CRC
Centre de recherche
sur les Risques et les Crises

POLITECNICO DI MILANO

Introduction to Monte Carlo Simulation

Enrico Zio

CONTENTS

- Sampling Random Numbers
- Simulation of system transport
- Simulation for reliability/availability analysis of a component
- Examples

The History of Monte Carlo Simulation

Buffon	*1707-88*
Kelvin	*1824-1907*
Gosset (Student)	*1908*
Fermi, von Neumann, Ulam	*1930-40's*
Neutron transport	*1950's*
System Transport (RAMS)	*1990's*

SAMPLING RANDOM NUMBERS

Example: Exponential Distribution

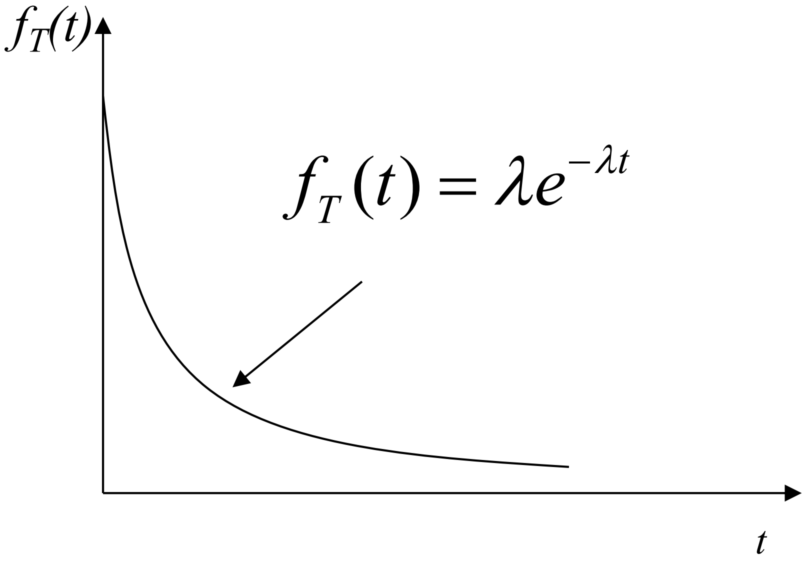
Probability density function:

$$f_T(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$
$$= 0 \qquad t < 0$$

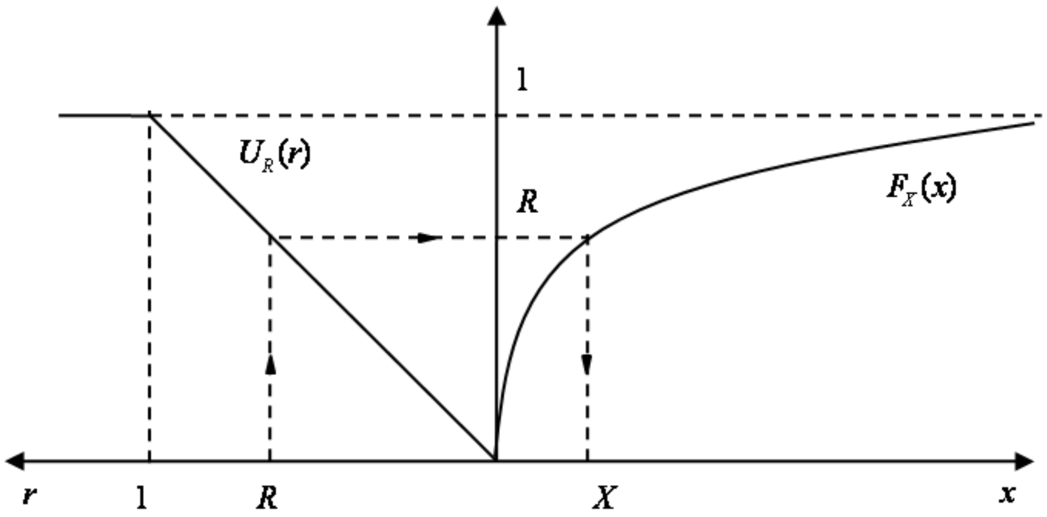
Expected value and variance:

$$E[T] = \frac{1}{\lambda}$$

$$Var[T] = \frac{1}{\lambda^2}$$



Sampling Random Numbers from $F_X(x)$



Sample R from $U_R(r)$ and find X:

$$X = F_X^{-1}(R)$$

Example: Exponential distribution

$$F_X(x) = 1 - e^{-\lambda x}$$

$$R = F_X(x) = 1 - e^{-\lambda x}$$

$$\Downarrow$$

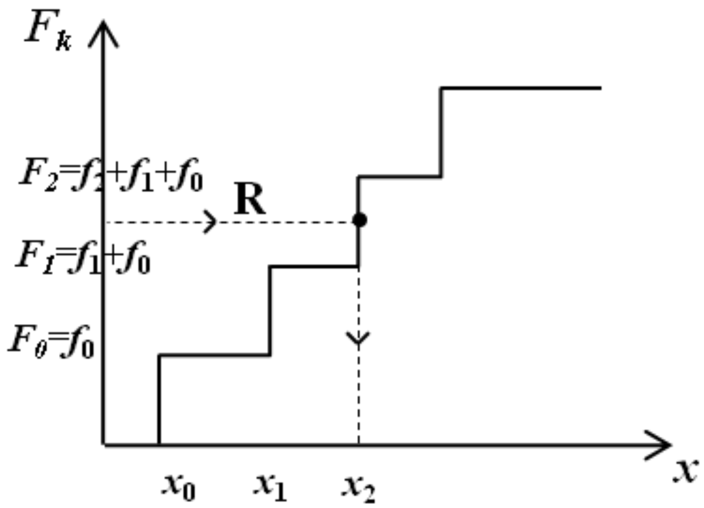
$$X = F_X^{-1}(R) = -\frac{1}{\lambda}\ln(1-R)$$

Sampling from discrete distributions

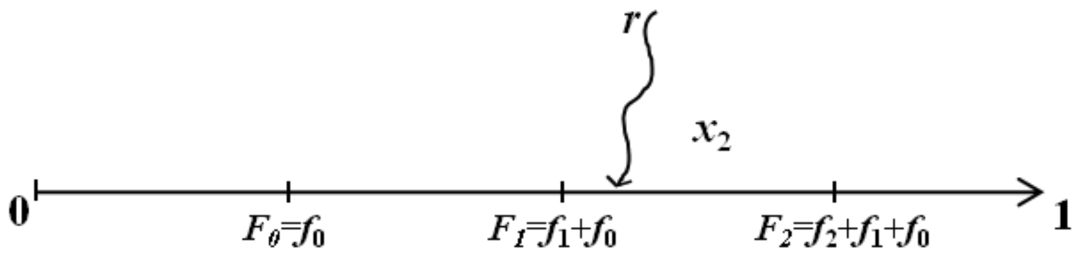
$$\Omega = \{x_0, x_1, \ldots, x_k, \ldots\}$$

$$F_k = P\{X \le x_k\} = \sum_{i=0}^{k} P[X = x_i]$$

sample an $R \sim U[0,1)$



Graphically:



SIMULATION OF SYSTEM TRANSPORT

Monte Carlo simulation for system reliability

PLANT = system of *Nc* suitably connected components.

