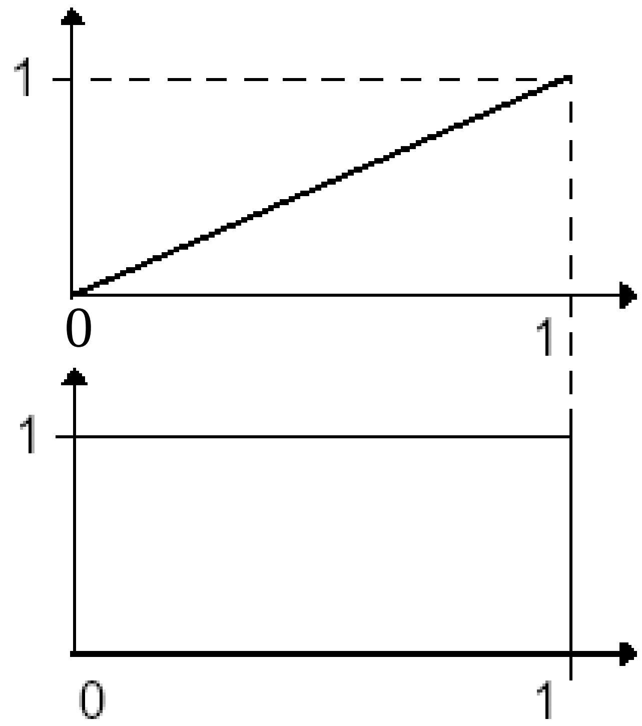
$$P = P\{Y \leq Lsin\theta\} = \int_0^{Lsin\theta} f_Y(y)dy = \int_0^{Lsin\theta} \frac{1}{D} dy = \frac{Lsin\theta}{D}$$



For a random value of $\theta \rightarrow$ joint pdf of (y, θ):

$$P = \int_{\theta=0}^{\pi} \int_{y=0}^{Lsin\theta} f_{Y,\Theta}(y,\theta)dyd\theta = \int_{\theta=0}^{\pi} \frac{1}{\pi} d\theta \int_{y=0}^{Lsin\theta} \frac{1}{D} dy = \int_{\theta=0}^{\pi} \frac{Lsin\theta}{\pi D} d\theta = \frac{2L}{\pi D}$$

Sampling (pseudo) Random Numbers Uniform Distribution



cdf : $U_R(r) = P\{R \le r\} = r$

pdf : $u_R(r) = \dfrac{dU_R(r)}{dr} = 1$

The Random Number Book (1955)

1 million random numbers

73735	45963	78134	63873
02965	58303	90708	20025
98859	23851	27965	62394
33666	62570	64775	78428
81666	26440	20422	05720

15838	47174	76866	14330
89793	34378	08730	56522
78155	22466	81978	57323
16381	66207	11698	99314
75002	80827	53867	37797

99982	27601	62686	44711
84543	87442	50033	14021
77757	54043	46176	42391
80871	32792	87989	72248
30500	28220	12444	71840

Sampling (pseudo) Random Numbers from Uniform Distribution: Linear Congruential Generator (LCG)

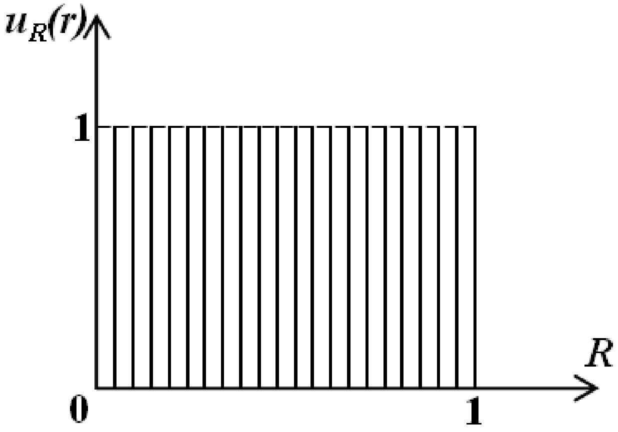
$$R \sim U[0,1)$$

$$x_i = \left(a x_{i-1} + c\right) \bmod m$$

where $a, c \in [0, m-1]$

$m \gg 1$

$$r_i = \frac{x_i}{m}$$



Example: $a = 5$, $c = 1$, $m = 16$

$$x_0 = 2 \Rightarrow r_0 = \frac{2}{16}$$

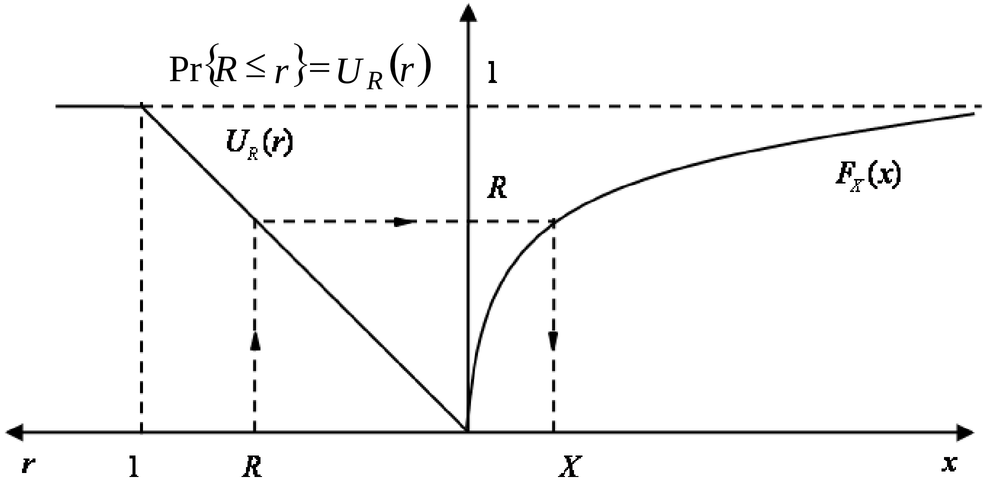
$$x_1 = (5 \cdot 2 + 1) \bmod 16 = 11 \Rightarrow r_1 = \frac{11}{16}$$

...

$$x_{15} = 13 \Rightarrow r_{15} = \frac{13}{16}$$

$$x_{16} = 2$$

Sampling (pseudo) random numbers from generic distribution: Inverse Transform Method



Sample R from U_R(r) and find X:

$$X = F_X^{-1}(R)$$

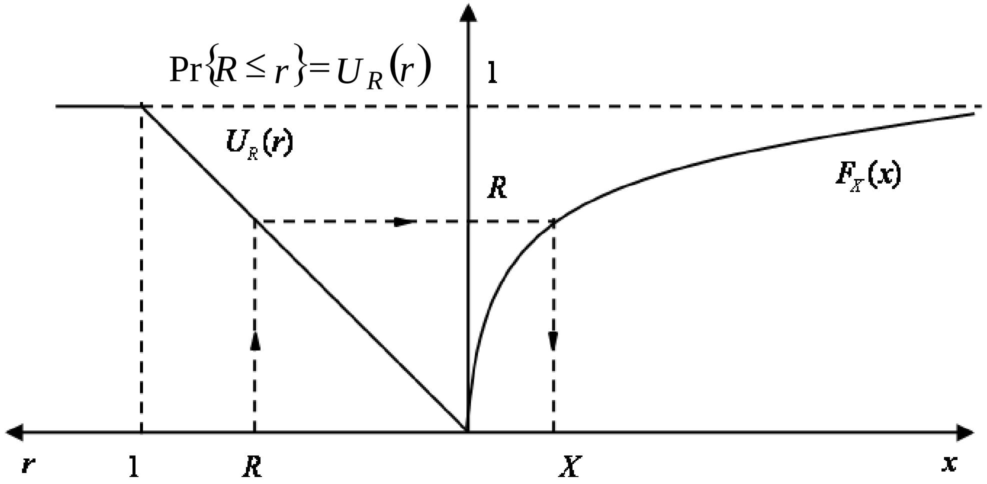
Question: which distribution does X obey?

$$P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}$$

$$F$$

$$P\{R \leq F_X(x)\}$$

Sampling (pseudo) random numbers from generic distribution: Inverse Transform Method



Sample R from $U_R(r)$ and find X:

$$X = F_X^{-1}(R)$$

<u>Question</u>: which distribution does X obey?

$$P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}$$

Application of the operator F_x to the argument of P above yields

$$P\{X \leq x\} = P\{R \leq F_X(x)\} = F_X(x)$$

<u>Summary</u>:
From an R ~ $U_R(r)$ we obtain an X ~ $F_X(x)$

Buffon's needle: MC simulation with inverse transform method

- Initialize the counter of the number of times the needle intercepts a line: $N_s = 0$
- Simulate N > > 1 needle throws by
  - Sampling Y from the uniform distribution in the interval $[0,D]$: $f_Y(y) = \frac{1}{D} \qquad y \in [0, D]$

    by using the inverse transform method:

$$F_Y(y) = \begin{cases} \frac{y}{D} & for\ 0 \le y \le D \\ 0 & for\ y < 0 \\ 1 & for\ y > D \end{cases} \Rightarrow R = F_Y(y) = \frac{Y}{D} \Rightarrow Y = R_1 D$$

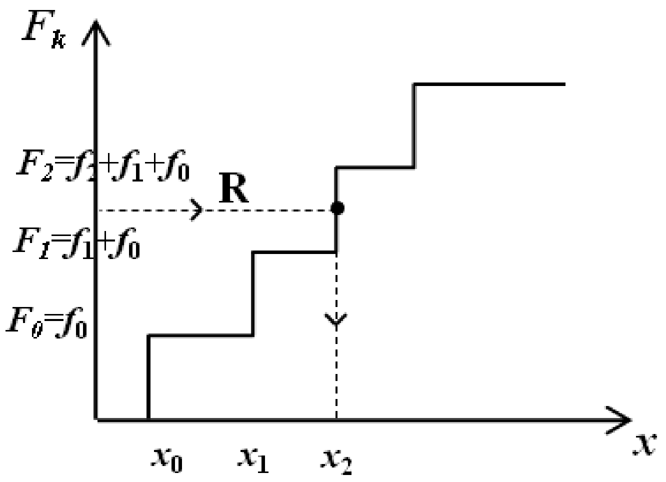
  - Sampling Θ from the uniform distribution in the interval $[0,\pi]$ by using the inverse transform method: $\Theta = R_2\pi$
  - If the needle intercepts a line, i.e. $Y \le Lsin\Theta$, set $N_s = N_s + 1$
- At the end of the procedure: $P = \frac{N_s}{N} \cong \frac{2L}{\pi \mathrm{D}}$

