POLITECNICO DI MILANO

Reliability of Simple Systems

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Introduction: reliability and availability

- **Reliability** and **availability**: important performance parameters of a system, with respect to its ability to fulfill the required mission in a given period of time

- Two different system types:
  - Systems which must satisfy a specified mission within an assigned period of time: **reliability** quantifies the ability to achieve the desired objective without failures
  - Systems maintained: **availability** quantifies the ability to fulfill the assigned mission at **any** specific moment of the life time

Basic definitions

- T = Time to failure of a component (random variable)
  - cdf = $F_T(t)$ = probability of failure before time t: P(T<t)
  - pdf = $f_T(t)$ = probability density function at time t:

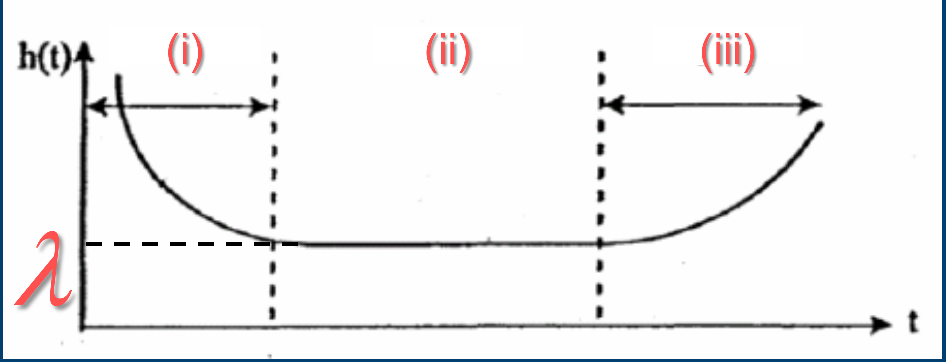
    $$f_T(t)dt = P(t<T<t+dt)$$

  - ccdf = $R(t)$ = 1- $F_T(t)$ = reliability at time t: P(T>t)
  - $h_T(t)$ = **hazard function** or failure rate at time t

$$\boxed{h_T(t)dt = P(t < T \leq t + dt \mid T > t) = \frac{P(t < T \leq t + dt)}{P(T > t)} = \frac{f_T(t)dt}{R(t)}}$$

Hazard function: the bath-tub curve

- Usually, the hazard function shows three distinct phases:
    i. Decreasing - *infant mortality* or *burn in period*
    ii. Constant - *useful life*
    iii. Increasing - *ageing*

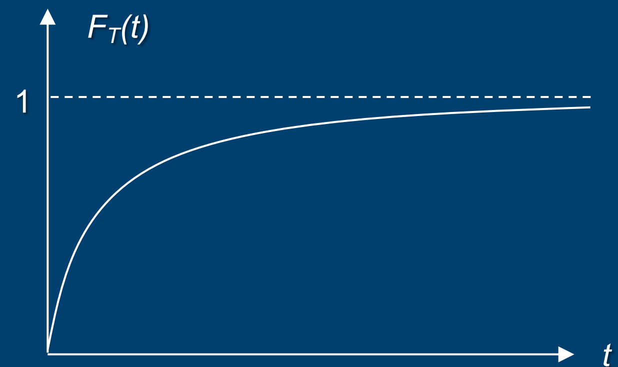


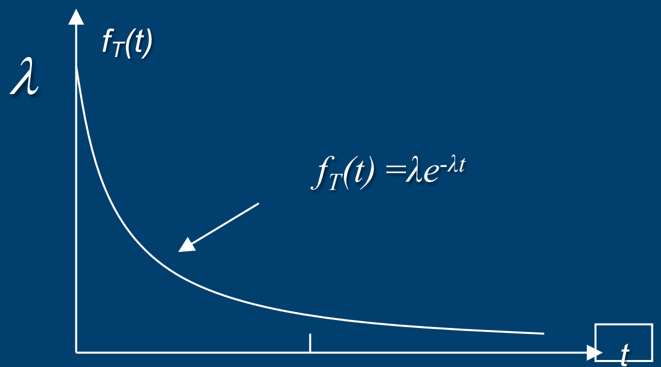
The exponential distribution (1)