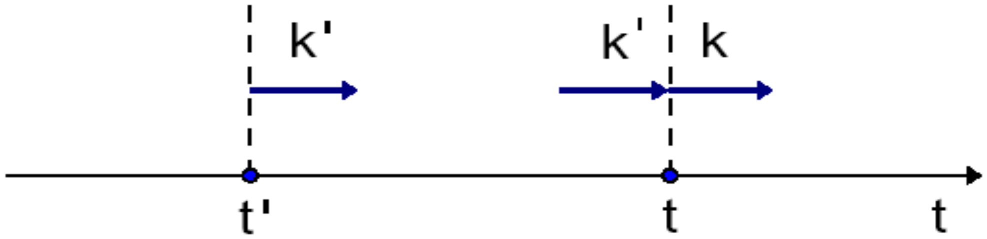
COMPONENT = a subsystem of the plant (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.

STATE of the PLANT at t = the set of the states in which the *Nc* components stay at t. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.

PLANT TRANSITION = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the n-th transition is called t_n and the plant state thereby entered is called k_n.

PLANT LIFE = stochastic process.

Stochastic Transitions: Governing Probabilities



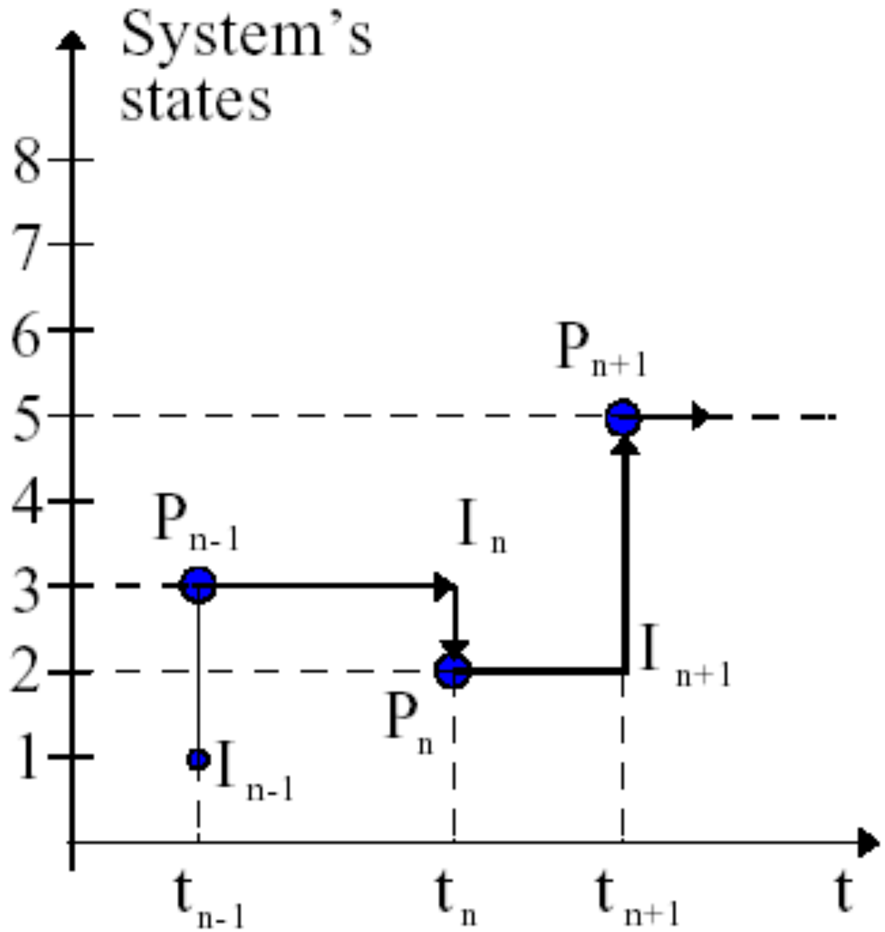
- $T(t\,/t';\,k')dt$ = conditional probability of a transition at t∈dt, given that the preceding transition occurred at *t'* and that the state thereby entered was *k'*.
- $C(k\;/\;k';\;t)$ = conditional probability that the plant enters state *k*, given that a transition occurred at time *t* when the system was in state *k'*. Both these probabilities form the "trasport kernel":

$$K(t;\,k\;/\;t';\,k')dt = T(t\;/\;t';\,k')dt\;C(k\;/\;k';\,t)$$

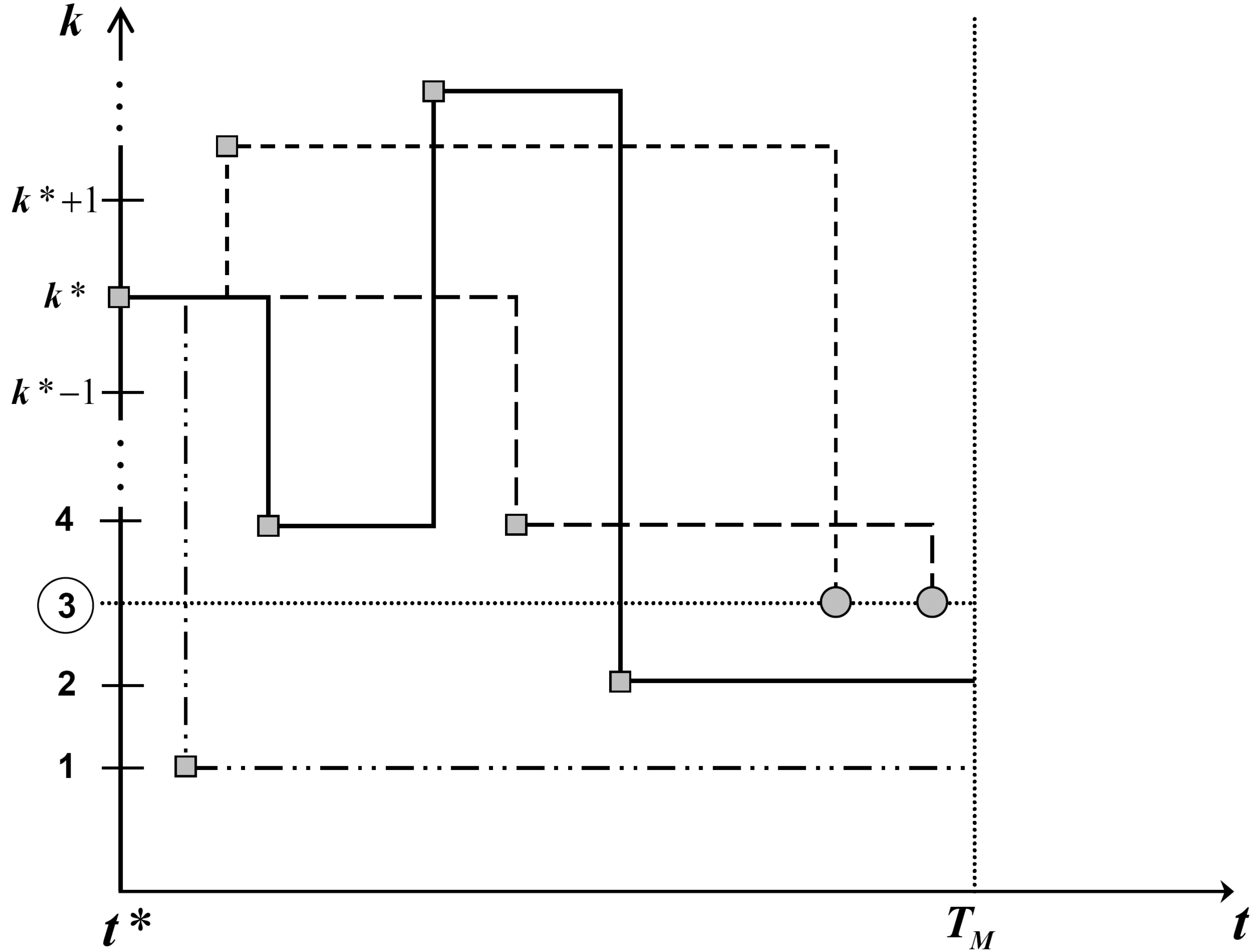
- $\psi(t;\,k)$ = ingoing transition density or probability density function (pdf) of a system transition at *t*, resulting in the entrance in state *k*