Sampling by the Inverse Transform Method: Discrete Distributions

$$\Omega = \{x_0, x_1, \ldots, x_k, \ldots\}$$

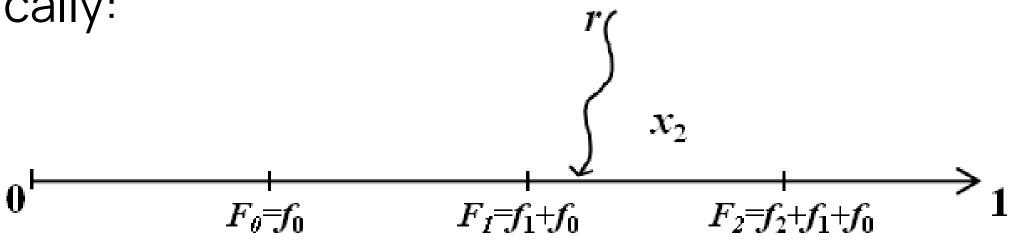
$$F_k = P\{X \le x_k\} = \sum_{i=0}^{k} P[X = x_i]$$

sample ar $R \sim U[0,1)$



- $P[F_{k-1} < R \le F_k] = F_R(F_k) - F_R(F_{k-1})$
- $R \sim U[0,1)$ and $F_R(r) = r$
  $\Rightarrow P[F_{k-1} < R \le F_k] = F_k - F_{k-1} = f_k = P[X = x_k]$

Graphically:



Sampling by the Inverse Transform Method: Exponential Distribution

- Markovian system with two states (good, failed)
- hazard rate, λ = constant
- cdf $F_T(t) = P\{T \leq t\} = 1 - e^{-\lambda t}$
- pdf $f_T(t) \cdot dt = P\{t \leq T < t + dt\} = \lambda e^{-\lambda t} \cdot dt$
- Sampling a failure time *T*

$$R \equiv F_R(r) = F_T(t) = 1 - e^{-\lambda t}$$

$$\Downarrow$$

$$T = F_T^{-1}(R) = -\frac{1}{\lambda} \ln(1 - R)$$

Sampling by the Inverse Transform Method: Weibull Distribution

cdf: $F_T(t) = P\{T \le t\} = 1 - e^{-\beta t^{\alpha}}$

pdf: $f_T(t) \cdot dt = P\{t \le T < t + dt\} = \alpha \beta t^{\alpha - 1} e^{-\beta t^{\alpha}} \cdot dt$

- Sampling a failure time *T*

$$R \equiv F_R(r) = F_T(t) = 1 - e^{-\lambda t^{\alpha}}$$

$$\Downarrow$$

$$T = F_T^{-1}(R) = \left(-\frac{1}{\beta} \ln(1 - R) \right)^{\frac{1}{\alpha}}$$

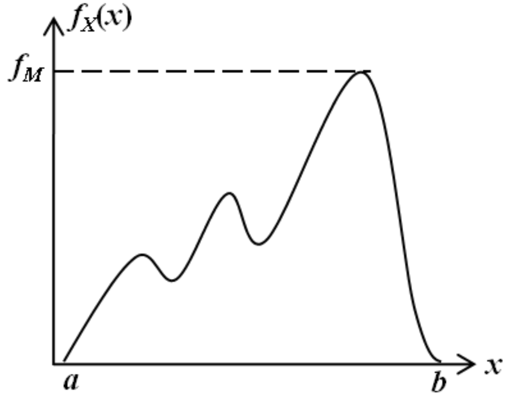
Sampling by the Rejection Method: von Neumann Algorithm

- Given a pdf $f_X(x)$ limited in (a,b), let

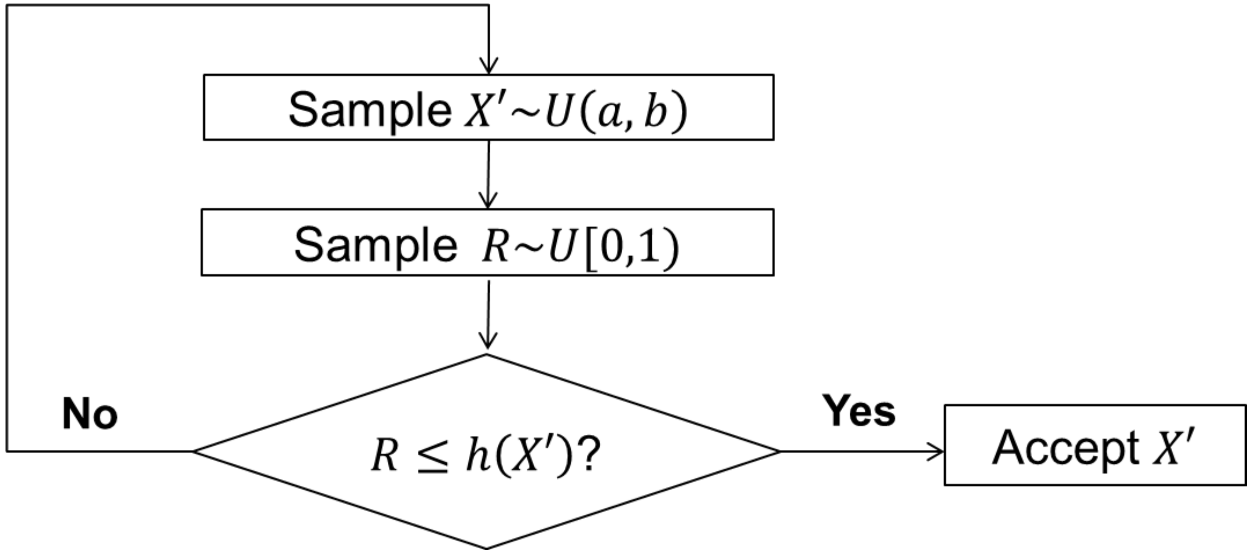
$$h\ (x) = \frac{f_X\,(x)}{f_M}$$

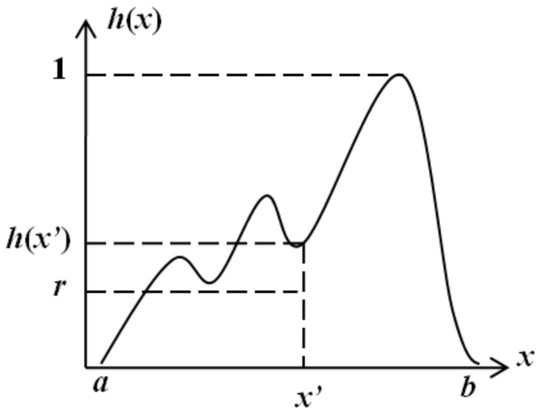
so that

$$0 \le h\ (x) \le 1, \forall x \in (a,b)$$



- The operative procedure





Sampling by the Rejection Method: von Neumann Algorithm

More generally:

$$X \sim f_X(x) = g_{X'}(x) \cdot H(x)$$

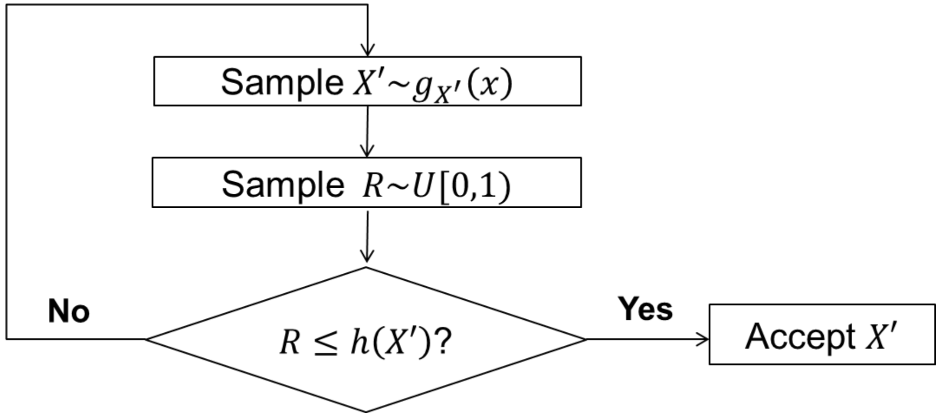
$$B_H : \max_x H(x)$$

$$h(x) = \frac{H(x)}{B_H}, \quad 0 \le h(x) \le 1$$

$g_{X'}(x)$ is a distribution it is easy to sample from (uniform, normal, …)
$H(x)$ is a shape correction factor $f(x)/g(x)$

The operative procedure:

- sample $X' \sim g_{X'}(x)$, and calculate $h(X')$
- sample $R \sim U[0,1)$. If $R <= h(X')$ the value X' is accepted; else start again.



We show that the accepted value is actually a realization of X sampled from $f_X(x)$

$$P\left[X' \le x \mid \text{accepted}\right] = \frac{P\left[X' \le x \cap \text{accepted}\right]}{P\left[\text{accepted}\right]} = \frac{P\left[X' \le x \cap R \le h(X')\right]}{P\left[\text{accepted}\right]}$$

Sampling by the Rejection Method: von Neumann Algorithm

2.

$$P\left[z \le X' \le z+dz \cap \text{accepted}\right] = P\left[z \le X' \le z+dz\right] P\left[R \le h(z)\right] =$$
$$= g_{X'}(z)dz \cdot h(z)$$

3.

$$P\left[X' \le x \cap R \le h(X')\right] = \int_{-\infty}^{x} g_{X'}(z)dz \cdot h(z)$$

4.

$$P\left[\text{accepted}\right] = \int_{-\infty}^{\infty} g_{X'}(z)dz \cdot h(z) =$$
$$= \frac{1}{B_H} \int_{-\infty}^{\infty} g_{X'}(z)dz \cdot H(z) = \frac{1}{B_H} \int_{-\infty}^{\infty} f_X(x)dx = \frac{1}{B_H}$$

Sampling by the Rejection Method: von Neumann Algorithm

$$P\left[X' \le x \middle| \text{accepted}\right] = \frac{P\left[X' \le x \cap R \le h(x')\right]}{P[\text{accepted}]} = \frac{\int_{-\infty}^{x} g_{X'}(z)dz \cdot h(z)}{\frac{1}{B_H}}$$

$$= \int_{-\infty}^{x} g_{X'}(z)dz \cdot H(z) = \int_{-\infty}^{x} f_X(z)dz = F_X(x)$$

The efficiency of the method is given by the probability of accepted:

$$\varepsilon = P[\text{accepted}] = \int_{-\infty}^{\infty} g_{X'}(z)h(z)dz = \frac{1}{B_H}$$

Sampling by the Rejection Method: von Neumann Algorithm Example

Sample from the pdf: $f_X(x) = \frac{2}{\pi} \cdot \frac{1}{(1+x)\sqrt{x}} \quad 0 \le x \le 1$

Sampling by the Rejection Method: von Neumann Algorithm Example

The operative procedure:

- sample $R_1 \sim U[0,1) \Rightarrow X' = R_1^2$ and $h(X') = \frac{1}{1+R_1^2}$
- sample $R_2 \sim U[0,1)$. If $R_2 \leq h(X')$ accept $X = X'$; else start again

The efficiency of the method is:

$$\varepsilon = \frac{1}{B_H} = \frac{\pi}{4} = 78.5\%$$

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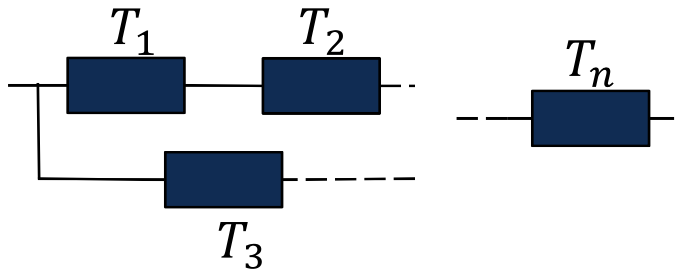
RAM quantities of interest are definite integrals

Some Examples:



- Unreliability:

$$F_T(t_{miss}) = P\{T \leq t_{miss}) = P\{q(T_1, \dots, T_n) \leq t_{miss}\} =$$
$$= \int_{(t_1,\dots,t_n):q(t_1,\dots,t_n)\leq t_{miss}} f_{T_1,\dots,T_n}(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

- $MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$

MC Evaluation of Definite Integrals (1D)

$$G = \int_a^b h(x)dx = \int_a^b g(x) f_X(x) dx$$

- x is a random variable with pdf $f_X(x)$:
- $g(x)$ is a random variable

$$\begin{cases} f_X(x) \geq 0 \\ \int_a^b f_X(x)\,dx = 1 \end{cases}$$

$$E[g(x)] = \int_a^b g(x) f_X(x) dx = G$$

MC Evaluation of Definite Integrals (1D)

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

Problem → Estimate $E[g(x)]$

Solution → Dart Game

1) for $i = 1,2, \ldots, N$
   - Sample X_i from $f_X(x)$ (the probability that a shot hits $x \in dx$ is $f(x)dx$)
   - Compute $g(X_i)$ (the award is $g(x)$)

   End

Consider N trials with results $\{x_1, x_2, \ldots, x_N\}$: the average award is: $G_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i) = \bar{g}$

Random variable!