- $h_T(t) = \lambda,\ t \geq 0$

$$F_T(t) = P(T \leq t) = 1 - e^{-\lambda t}$$

$$\begin{cases} f_T(t) = \lambda e^{-\lambda t} & t \geq 0 \\ \quad\quad = 0 & t < 0 \end{cases}$$





- Only distribution characterized by a constant hazard rate
- Widely used in reliability practice to describe the constant part of the bath-tub curve

The exponential distribution (2)

- The expected value and variance of the distribution are:

$$E[T] = \frac{1}{\lambda} = MTTF \qquad ; \qquad Var[T] = \frac{1}{\lambda^2}$$

- Failure process is **memoryless**

$$P(t_1 < T < t_2 \mid T > t_1) = \frac{P(t_1 < T < t_2)}{P(T > t_1)} = \frac{F_T(t_2) - F_T(t_1)}{1 - F_T(t_1)} =$$

$$= \frac{e^{-\lambda t_1} - e^{-\lambda t_2}}{e^{-\lambda t_1}} = 1 - e^{-\lambda (t_2 - t_1)}$$

The Weibull distribution

- In practice, the age of a component influences its failure process so that the hazard rate does not remain constant throughout the lifetime

$$F_T(t) = P(T \le t) = 1 - e^{-\lambda t^{\alpha}}$$

$$\begin{cases} f_T(t) = \lambda \alpha t^{\alpha - 1} e^{-\lambda t^{\alpha}} & t \ge 0 \\ \quad = 0 & t < 0 \end{cases}$$

$$E[T] = \frac{1}{\lambda} \Gamma\left(\frac{1}{\alpha} + 1\right) \quad ; \quad Var[T] = \frac{1}{\lambda^2}\left(\Gamma\left(\frac{2}{\alpha}+1\right) - \Gamma\left(\frac{1}{\alpha}+1\right)\right)^2$$

$$\Gamma(k) = \int_0^{\infty} x^{k-1} e^{-x} dx \qquad k > 0$$

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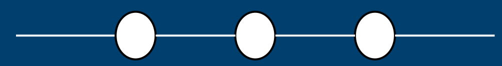
Definition of the problem

- **Objective**:
  - Computation of the system reliability $R(t)$
- **Hypotheses**:
  - N = number of system components
  - The components' reliabilities $R_i(t)$, i = 1, 2, …, N are known
  - The system **configuration** is known

Series system



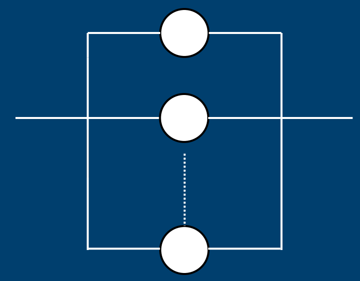
- All components must function for the system to function

$$\boxed{R(t) = \prod_{i=1}^{N} R_i(t)}$$

- For N **exponential** components:

$$\boxed{R(t) = e^{-\lambda t}} \longrightarrow \begin{cases} \lambda = \sum_{i=1}^{N} \lambda_i & \longrightarrow \text{System failure rate} \\ E[T] = \dfrac{1}{\lambda} & \longrightarrow \textit{MTTF} \end{cases}$$

Parallel system



- All components must fail for the system to fail

$$R(t) = 1 - \prod_{i=1}^{N} \left[1 - R_i(t)\right]$$

- For *N* **exponential** components:

$$R(t) = 1 - \prod_{i=1}^{N} \left[1 - e^{-\lambda_i t}\right] \longrightarrow \left\{ \begin{aligned} MTTF = & \sum_{i=1}^{N} \frac{1}{\lambda_i} - \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{\left[\lambda_i + \lambda_j\right]} + \\ & + \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{k=j+1}^{N} \frac{1}{\left[\lambda_i + \lambda_j + \lambda_k\right]} - \ldots + (-1)^{N-1} \frac{1}{\sum_{i=1}^{N} \lambda_i} \end{aligned} \right.$$