Plant life: random walk

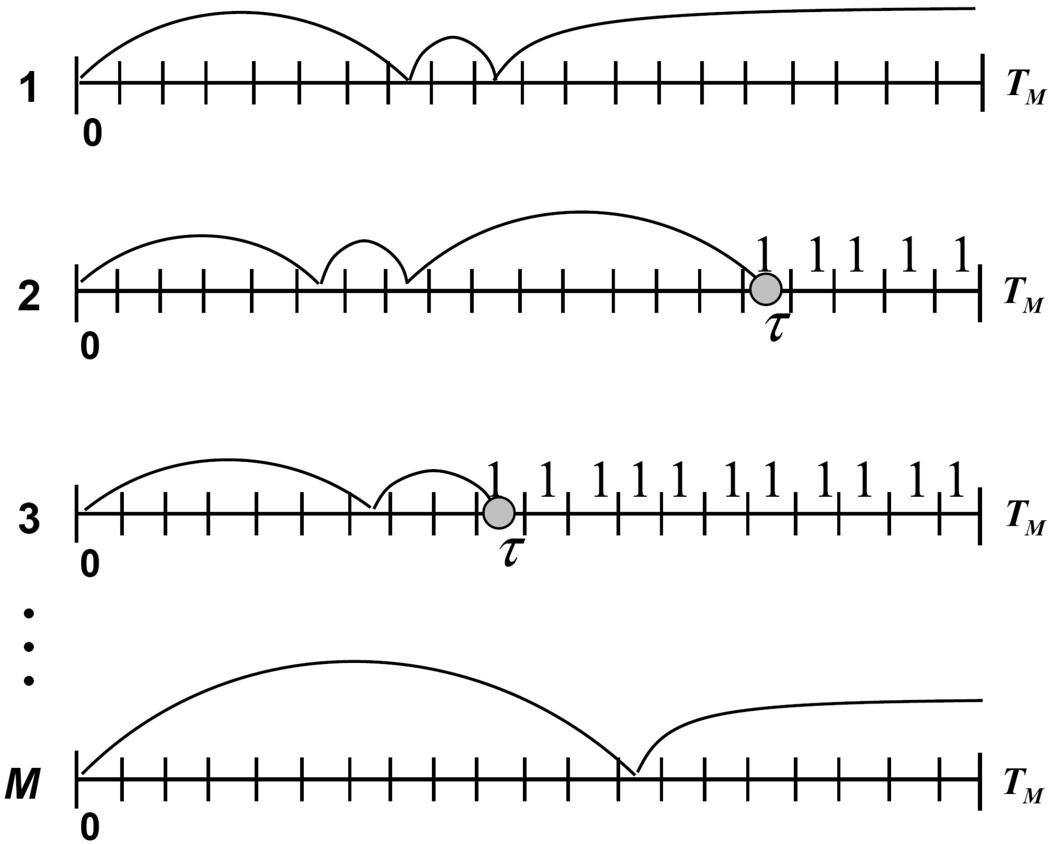
Random walk = realization of the system life generated by the underlying state-transition stochastic process.



Phase Space



Example: System Reliability Estimation



$$C^R(t) = C^R(t) \quad t \in [0, T_M]$$

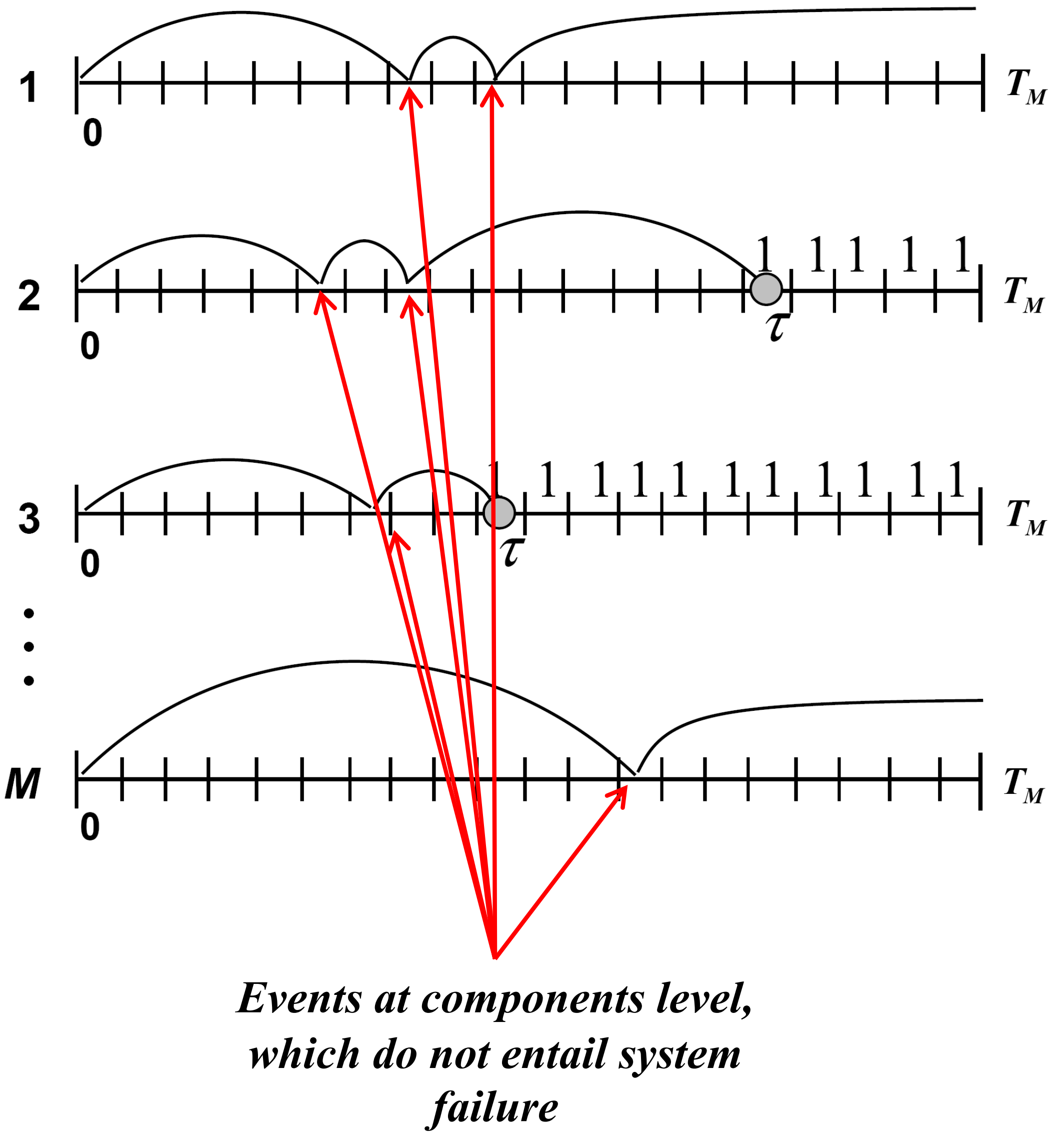
$$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$$