$E[G_N]$ ⟵ ⟶ $Var[G_N]$

Is G_N a good estimator of $E[g(x)]$?

MC Evaluation of Definite Integrals (1D): Why G_N is a good estimator of *G?*

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$

⬇

G_N is a random variable with:

$$E[G_N] = E\left[\frac{1}{N}\sum_{i=1}^{N} g(x_i)\right] = \frac{1}{N}\sum_{i=1}^{N} E[g(x)] = G$$

$$Var[G_N] = Var\left[\frac{1}{N}\sum_{i=1}^{N} g(x_i)\right] = \frac{1}{N^2}\sum_{i=1}^{N} Var[g(x)] = \frac{1}{N} Var[g(x)]$$

⬇

G_N is an unbiased estimator of *G:* $E[G_N] = G$

G_N is a consistent estimator of *G:* $\lim_{N\to\infty} Var[G_N] = 0$

MC Evaluation of Definite Integrals (1D): Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2} x\right) dx = \frac{2}{\pi} = 0.6366$$

How can we write the integral for MC estimation?

$f(x) = ?$

$g(x) = ?$

MC Evaluation of Definite Integrals (1D): Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2} x\right) dx = \frac{2}{\pi} = 0.6366$$

By setting:

$$f(x) = 1 \quad for \quad x \in [0,1]$$

$$g(x) = \cos\left(\frac{\pi}{2} x\right)$$

We perform N=10^4 trials:

$$x_i \rightarrow U[0,1)$$

$$g(x_i) = \cos\left(\frac{\pi}{2} x_i\right)$$

$$\Rightarrow$$

$$G_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i) = \frac{1}{N} \sum_{i=1}^{N} cos\left(\frac{\pi}{2} x_i\right) = 0,6342$$

MC Evaluation of Definite Integrals (1D): Example

$$G=\int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

We perform N=10^4 trials:

$$x_i \rightarrow U[0,1)$$

$$g(x_i)=\cos(\frac{\pi}{2}x_1)$$

$$N=10^4$$

$$G_N=\frac{1}{N}\sum_{i=1}^{N} g(x_i)=\frac{1}{N}\sum_{i=1}^{N}\cos\left(\frac{\pi}{2}x_i\right)=0.6342$$

$$Var[G_N]=\frac{1}{N}Var[g(x)]=\frac{1}{N}\left(E[g^2(x)]-(E[g(x)])^2\right)=\frac{1}{N}\left(E[g^2(x)]-G^2\right)$$

Unknown in a practical case!

MC Evaluation of Definite Integrals (1D): Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

We perform N=10^4 trials:

$$x_i \rightarrow U[0,1)$$
$$g(x_i) = \cos(\frac{\pi}{2}x_1)$$
$$N = 10^4$$

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(x_i) = \frac{1}{N}\sum_{i=1}^{N}\cos\left(\frac{\pi}{2}x_i\right) = 0.6342$$

$$Var[G_N] = \frac{1}{N}Var[g(x)] = \frac{1}{N}\left(E[g^2(x)] - (E[g(x)])^2\right) = \frac{1}{N}\left(E[g^2(x)] - G^2\right)$$

Unknown in a practical case!

$$G = E[g(x)] = \frac{2}{\pi}$$

$$E[g^2(x)] = \int_0^1 \cos^2\left(\frac{\pi}{2}x\right)dx = \frac{1}{2}$$

$$Var[G_N] = \frac{1}{10^4}\left(\frac{1}{2} - \left(\frac{2}{\pi}\right)^2\right) = 9.47 \cdot 10^{-6}$$

MC Evaluation of Definite Integrals (1D): Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

We perform N=10^4 trials:

$$x_i \to U[0,1)$$
$$g(x_i) = \cos(\frac{\pi}{2}x_1)$$
$$N = 10^4$$

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(x_i) = \frac{1}{N}\sum_{i=1}^{N}\cos\left(\frac{\pi}{2}x_i\right) = 0.6342$$

$$Var[G_N] = \frac{1}{N}Var[g(x)] = \frac{1}{N}\left(E[g^2(x)] - (E[g(x)])^2\right) = \frac{1}{N}\left(E[g^2(x)] - G^2\right)$$

$$G \approx G_N = 0.6342$$
$$E[g^2(x)] \approx \frac{1}{N}\sum_{i=1}^{N} g^2(x_i)$$

They can be computed during the MC simulation!

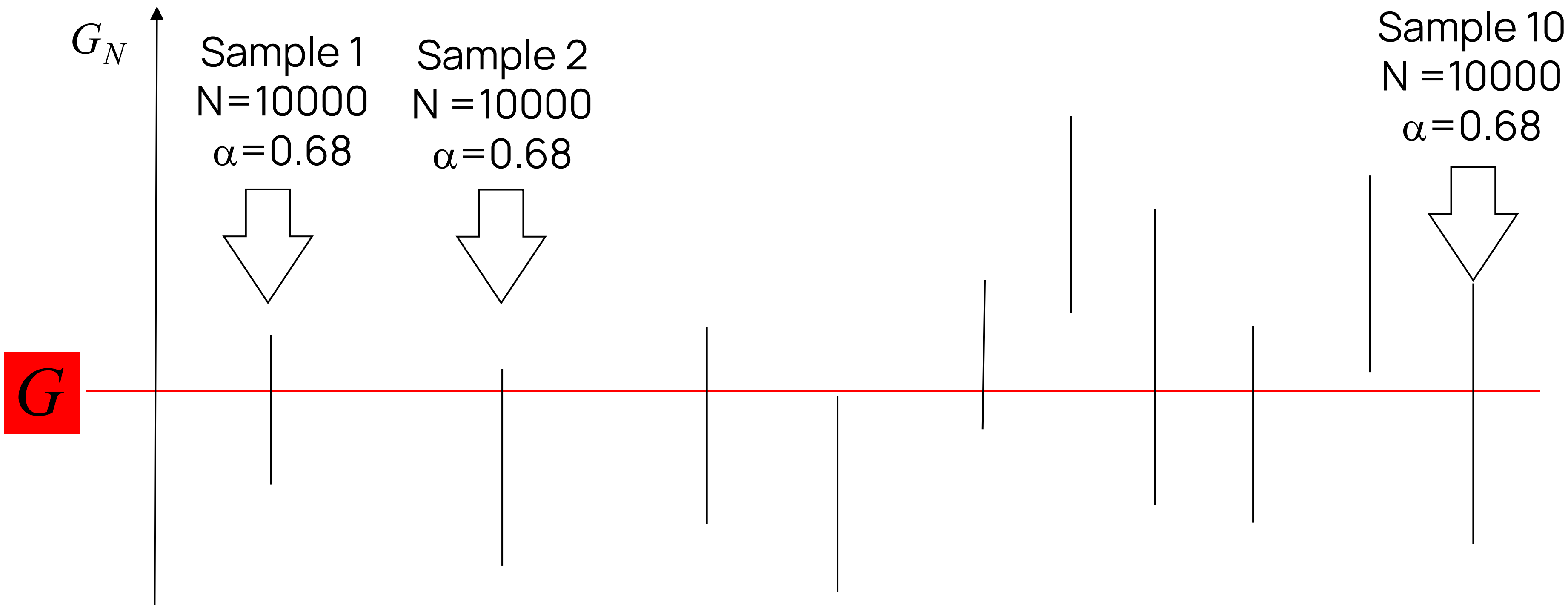
$$Var[G_N] \approx \frac{1}{N}\left(\overline{g^2} - G_N^{\ 2}\right) = 9.6 \cdot 10^{-6}$$

Estimated Variance!

$$G = 0.6342 \pm \sqrt{9.6 \cdot 10^{-6}} = 0.6342 \pm 0.0031$$

True value is 0.6366

MC Integral: interpretation of the variance



unknown

$$P\left(G_N \in \left[G-\sigma_{G_N}; G+\sigma_{G_N}\right]\right) \approx 0.68$$

$$P\left(G \in \left[G_N-\sigma_{G_N}; G_N+\sigma_{G_N}\right]\right) \approx 0.68$$

Definite Integral – Monte Carlo Vs Deterministic Numerical Integration

Why Monte Carlo instead of deterministic numerical integration?

Because the latter suffers from two major issues when dealing with highly multidimensional problems:

1. The number of function evaluations (grid) increases combinatorially with the number of dimensions

2. The boundaries of the multidimensional integration domain *D* become intractable

Estimation error – variance reduction

- The estimate G_N becomes more precise (less uncertain) as the estimator variance $Var[G_N]$ decreases!

- How can we achieve lower $Var[G_N] = \frac{1}{N} Var[g(x)]$?