Parallel system: an example

- Two exponential units with failure rates λ_1 and λ_2

$$R(t) = 1-(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t}) = \underbrace{e^{-\lambda_1 t}}_{R_1} + \underbrace{e^{-\lambda_2 t}}_{R_2} - e^{-(\lambda_1+\lambda_2)t} > R_1(t), \quad R_2(t) > e^{-\lambda t} = e^{-(\lambda_1+\lambda_2)t} \ (series)$$

$$MTTF = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{[\lambda_1+\lambda_2]}$$

- For N identical elements, compare series and parallel

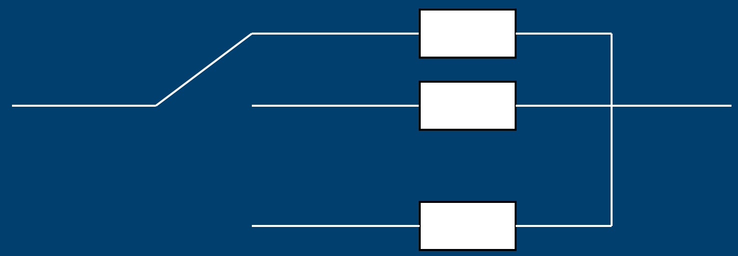
$$\left.\begin{array}{ll} parallel & MTTF = \sum_{n=1}^{N} \frac{1}{n\lambda} \\ series & MTTF = \frac{1}{N\lambda} \end{array}\right\} \longrightarrow \lambda \cdot MTTF_{series} = \frac{1}{N} < \sum_{n=1}^{N} \frac{1}{n} = \lambda \cdot MTTF_{parallel}$$

r-out-of-*N* system

- *N* identical components function in parallel but only *r* are needed (parallel system: *r* = 1)
- For *N* identical **exponential** components:

$$R(t) = \sum_{k=r}^{N} \binom{N}{k} e^{-\lambda k t} \left(1 - e^{-\lambda t}\right)^{N-k} \longrightarrow MTTF = \sum_{k=r}^{N} \frac{1}{k\lambda}$$

Standby system



- One component is functioning and when it fails it is replaced immediately by another component (sequential operation of one component at a time)
- The system configuration is **time-dependent** $\Rightarrow$ the story of the system from t = 0 must be considered
- Two types of standby:
  - **Cold**: the standby unit cannot fail until it is switched on
  - **Hot**: the standby unit can fail also while in standby

Cold standby (1)

- Since the components are operated sequentially, the system fails at time $T=\sum_{i=1}^{N} T_i$, which is a random variable sum of *N* independent random variables

- **Convolution** theorem

Example: 2 components

$$\left.\begin{array}{l} T_1, f_{T1}(t) \\ T_2, f_{T2}(t) \end{array}\right\} \Rightarrow T = T_1 + T_2, f_T(t) = f_{T1}(t) * f_{T2}(t) = \int_{-\infty}^{\infty} f_{T1}(x) f_{T2}(t-x)dx$$

$$L[f(x)] = \tilde{f}(s) = \int_0^{\infty} e^{-s \cdot x} f(x)dx \quad \tilde{f}_T(s) = L[f_{T1}(t) * f_{T2}(t)] = \tilde{f}_{T1}(s)\tilde{f}_{T2}(s)$$

Cold standby (2)

Example: N components

$$\tilde{f}_T(s) = \prod_{i=1}^{N} \tilde{f}_{Ti}(s)$$

$$R(t) = 1 - \int_0^t f_T(x)dx$$

Cold standby: an example

- Consider a "cold" standby system of two units
- The on-line unit has an *MTTF* of 2 years
- When it fails, the standby unit comes on line and its *MTTF* is 3 years
- Assume that each component has an exponential failure times distribution

(1) What is the probability density function of the system failure time? What is the *MTTF* of the system?
(2) Repeat assuming that the two components are in parallel in a one-out-of-two configuration