$$C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M]$$

$$C^R(t) = C^R(t) \quad t \in [0, T_M]$$

$$\hat{F}_T(t) = \frac{C^R(t)}{M}$$

Example: System Reliability Estimation



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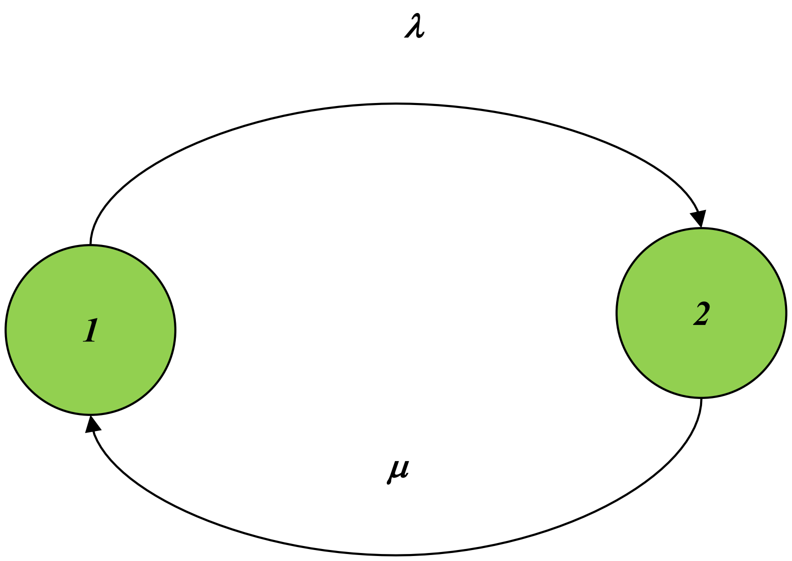
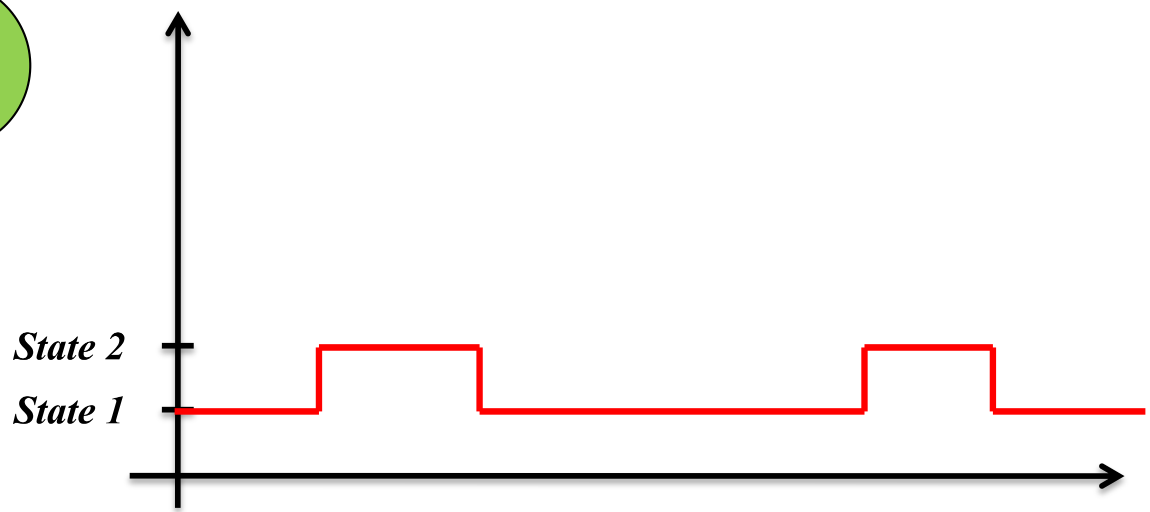
SIMULATION OF COMPONENT STOCHASTIC STATE TRANSITION PROCESS FOR AVAILABILITY / RELIABILITY ESTIMATION