1. Increasing the number N of MC trials $\Rightarrow$ “brute force”

2. Decreasing $Var[g(x)] \Rightarrow$ variance reduction techniques

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx \equiv \int_D g_1(x) f_1(x) dx$$

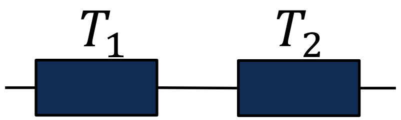
Forced (biased) MC simulation

MC estimation of RAM quantities of interest: Unreliability estimation example

$T = System\ failure\ time$

$T_i \approx f_{T_i}(t_i) = Component\ failure\ time$

$F_T(t)???$



$T_1 \approx f_{T_1}(t_1)$

$T_2 \approx f_{T_2}(t_2)$

$$F_T(t_{miss}) = P\{T \leq t_{miss}) =$$

$$= \int_0^{t_{miss}} f_T(t)dt = \int_0^{+\infty} I_g(t) f_T(t)dt$$

with

$$I_g(t) = \begin{cases} 1 & if\ t \leq t_{miss} \\ 0 & otherwise \end{cases}$$

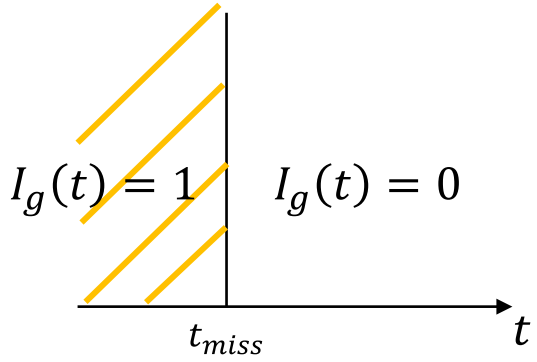
$$G = \int_a^b g(x)f(x)dx = E[g(x)]$$

$$G = F_T(t_{miss}) \quad g(x) = I_g(t) \quad f(x) = f_T(t)$$

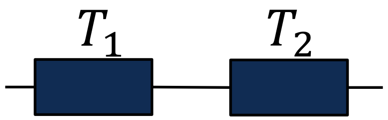
$T\ is\ evaluated\ by\ means\ of\ MC\ simulation$

$$F_T(t_{miss}) = P\{T \leq t_{miss}) =$$

$$= \int_0^{t_{miss}} f_T(t)dt = \int_0^{+\infty} I_g(t) f_T(t)dt$$



MC estimation of RAM quantities of interest: Unreliability estimation example

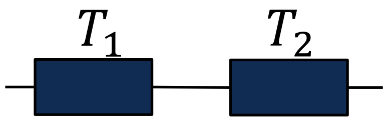


$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$
$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

t_{miss} = 8760 h
λ_1 = 2 10^{-4} h^{-1}
λ_2 = 5 10^{-3} h^{-1}

OBJECTIVE:
MC Estimate System Unreliability at the Mission Time

MC estimation of RAM quantities of interest: Unreliability estimation example



$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$

$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$

t_{miss} = 8760 h

λ_1 = 2 10^{-4}h^{-1}

λ_2 = 5 10^{-3} h^{-1}

OBJECTIVE:
MC Estimate System Unreliability at the Mission Time

$$N = 10000 \rightarrow \lambda_{sys} = \lambda_1 + \lambda_2 \rightarrow for\ i = 1, \dots, N \rightarrow t_i = -\frac{1}{\lambda_{sys}} \ln(1 - r_i), r_i \rightarrow U[0,1)$$

$$F_N(t_{miss}) = \frac{1}{N}\sum_{i=1}^{N} I_g(t_i) = 0{,}9891 \quad where \quad I_g(t_i) = \begin{cases} 1 & if\ t_i < t_{miss} \\ 0 & otherwise \end{cases}$$

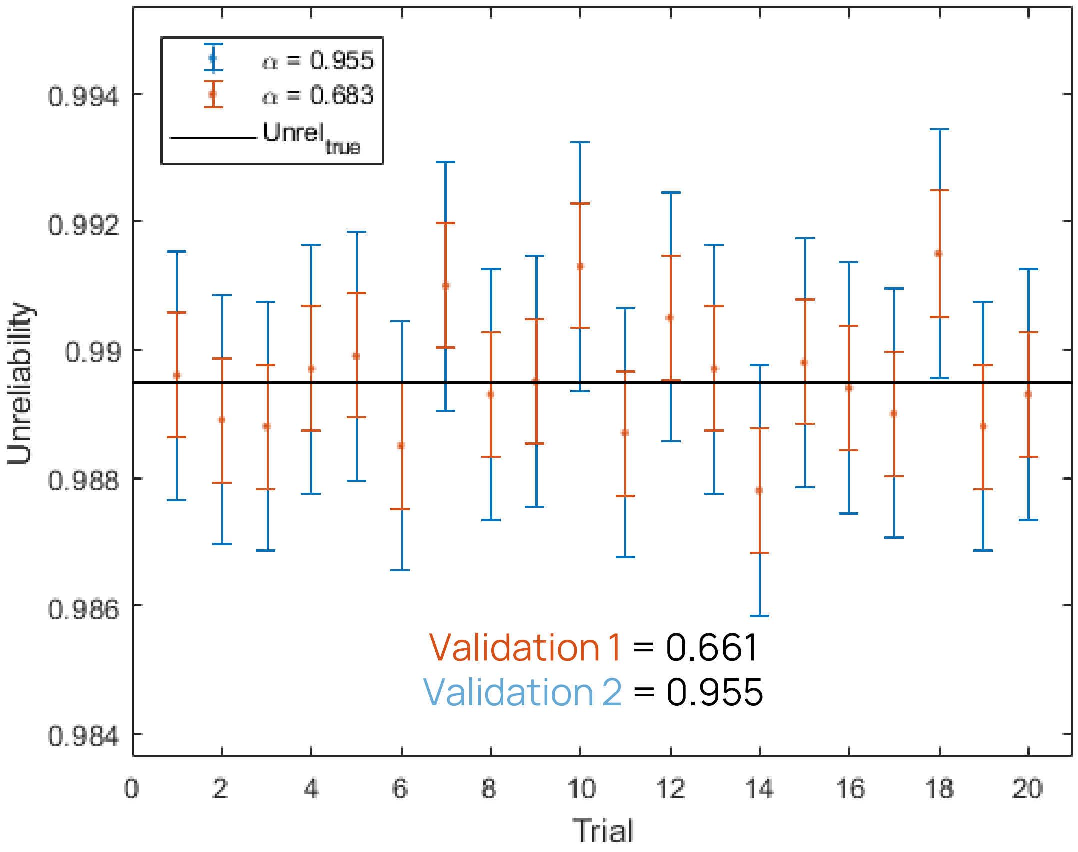
$$Var[F_N(t_{miss})] \approx \frac{1}{N}\left(\left(\frac{1}{N}\sum_{i=1}^{N}\left(I_g(t_i)\right)^2\right) - F_N^2(t_{miss})\right) \approx \frac{1}{N}\left(F_N(t_{miss}) - F_N^2(t_{miss})\right) = 1{,}08\ 10^{-6}$$

MC ESTIMATION OF SYSTEM UNRELIABILITY $= F_N(t_{miss}) \pm \sqrt{Var[F_N(t_{miss})]} = 0{,}9891 \pm 1{,}0\ 10^{-3}$

TRUE VALUE OF SYSTEM UNRELIABILITY $= 1 - e^{-(\lambda_1+\lambda_2)t_{miss}} = 0{,}9895$

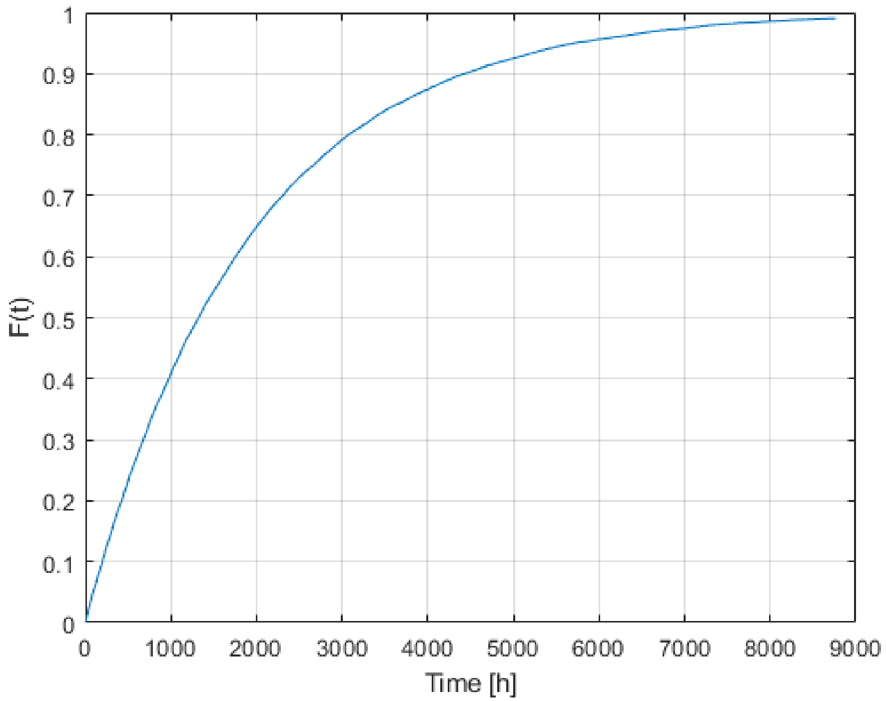
MC estimation of RAM quantities of interest: Unreliability estimation example

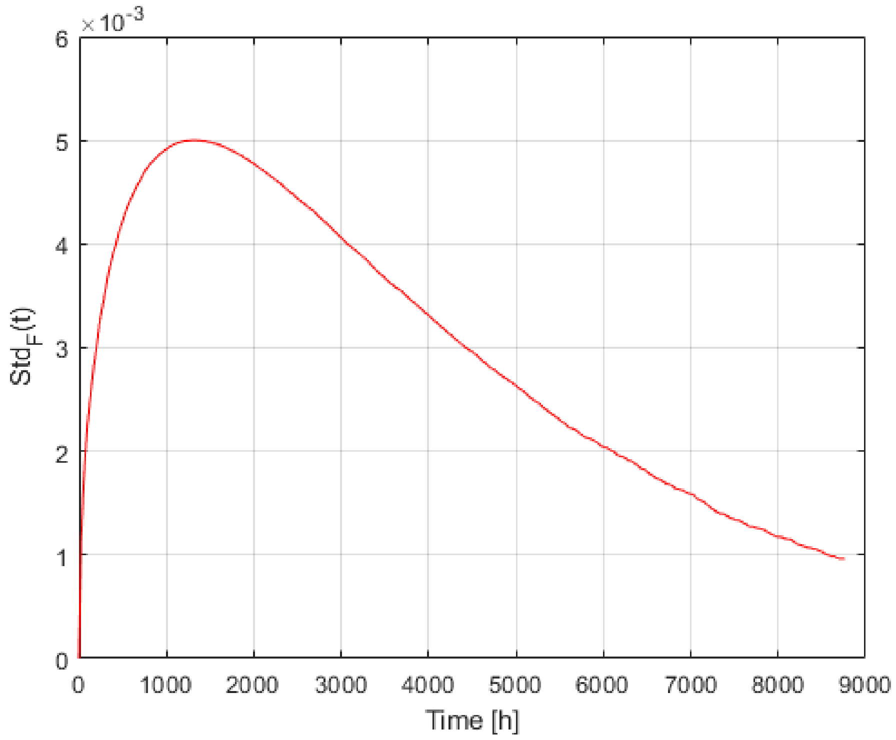
Repeating the system unreliability estimation 1000 times …



MC estimation of RAM quantities of interest: Unreliability estimation example

Time evolution





MTTF estimation example

$$MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)] \qquad G = MTTF \qquad g(x) = t \qquad f(x) = f_T(t)$$

Exponential failure time T

$f_T(t) = \lambda e^{-\lambda t}$

λ = 0,2 h^{-1}

OBJECTIVE:
MC Estimate System MTTF

MTTF estimation example

$$MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)] \qquad G = MTTF \qquad g(x) = t \qquad f(x) = f_T(t)$$