Cold standby: an example – solution (1)

$$MTTF_i = \int_0^\infty t f_{Ti}(t)dt = \int_0^\infty \lambda_i t e^{-\lambda_i t} dt = \frac{1}{\lambda_i}$$

$$\lambda_1 = \frac{1}{2yrs}$$

$$\lambda_2 = \frac{1}{3yrs}$$

- T_1 and T_2 are independent random variables denoting the times when the on-line and standby units are operating, respectively
- The system failure time is also a random variable, $T = T_1 + T_2$

$$f_T(t) = \int_0^t \lambda_1 e^{-\lambda_1 \tau} \lambda_2 e^{-\lambda_2 (t-\tau)} d\tau = \lambda_1 \lambda_2 \int_0^t e^{-(\lambda_1-\lambda_2)\tau} e^{-\lambda_2 t} d\tau = \lambda_1 \lambda_2 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1-\lambda_2)\tau} d\tau =$$

$$= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_2 t} \left(e^{-(\lambda_1-\lambda_2)\tau} \right)_0^t = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} \left(e^{-\lambda_1 t} - e^{-\lambda_2 t} \right) = e^{-t/3yrs} - e^{-t/2yrs}$$

Cold standby: an example – solution (2)

$$u = \frac{t}{3yrs} \int_o^\infty t f_T(t) dt = \int_0^\infty (te^{-t/3yrs} - te^{-t/2yrs}) dt$$

$$u = \frac{t}{3yrs} \qquad \xi = \frac{t}{2yrs}$$

$$MTTF = (3yrs)^2 \left[-ue^{-u} - e^{-u} \right]_0^\infty - (2yrs)^2 \left[-\xi e^{-\xi} - e^{-\xi} \right]_0^\infty =$$

$$= (3yrs)^2 \left(\frac{1}{yr} \right) - (2yrs)^2 \left(\frac{1}{yr} \right) = 5yrs$$

(3.8*yrs* for parallel!)

Hot standby (1)

- The convolution theorem can no longer be used to calculate the reliability of the system, because there is no independence of failures any more
- Simple case of two components: the system will perform its task in the interval (0, t) in either of the two mutually exclusive ways:
  - the online component 1 does not fail in (0, t)
    [probability = $R_1(t)$]
  - the online component fails in (0, τ) [probability = $f_{T1}(\tau)d\tau$];
    the standby component 2 does not fail in (0, τ) [probability $R_s(\tau)$]

    and it operates successfully from τ to t [probability $R_2(t-\tau)$]

Hot standby (2)

- The system reliability is given by the sum of the probabilities of the two mutually exclusive events:

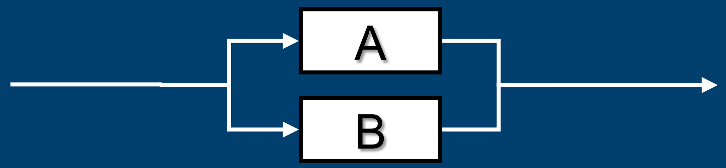
$$R(t) = R_1(t) + \int_0^t f_1(\tau)d\tau \, R_s(\tau) \, R_2(t-\tau)$$

- For 2 **exponential** components:

$$R(t) = e^{-\lambda_1 t} + \int_0^t \lambda_1 e^{-\lambda_1 \tau} e^{-\lambda_s \tau} e^{-\lambda_2 (t-\tau)} d\tau =$$

$$= e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_s - \lambda_2}\left[e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_s)t}\right]$$

Time-dependent systems: an example



- When both A and B are fully energized they share the total load and the failure densities are $f_A(t)$ and $f_B(t)$
- If either one fails, the survivor must carry the full load and its failure density becomes $g_A(t)$ or $g_B(t)$

Find the reliability *R(t)* of the system if

$$f_A(t) = f_B(t) = \lambda e^{-\lambda t}$$

$$g_A(t) = g_B(t) = k\lambda e^{-k\lambda t} \qquad k > 1$$

Time-dependent systems: an example - solution

- *R(t)* = P{system survives up to *t*} = P{neither component fails before *t*}+P{one fails at some time $\tau < t$, the other one survives up to τ, with *f(t)*, and from τ to *t* with *g(t)*} =

$$= e^{-2\lambda t} + 2\int_0^t \left(\lambda e^{-\lambda\tau} d\tau\right)\left(e^{-\lambda\tau}\right)\left(e^{-k\lambda(t-\tau)}\right) = e^{-2\lambda t} + 2\lambda e^{-k\lambda t}\int_0^t e^{-\lambda(2-k)\tau} d\tau =$$

$$= \frac{2e^{-k\lambda t} - ke^{-2\lambda t}}{2-k}$$