One component with exponential distribution of the failure time

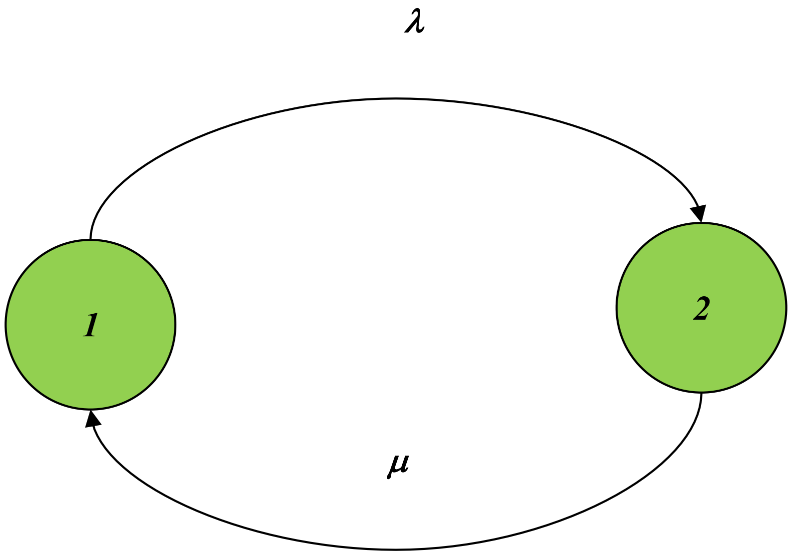


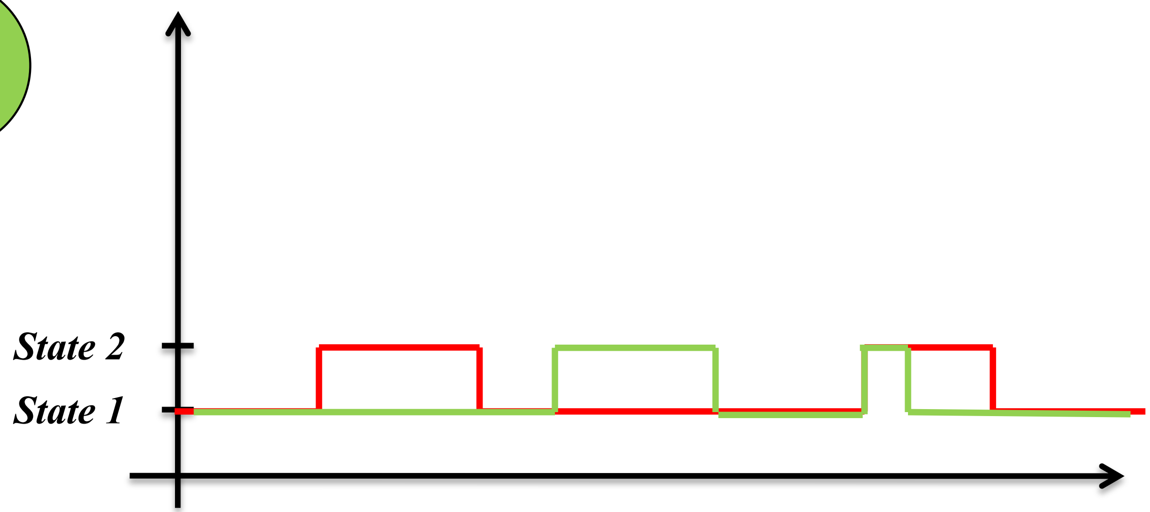


State X=1 → ON

State X=2 → OFF

One component with exponential distribution of the failure time





State X=1 → ON

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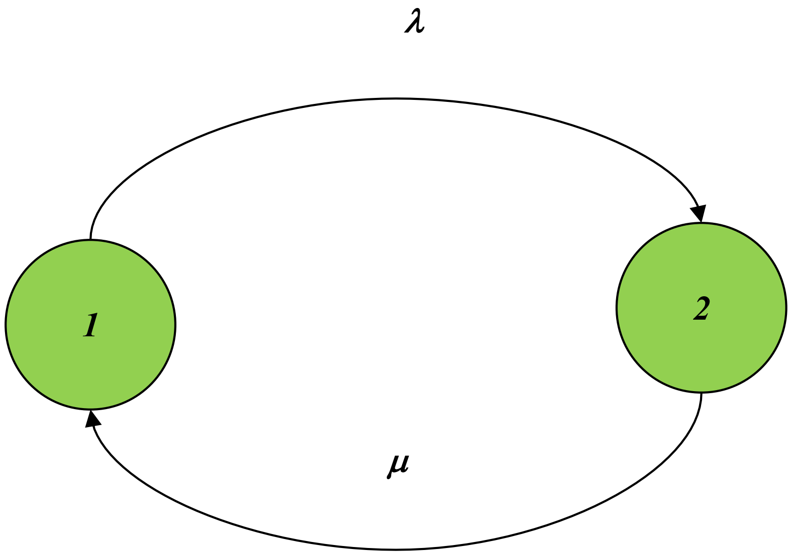
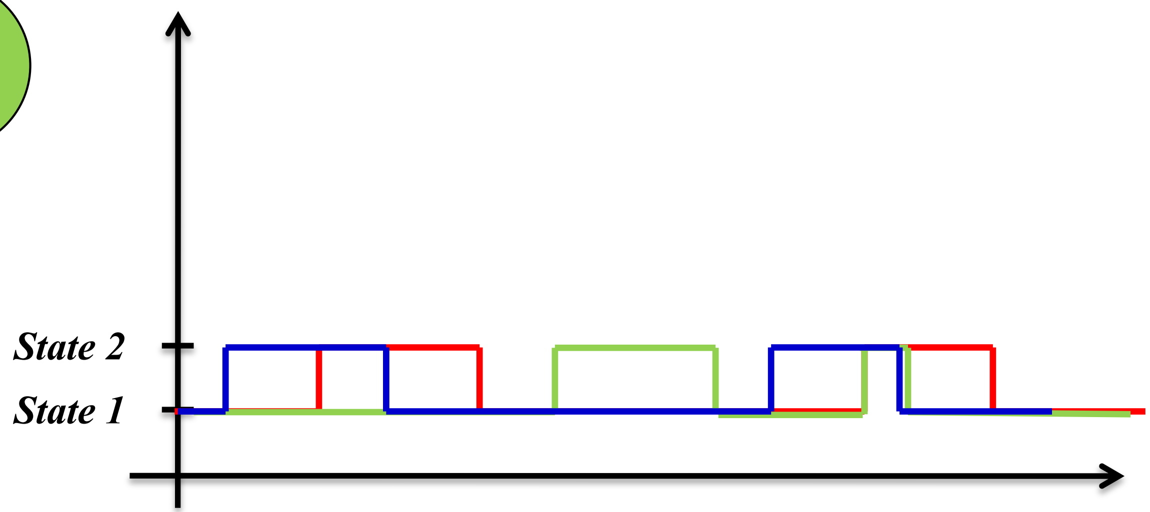
One component with exponential distribution of the failure time