Exponential failure time T

$f_T(t) = \lambda e^{-\lambda t}$ $\qquad \lambda = 0{,}2\ h^{-1}$

OBJECTIVE:
MC Estimate System MTTF

Considering $N = 10000$ trials:

$$MTTF_N = \frac{1}{N}\sum_{i=1}^{N} T_i = 4{,}98\ h$$

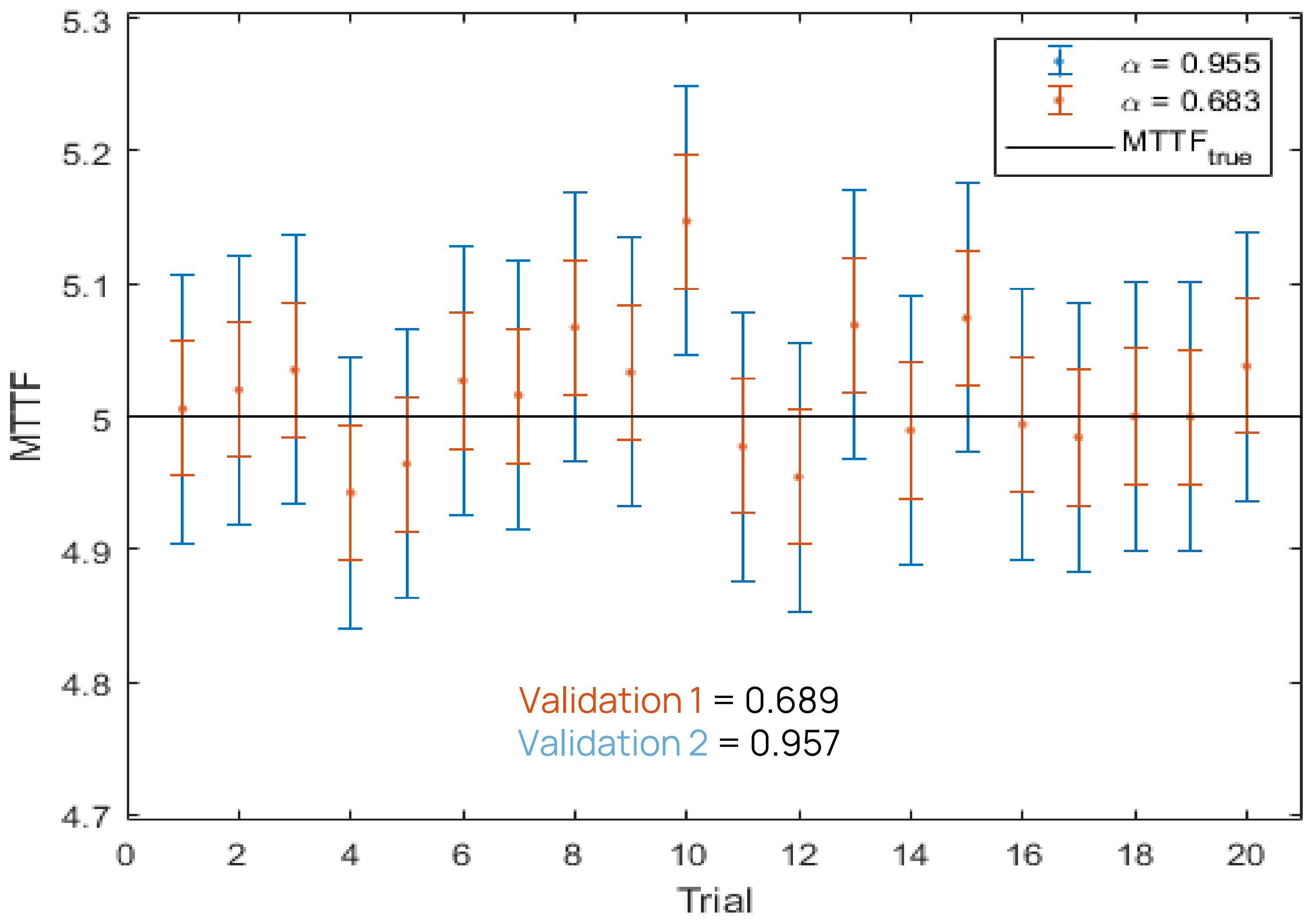
$$Var[MTTF_N] \approx \frac{1}{N}\left(\left(\frac{1}{N}\sum_{i=1}^{N} T_i^2\right) - MTTF_N^2\right) = 0{,}0024\ h$$

MC ESTIMATION OF SYSTEM $MTTF = MTTF_N \pm \sqrt{Var[MTTF_N]} = 4{,}98 \pm 0{,}049$

TRUE VALUE OF SYSTEM $MTTF = \frac{1}{\lambda} = 5\ h$

MTTF estimation example

Repeating the system MTTF estimation 1000 times ...



CONTENTS

Monte Carlo simulation for system reliability

SYSTEM = system of Nc suitably connected components.

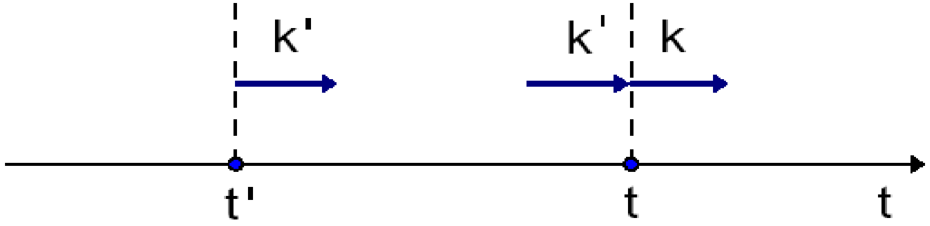
COMPONENT = a subsystem of the system (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.

STATE of the SYSTEM at t = the set of the states in which the Nc components stay at t. The states of the system are labeled by a scalar which enumerates all the possible combinations of all the component states.

SYSTEM TRANSITION = when any one of the plant components performs a state transition we say that the system has performed a transition. The time at which the system performs the n-th transition is called t_n and the system state thereby entered is called k_n.

SYSTEM LIFE = stochastic process.

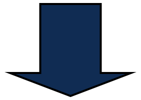
Stochastic Transitions: Governing Probabilities



T(t | t'; k')dt = conditional probability of a transition at t ∈ dt, given that the preceding transition occurred at t' and that the state thereby entered was k'.

C(k | k'; t) = conditional probability that the system enters state k, given that a transition occurred at time t when the system was in state k'.

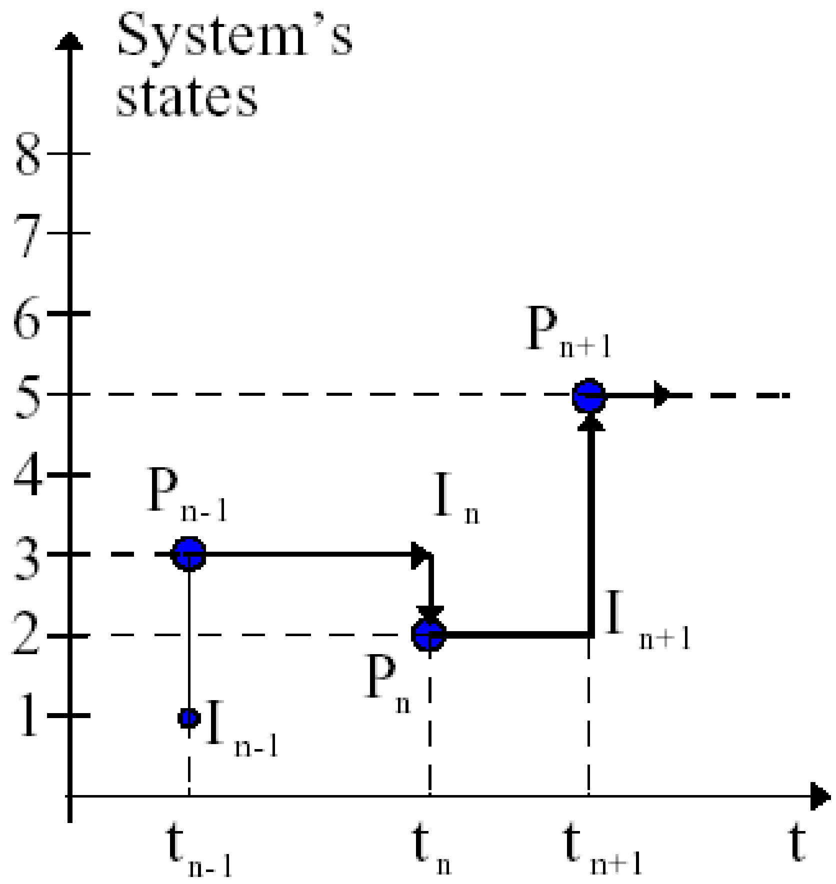
Both these probabilities form the "trasport kernel" :
K(t; k | t'; k')dt = T(t | t'; k')dt C(k | k'; t)



ψ(t; k) = ingoing transition density or probability density function (pdf) of a system transition at t, resulting in the entrance in state k

System life: random walk

Random walk = realization of the system life generated by the underlying state-transition stochastic process.



The von Neumann's Approach and the Transport Equation

The transition density $\psi(t; k)$ is expanded in series of the partial transition densities:

$\psi^n(t; k)$ = pdf that the system performs the n–th transition at t, entering the state k.

Then,

$$\psi(t,k) = \sum_{n=0}^{\infty} \psi^n(t,k) =$$

$$= \psi^0(t,k) + \sum_{k'} \int_{t_0}^{t} dt' \psi(t',k') K(t,k \mid t',k')$$

Transport equation for the plant states

Monte Carlo Solution to the Transport Equation (1)

Von Neumann approach:

- Initial Conditions: $t_0=t^*$, $k_0=k^*$, $P_0\equiv P^*$
- The subsequent transition densities in the random walk:

$$\psi^1(t_1,k_1)=K(t_1,k_1|t_0,k_0)$$

$$\psi^2(t_2,k_2)=\sum_{k_1}\int_{t^*}^{t_2}\psi^1(t_1,k_1)dt_1K(t_2,k_2|t_1,k_1)$$

……

- In general:

$$\psi^n(t_n,k_n)=\sum_{k_{n-1}}\int_{t^*}^{t_n}\psi^{n-1}(t_{n-1},k_{n-1})dt_{n-1}K(t_n,k_n|t_{n-1},k_{n-1})$$

$$t_n\to t \qquad k_{n-1}\to k'$$

$$t_{n-1}\to t' \qquad k_n\to k$$

Monte Carlo Solution to the Transport Equation (2)

$$\psi^{n}(t,k)=\sum_{k'}\int_{t^*}^{t}\psi^{n-1}(t',k')dt'K(t,k|t',k')$$

$$\Rightarrow\psi(t,k)=\sum_{n=0}^{\infty}\psi^{n}(t,k)=\psi^{0}(t,k)+$$

$$+\sum_{k'}\int_{t^*}^{t}\underbrace{\sum_{n-1=0}^{\infty}\psi^{n-1}(t',k')}_{\psi(t',k')}dt'K(t,k|t',k')$$

$$\left(\sum_{n-1=0}^{\infty}\psi^{n-1}(t',k')=\psi(t',k')\right)$$

Monte Carlo Solution to the Transport Equation (3)