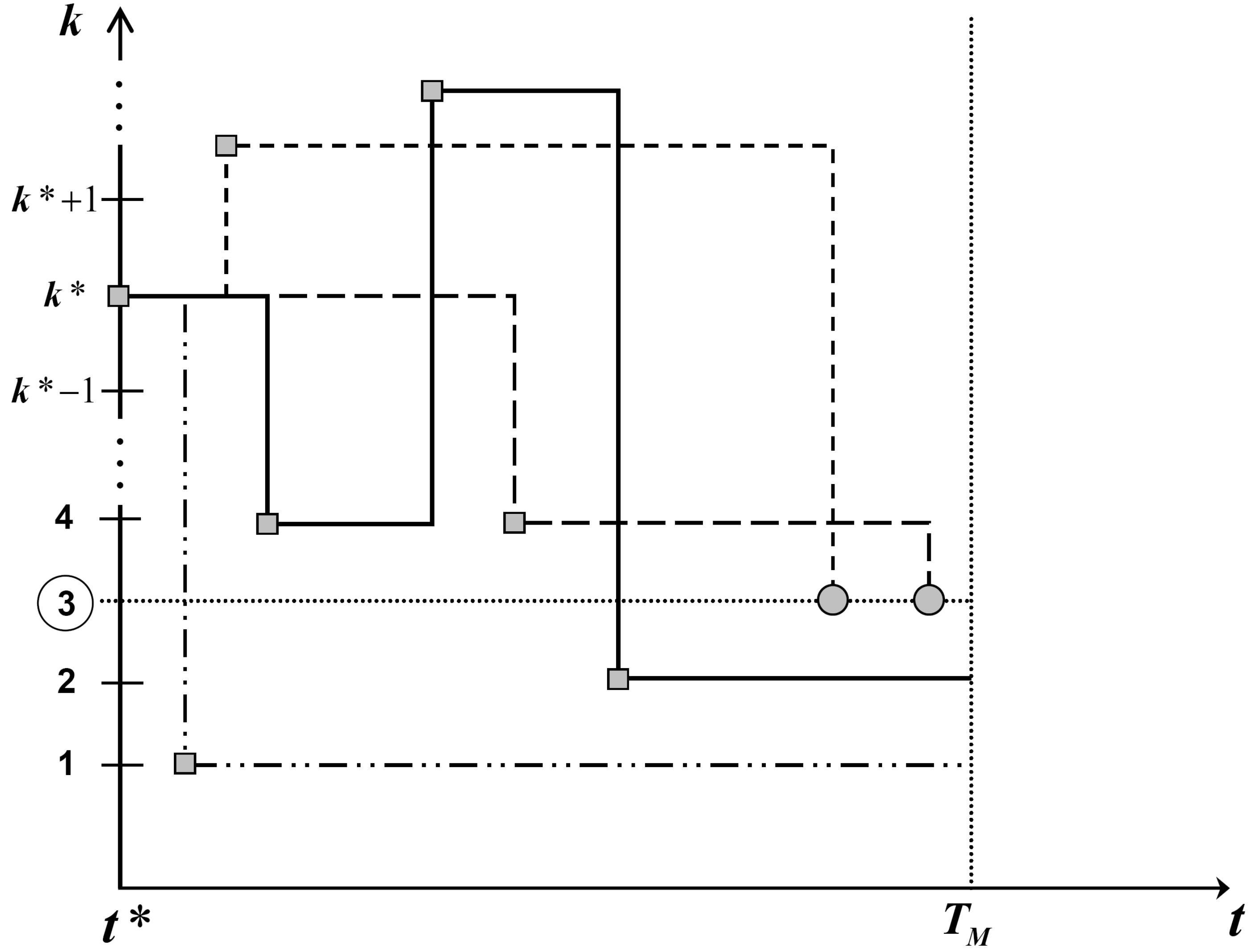


State X=1 → ON

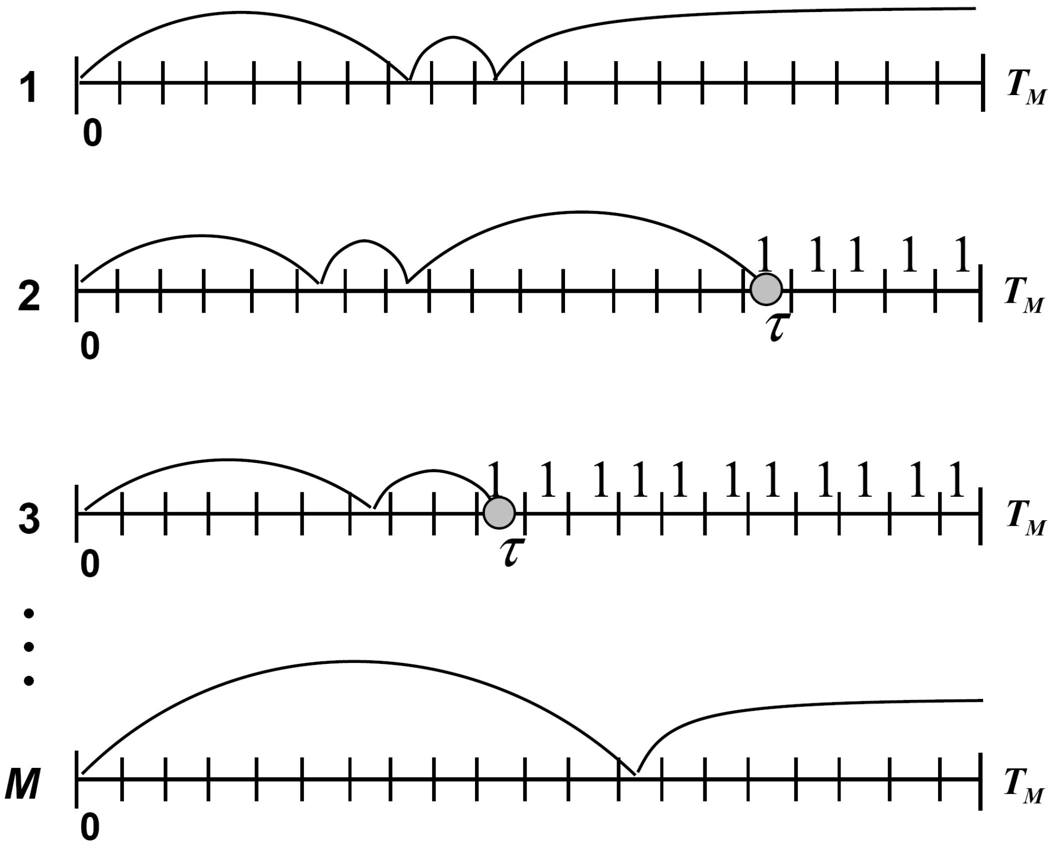
State X=2 → OFF

SIMULATION OF SYSTEM STOCHASTIC STATE TRANSITION PROCESS FOR AVAILABILITY / RELIABILITY ESTIMATION

Phase Space



Example: System Reliability Estimation



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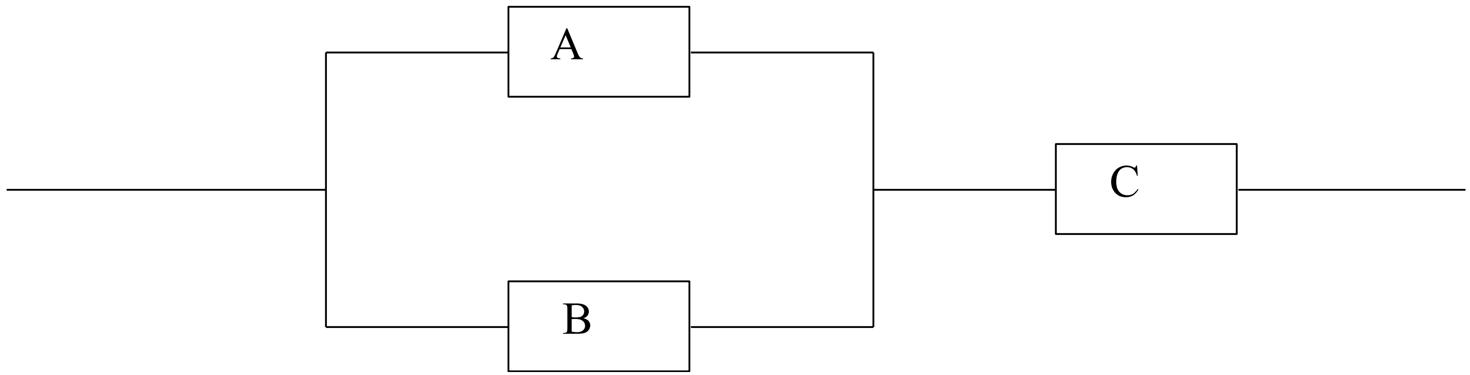
$$C^R(t) = C^R(t) \quad t \in [0, T_M]$$

$$\hat{F}_T(t) = \frac{C^R(t)}{M}$$

Indirect Monte Carlo: Example (1)



Components' times of transition between states are exponentially distributed

($\lambda^{i}_{j_i \to m_1}$ = rate of transition of component i going from its state j_i to the state m_i)

Initial \ Arrival	1	2	3
1	-	$\lambda_{1\to 2}^{A(B)}$	$\lambda_{1\to 3}^{A(B)}$
2	$\lambda_{2\to 1}^{A(B)}$	-	$\lambda_{2\to 3}^{A(B)}$
3	$\lambda_{3\to 1}^{A(B)}$	$\lambda_{3\to 2}^{A(B)}$	-

Indirect Monte Carlo: Example (2)

		Arrival			
		1	2	3	4
Initial	1	-	$\lambda^C_{1\to 2}$	$\lambda^C_{1\to 3}$	$\lambda^C_{1\to 4}$
	2	$\lambda^C_{2\to 1}$	-	$\lambda^C_{2\to 3}$	$\lambda^C_{2\to 4}$
	3	$\lambda^C_{3\to 1}$	$\lambda^C_{3\to 2}$	-	$\lambda^C_{3\to 4}$
	4	$\lambda^C_{4\to 1}$	$\lambda^C_{4\to 2}$	$\lambda^C_{4\to 3}$	-

- The components are initially (t=0) in their nominal states (1,1,1)
- One minimal cut set of order 1 (C in state 4:(*,*,4)) and one minimal cut set of order 2 (A and B in 3: (3,3,*)).

Analog Monte Carlo Trial

SAMPLING THE TIME OF TRANSITION

The rate of transition of component A(B) out of its nominal state 1 is:

$$\lambda_1^{A(B)} = \lambda_{1\to 2}^{A(B)} + \lambda_{1\to 3}^{A(B)}$$

- The rate of transition of component C out of its nominal state 1 is:

$$\lambda_1^{C} = \lambda_{1\to 2}^{C} + \lambda_{1\to 3}^{C} + \lambda_{1\to 4}^{C}$$

- The rate of transition of the system out of its current configuration (1, 1, 1) is:

$$\lambda^{(1,1,1)} = \lambda_1^{A} + \lambda_1^{B} + \lambda_1^{C}$$

- We are now in the position of sampling the first system transition time t_1, by applying the inverse transform method:

$$t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}} \ln(1 - R_t)$$

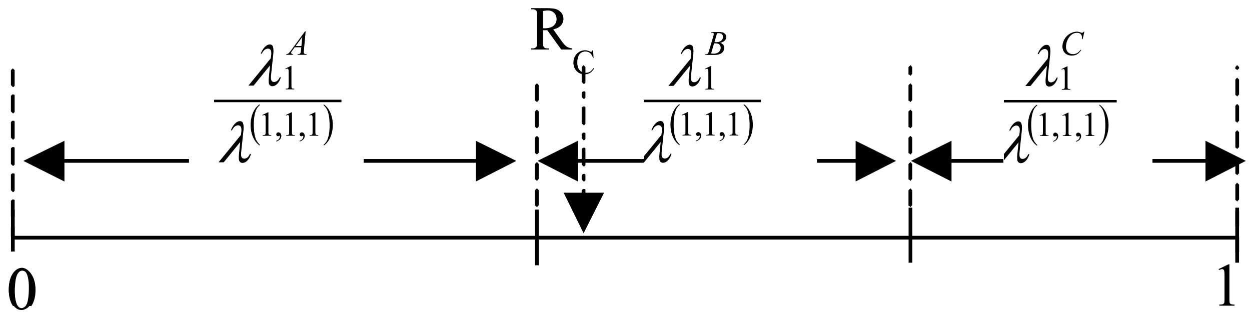
where $R_t \sim U[0,1)$

Sampling the Kind of Transition (1)

- Assuming that $t_1 < T_M$ (otherwise we would proceed to the successive trial), we now need to determine which transition has occurred, i.e. which component has undergone the transition and to which arrival state.
- The probabilities of components A, B, C undergoing a transition out of their initial nominal states 1, given that a transition occurs at time t_1, are:

$$\frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}}$$

- Thus, we can apply the inverse transform method to the discrete distribution

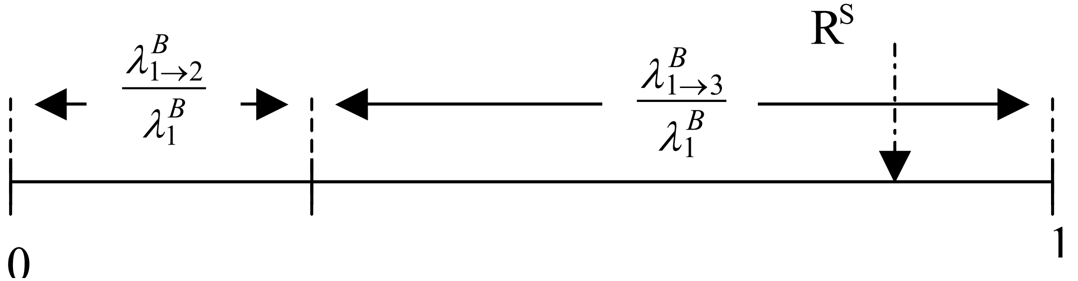


Sampling the Kind of Transition (2)

- Given that at t_1 component B undergoes a transition, its arrival state can be sampled by applying the inverse transform method to the set of discrete probabilities

$$\left\{ \frac{\lambda_{1\to 2}^{B}}{\lambda_1^{B}}, \frac{\lambda_{1\to 3}^{B}}{\lambda_1^{B}} \right\}$$

of the mutually exclusive and exhaustive arrival states



$R_S \sim U[0,1)$

- As a result of this first transition, at t_1 the system is operating in configuration (1,3,1).
- The simulation now proceeds to sampling the next transition time t_2 with the updated transition rate

$$\lambda^{(1,3,1)} = \lambda_1^{A} + \lambda_3^{B} + \lambda_1^{C}$$

Sampling the Next Transition

- The next transition, then, occurs at

$$t_2 = t_1 - \frac{1}{\lambda^{(1,3,1)}} \ln(1 - R_t)$$

where $R_t \sim U[0,1)$.

- Assuming again that $t_2 < T_M$, the component undergoing the transition and its final state are sampled as before by application of the inverse trasform method to the appropriate discrete probabilities.

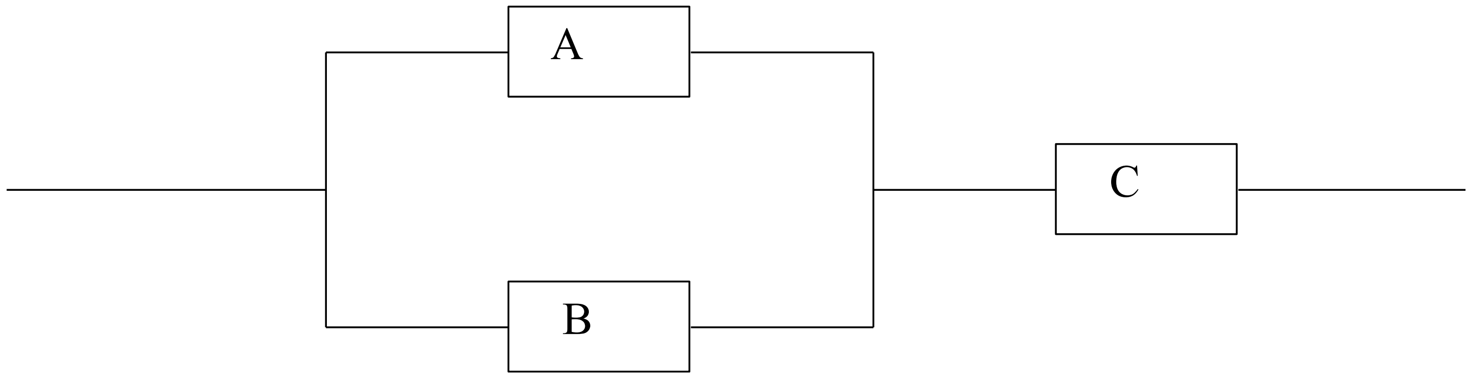
- The trial simulation then proceeds through the various transitions from one system configuration to another up to the mission time T_M.