Initial Conditions: (t^*, k^*)
Formally rewrite the partial transition densities:

$$\psi^1(t_1,k_1)=\sum_{k_0}\int_{t^*}^{t_1} dt_0\psi^0(t_0,k_0)K(t_1,k_1|t_0,k_0)=K(t_1,k_1|t^*,k^*)$$

$$\psi^2(t_2,k_2)=\sum_{k_1}\int_{t^*}^{t_2} dt_1\psi^1(t_1,k_1)K(t_2,k_2|t_1,k_1)=$$

$$=\sum_{k_1}\int_{t^*}^{t_2} dt_1 K(t_1,k_1|t^*,k^*)K(t_2,k_2|t_1,k_1)$$

……

$$\psi^n(t,k)=\sum_{k_1,k_2,\ldots,k_{n-1}}\int_{t^*}^{t_n} dt_{n-1}\int_{t^*}^{t_{n-1}} dt_{n-2}\ldots$$

$$\ldots\int_{t^*}^{t_2} dt_1 K(t_1,k_1|t^*,k^*)K(t_2,k_2|t_1,k_1)\cdots K(t,k|t_{n-1},k_{n-1})$$

CONTENTS

Monte Carlo Simulation in RAMS

$$G(t) = \sum_{k \in \Gamma} \int_0^t \psi(\tau,k) R_k(\tau,t) d\tau$$ Expected value

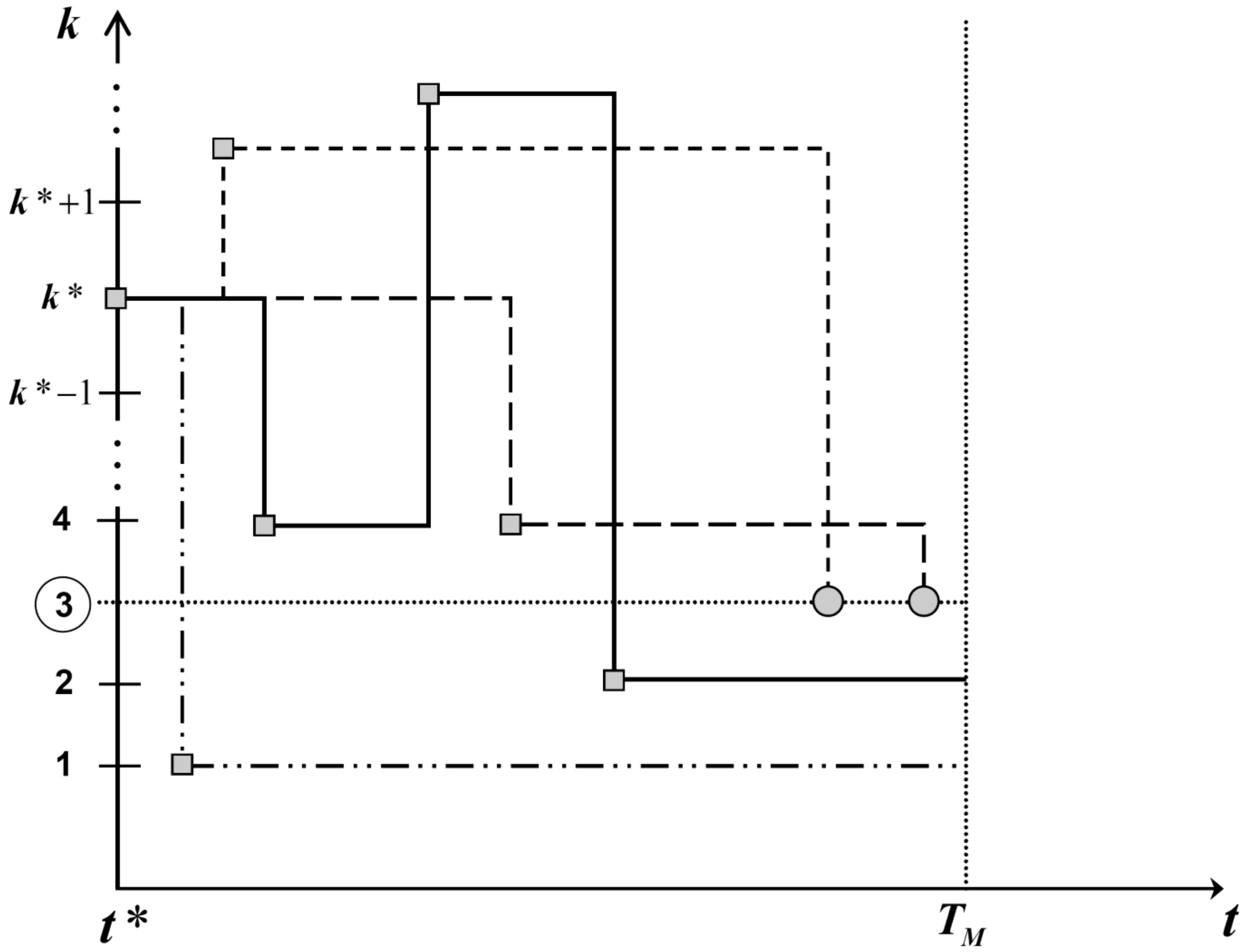
- $G(t)$: **expected value** — representing unavailability or unreliability.
- Γ: Subset of **system failure states**.
- $\psi(\tau, k)$: Probability density of entering state k at time τ.
- $R_k(\tau, t)$: **Residual probability** — probability the system stays in failure state k until time t, after entering at τ

- $R_k(\tau, t) = 1 \Rightarrow G(t)$ = unreliability
- $R_k(\tau, t)$ = prob. system not exiting before t from the state k entered at $\tau < t$ $\Rightarrow G(t)$ = unavailability

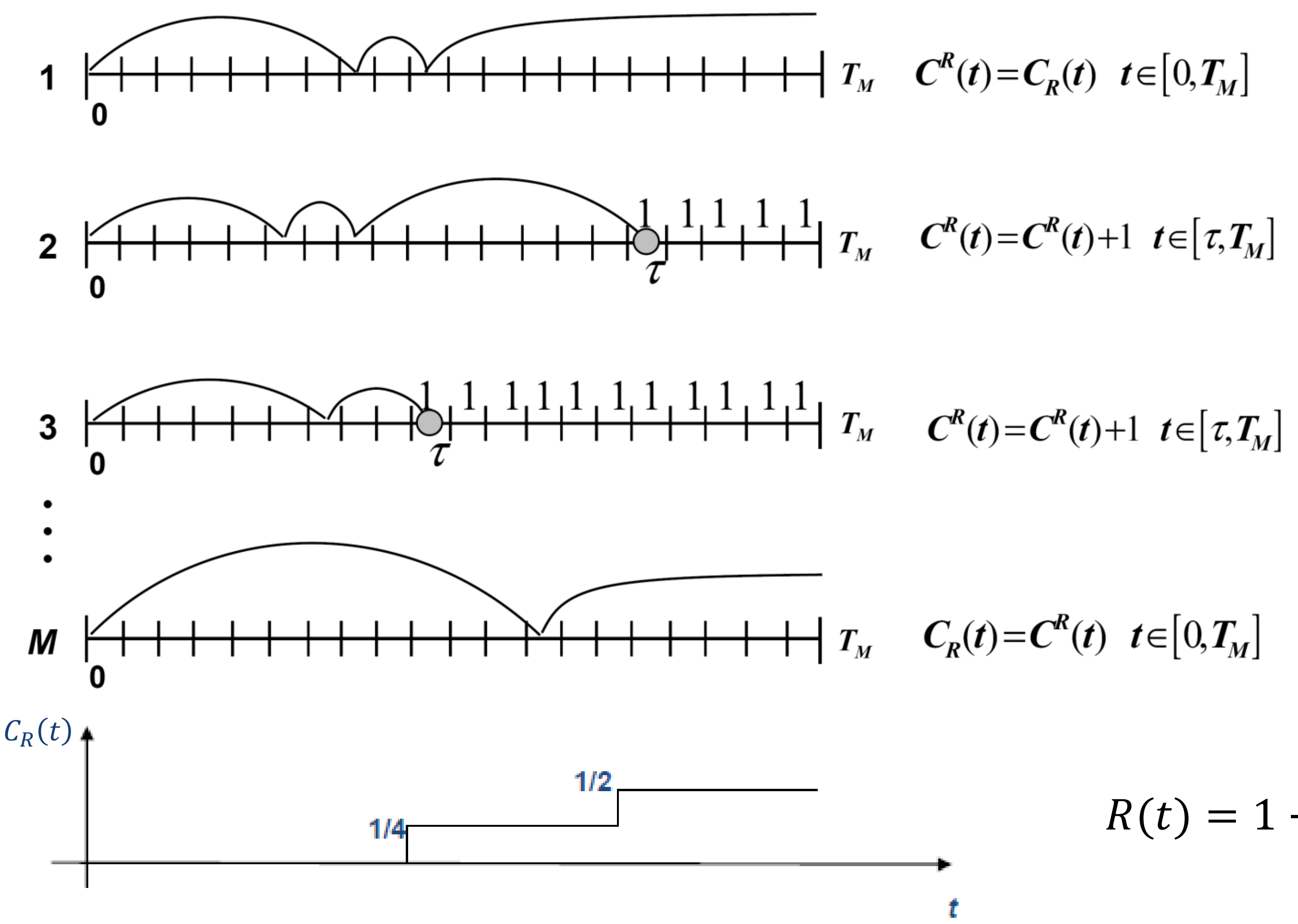
Monte Carlo solution of a definite integral: expected value ≅ sample mean

1. Randomly sampling system lives (random walks)
2. Estimating $G(t)$ as the mean over those samples

Phase space



System reliability estimation



$$R(t) = 1 - \frac{\Sigma_M C_R(t)}{M}$$

System availability estimation

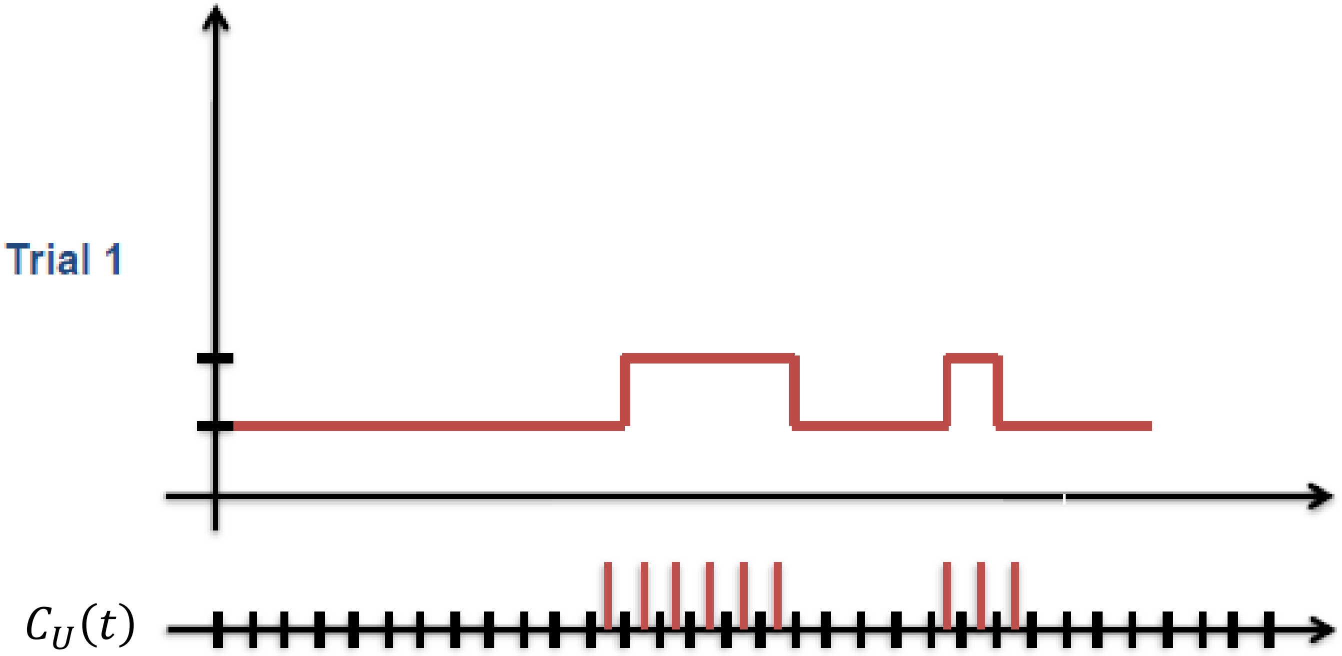
- Divide the mission time, TM,in bins and associate a counter (of the system failure) to each bin:

$$C_U(1), C_U(2), \dots, C_U(\tau)$$

- Initialize each counter to 0:

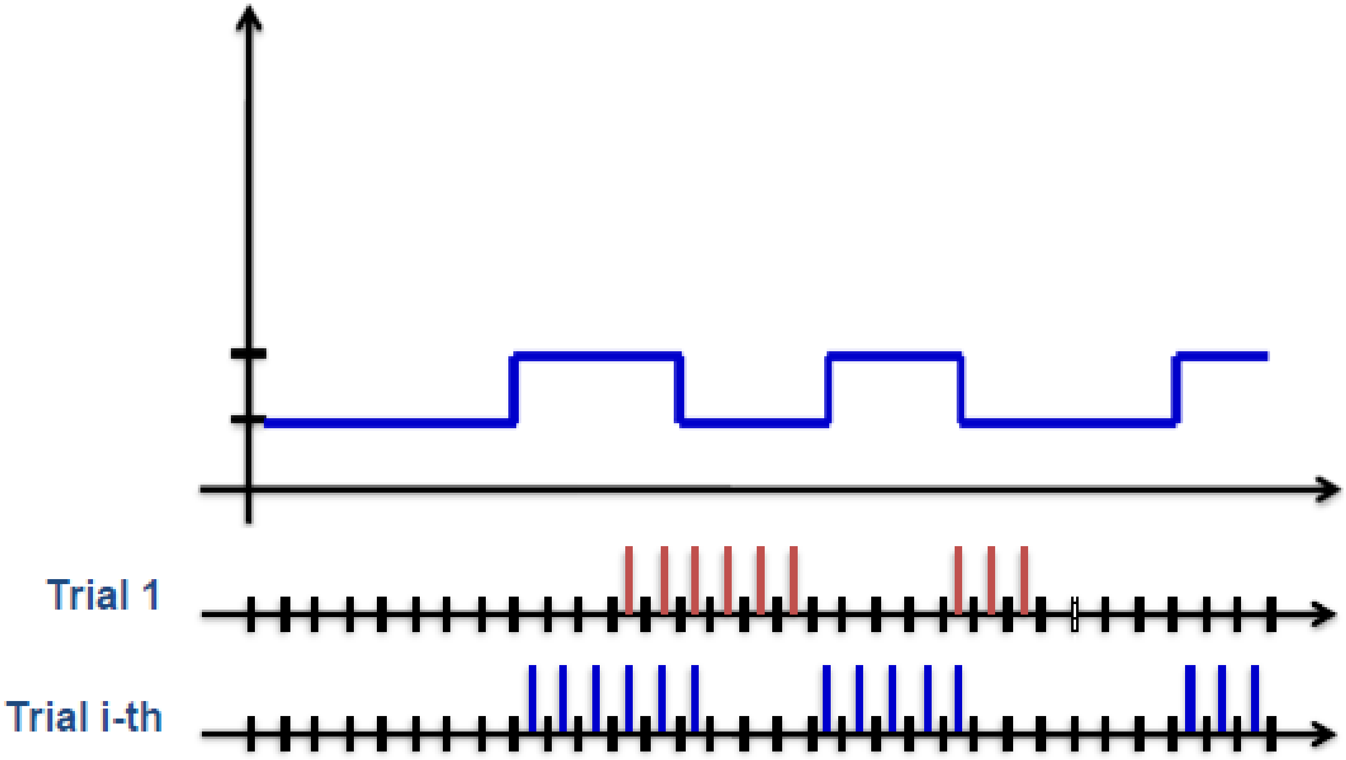
$$C_U(1), C_U(2), \dots, C_U(\tau) = 0$$

- If the component is failed in $(t_j, t_j + \Delta t)$, the corresponding counter increases $C_U(t_j) = C_U(t_j) + 1$, otherwise $C_U(t_j) = C_U(t_j)$



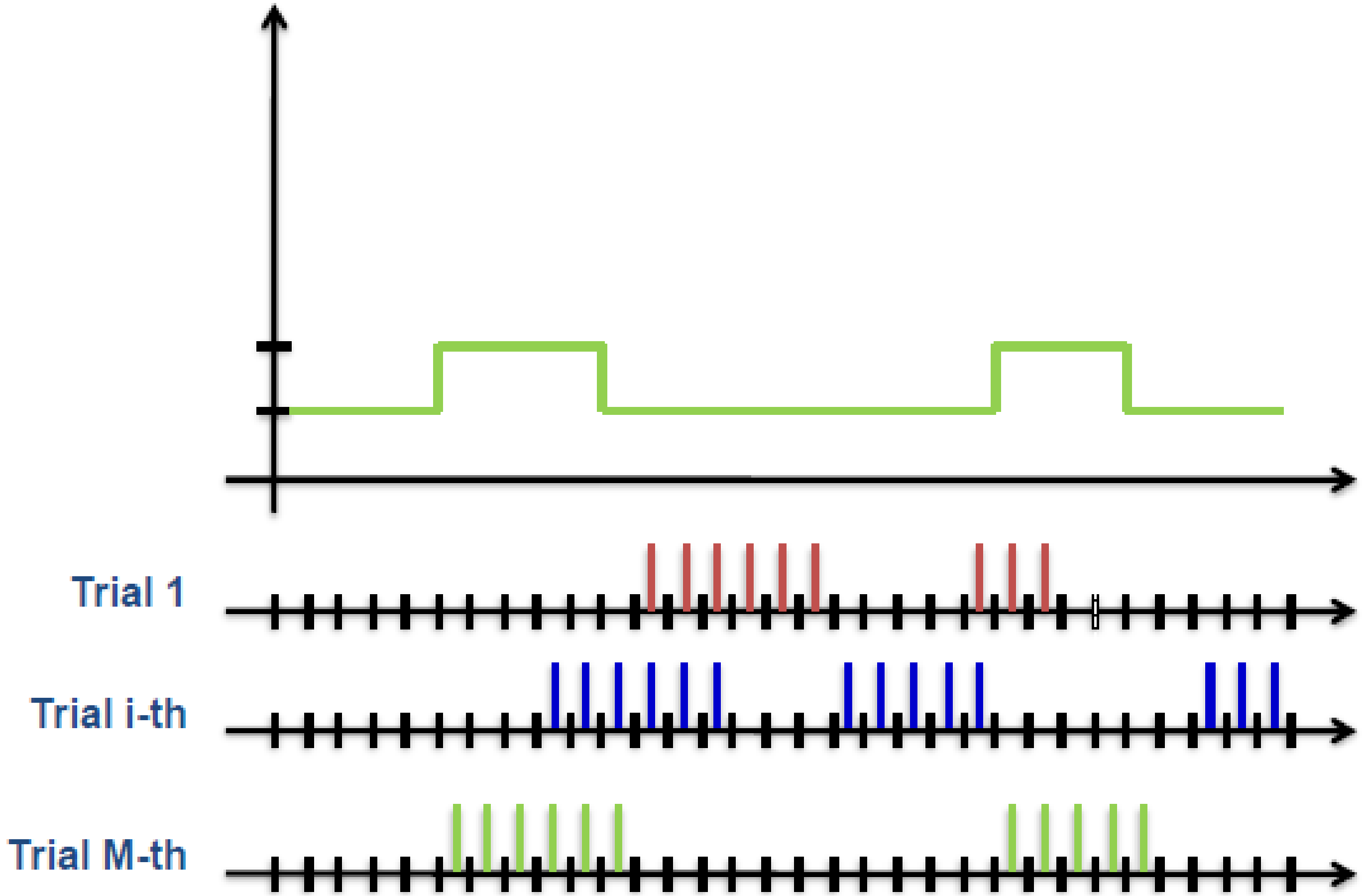
System availability estimation

Another trial



System availability estimation

Another trial



$$A(t) = 1 - \frac{\Sigma_M C_U(t)}{M}$$

Monte Carlo Simulation Approaches

- Each trial of a Monte Carlo simulation consists in generating a random walk which guides the system from one configuration to another, at different times.

- During a trial, starting from a given system configuration *k'* at *t'*, we need to determine when the next transition occurs and which is the new configuration reached by the system as a consequence of the transition.

- This can be done in two ways which give rise to the so called "<u>indirect</u>" and "<u>direct</u>" Monte Carlo approach.

Indirect Monte Carlo

The indirect approach consists in:

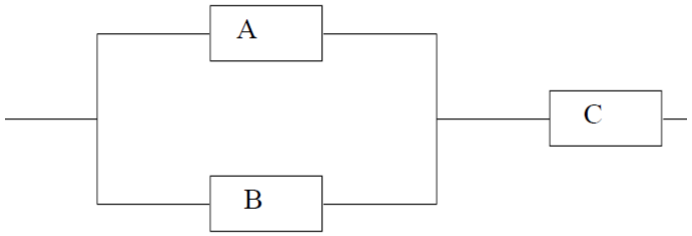
1. Sampling first the time t of a system transition from the corresponding conditional probability density $T(t|t', k')$ of the system performing one of its possible transitions out of k' entered at time t'.

2. Sampling the transition to the new configuration k from the conditional probability $C(k|t, k')$ that the system enters the new state k given that a transition has occurred at t starting from the system in state k'.

3. Repeating the procedure from k' at time t to the next transition.

Indirect Monte Carlo: Example

SAMPLING THE TIME OF TRANSITION



Transition rate exponentially distributed
$\lambda_1^A = \lambda_1^B \neq \lambda_1^C$

The rate of transition of component A(B) out of its nominal state 1 is:

$$\lambda_1^A = \lambda_1^B$$

- The rate of transition of component C out of its nominal state 1 is:

$$\lambda_1^C$$

- The rate of transition of the system out of its current configuration (1, 1, 1) is:

$$\lambda^{(1,1,1)} = \lambda_1^A + \lambda_1^B + \lambda_1^C$$

- We are now in the position of sampling the first system transition time t_1, by applying the inverse transform method:

$$t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}} \ln(1 - R_t)$$

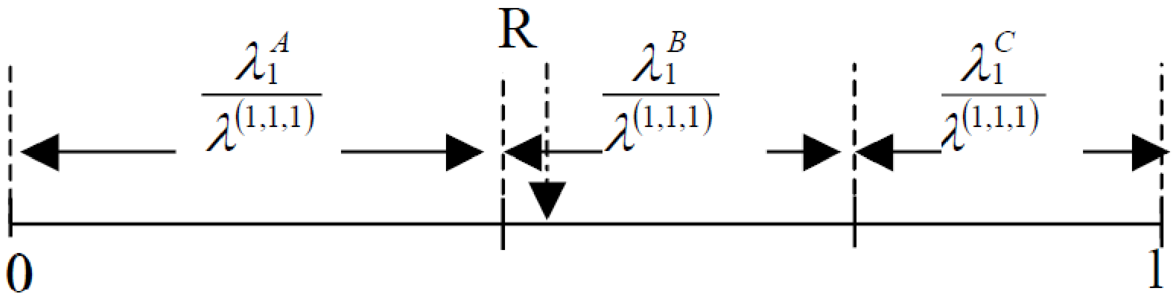
where $R_t \sim U[0,1)$

Indirect Monte Carlo: Example

- Assuming that $t_1 < T_M$ (otherwise we would proceed to the successive trial), we now need to determine which transition has occurred, i.e. which component has undergone the transition and to which arrival state.
- The probabilities of components A, B, C undergoing a transition out of their initial nominal states 1, given that a transition occurs at time t_1, are:

$$\frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}}$$

- Thus, we can apply the inverse transform method to the discrete distribution



Direct Monte Carlo (1)

The direct approach differs from the previous one in that the system transitions are not sampled by considering the distributions for the whole system but rather by sampling directly the times of all possible transitions of all individual components of the system and then arranging the transitions along a timeline, in accordance to their times of occurrence. Obviously, this timeline is updated after each transition occurs, to include the new possible transitions that the transient component can perform from its new state. In other words, during a trial starting from a given system configuration *k'* at *t'*:

1. We sample the times of transition $t^{i}_{j' \to m_i}, m_i = 1,2,\dots,N_{S_i}$, of each component *i, i = 1, 2,…, N_c* leaving its current state j'_i and arriving to the state m_i from the corresponding transition time probability distributions $f_T^{i,j_i \to m_i}(t|t')$.

2. The time instants $t^{i}_{j_i \to m_i}$ thereby obtained are arranged in ascending order along a timeline from t_{min} to $t_{max} \leq T_M$.

Direct Monte Carlo (2)

3. The clock time of the trial is moved to the first occurring transition time $t_{min} = t^*$ in correspondence of which the system configuration is changed, i.e. the component i^* undergoing the transition is moved to its new state m_i^*.

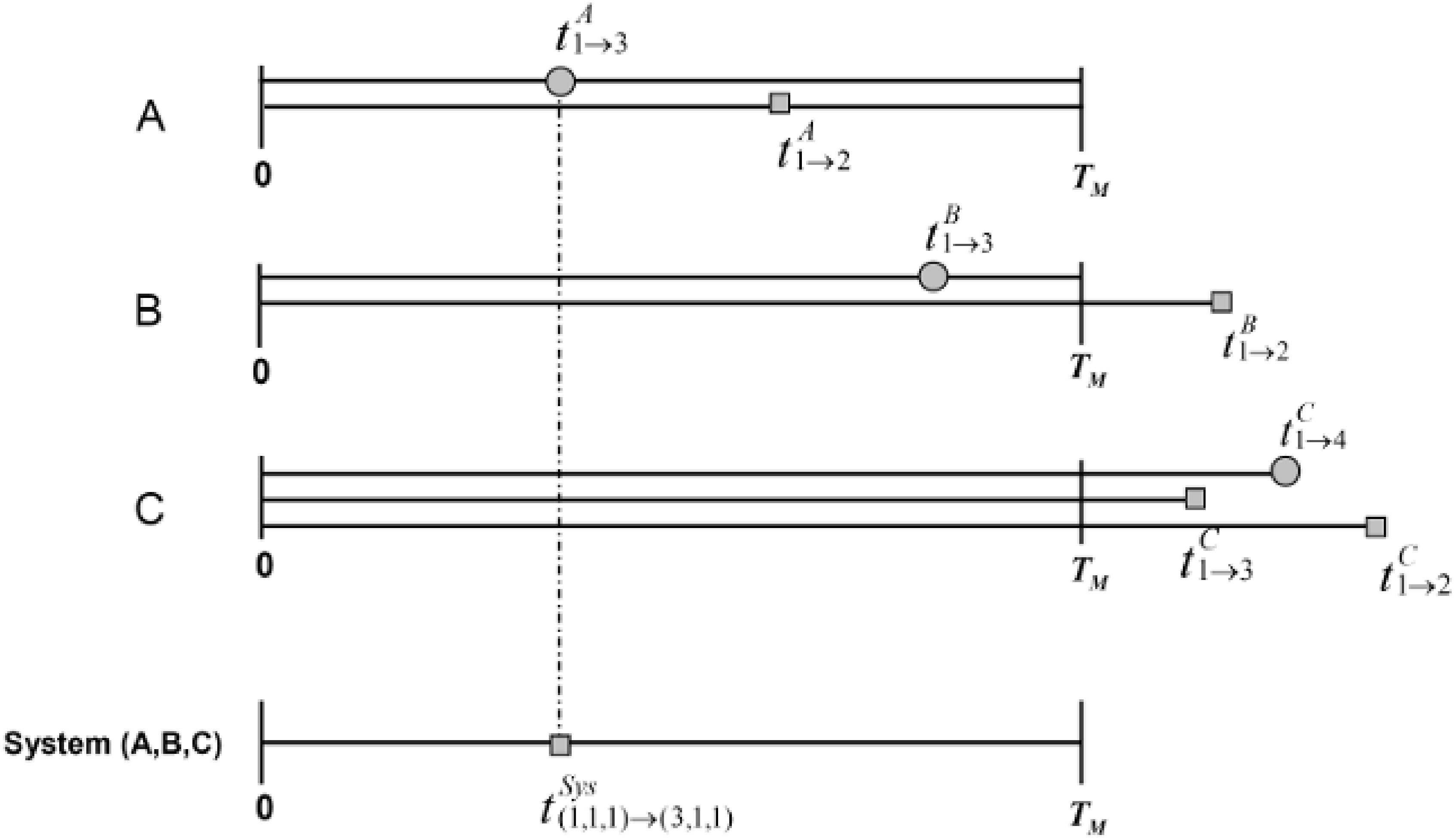
4. At this point, the new times of transition $t^{i^*}_{m_i^* \to l_i^*}, l_i^* = 1,2, \dots, N_S^{i^*}$, of component i^* out of its current state m_i^* are sampled from the corresponding transition time probability distributions, $f_T^{i^*, m_i^* \to l_i^*}(t|t^*)$, and placed in the proper position of the timeline.

5. The clock time and the system are then moved to the next first occurring transition time and corresponding new configuration, respectively.

6. The procedure repeats until the next first occurring transition time falls beyond the mission time, i.e. $t_{min} > T_M$.

Compared to the previous indirect method, the direct approach is more suitable for **systems whose components' failure and repair behaviours are represented by different stochastic distribution laws**.

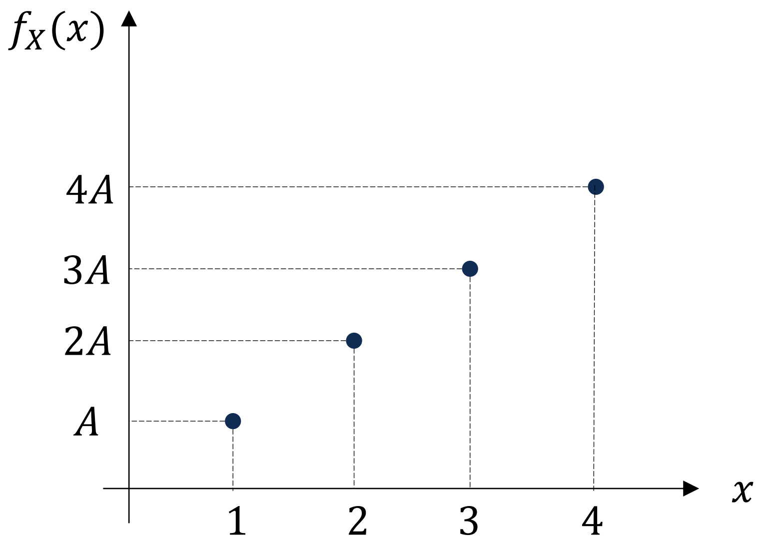
Direct Monte Carlo: Example



Exercise

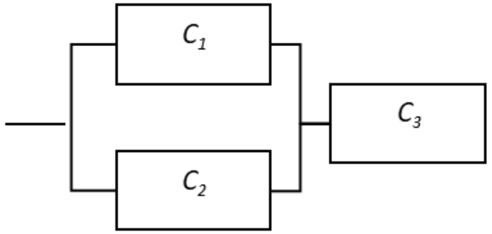
Consider the discrete probability distribution $f_X(x)$ in the graph:

1) Identify the value of the parameter A;
2) Compute the corresponding cumulative distribution;
3) Write an algorithm to sample N=1000 values from $f_X(x)$ and evaluate the distribution of the obtained samples;



Exercise: Indirect Monte Carlo

Consider the system in the Figure, which is made up of the three binary components.



λ	10^{-3} h^{-1}
μ	10^{-1} h^{-1}

The (failure) transition from the state 'WORKING' to the state 'FAILED' is described by the constant failure rate λ, whereas the (repair) transition from the state 'FAILED' to the state 'WORKING' by the constant repair rate μ, whose numerical values are reported in the Table below. The system mission time is Tm=1000 h and the components initial state at time t = 0 h are C1='WORKING', C2=' FAILED', C3='WORKING'.

1) Draw a possible life of the system in the phase space and indicate the states of the system which correspond to a system failure.
2) Compute the first transition time using the inverse transform method. Use R_1=0.232 as random number sampled from an uniform distribution in the range [0,1).
3) Find the state entered by the system as a result of the first transition. Use R_2=0.787 as random number sampled from an uniform distribution in the range [0,1).
4) Simulate one entire plant life using the random numbers attached, sampled from an uniform distribution in the range [0,1). Start the simulation from the time and the state found in point Q1.3) and Q1.4), respectively.

1) 0.8929
2) 0.3320
3) 0.8212
4) 0.0417
5) 0.1077
6) 0.5951
7) 0.5298
8) 0.4188
9) 0.3354
10) 0.6225
11) 0.4381
12) 0.7359
13) 0.5180
14) 0.5789
15) 0.6454
16) 0.9902
17) 0.8199
18) 0.4132
19) 0.8763
20) 0.8238
21) 0.0545
22) 0.7186
23) 0.8022
24) 0.7364
25) 0.7091
26) 0.5409
27) 0.1248
28) 0.9576
29) 0.4033
30) 0.2170
31) 0.7173

References

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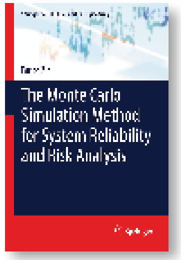

2013, 2013, XIV, 198 p. 69 illus., 24 in color.

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The Monte Carlo Simulation Method for System Reliability and Risk Analysis

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Thank you for your kind attention

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