Unreliability and Unavailability Estimation

- When the system enters a failed configuration (*,*,4) or (3,3,*), where the * denotes any state of the component, tallies are appropriately collected for the unreliability and instantaneous unavailability estimates (at discrete times $t_j \in [0, T_M]$);

- After performing a large number of trials M, we can obtain estimates of the system unreliability and instantaneous unavailability by simply dividing by M, the accumulated contents of $C^R(t_j)$ and $C_A(t_j)$, $t_j \in [0,T_M]$

Direct Monte Carlo: Example (1)



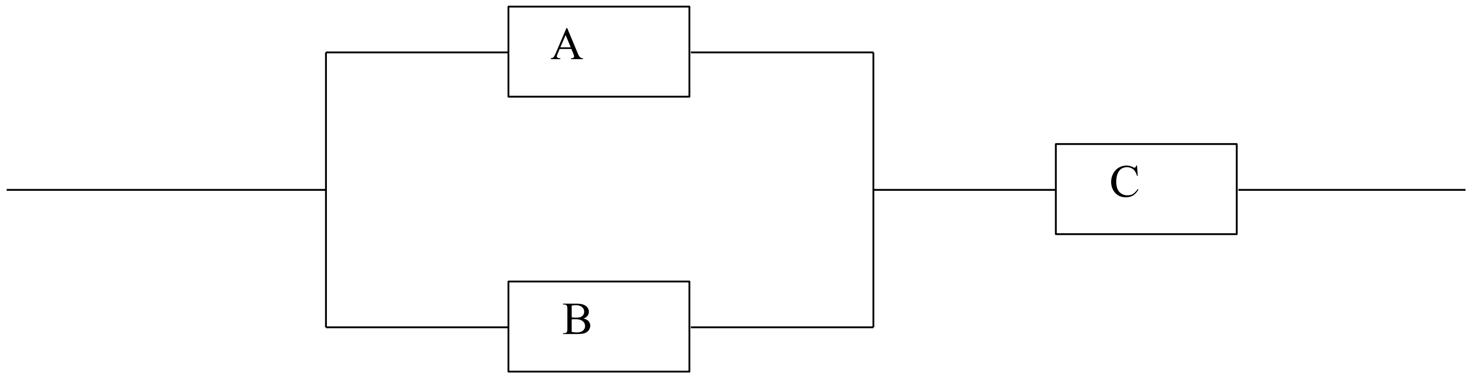
For any arbitrary trial, starting at t=0 with the system in nominal configuration (1,1,1) we would sample all the transition times:

$$t^i_{1\to m_i} = t_0 - \frac{1}{\lambda^i_{1\to m_i}} \ln(1 - R^i_{t,1\to m_i}) \qquad \left.\begin{array}{ll} i = A, B, C & \\ m_i = 2,3 & \text{for } i = A, B \\ m_i = 2,3,4 & \text{for } \mathrm{i} = C \end{array}\right\}$$

where $R^i_{t,1\to m_i} \sim U[0,1)$

These transition times would then be ordered in ascending order from t_{min} to $t_{max} \leq T_M$. Let us assume that t_{min} corresponds to the transition of component A to state 3 of failure. The current time is moved to $t_1 = t_{min}$ in correspondence of which the system configuration changes, due to the occurring transition, to (3,1,1) still operational.

Direct Monte Carlo: Example (2)



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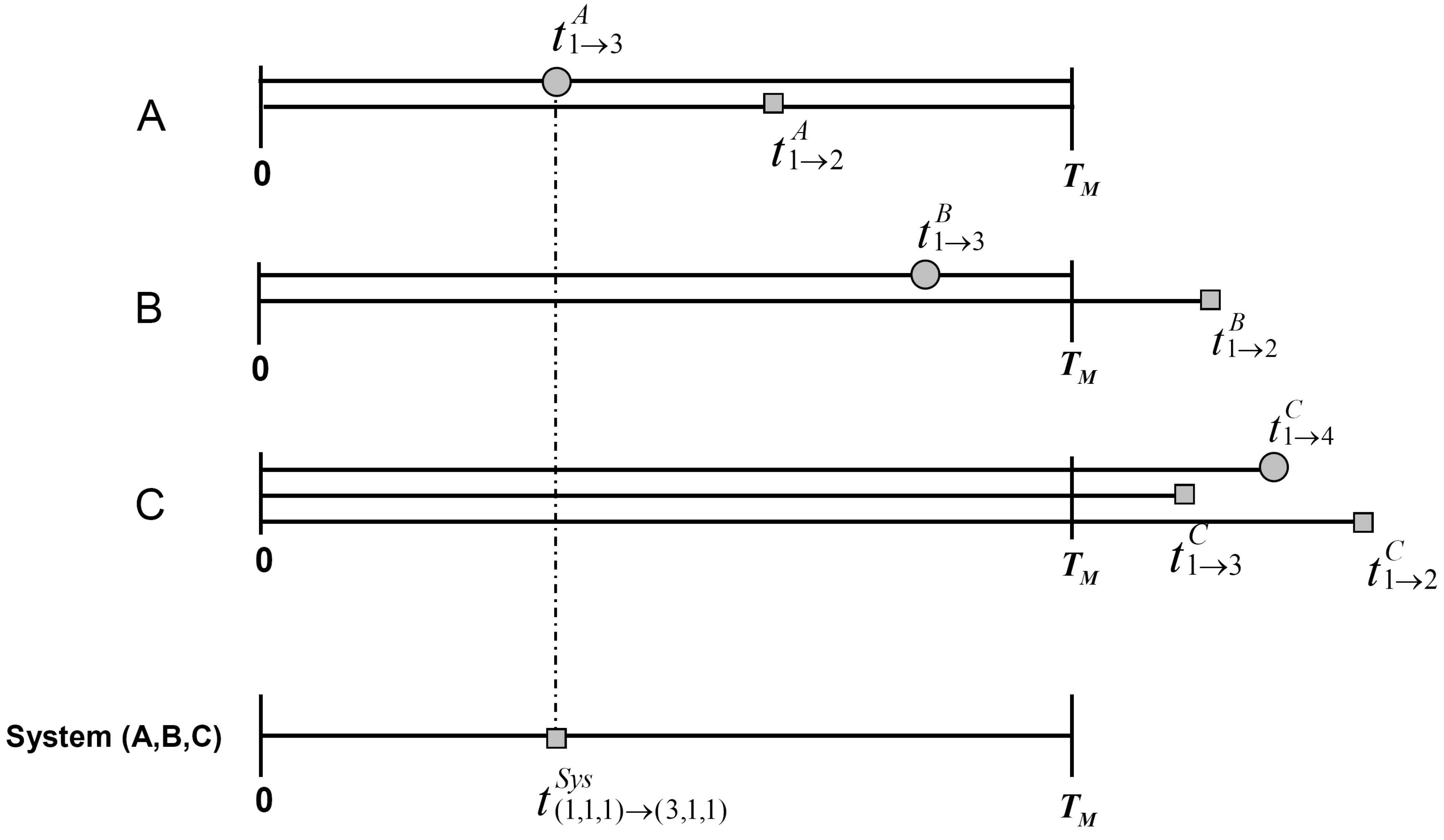
Direct Monte Carlo: Example (3)



Example (2)

The new transition times of component A are then sampled

$$t^A_{3\to m_A} = t_1 - \frac{1}{\lambda^A_{3\to m_A}} \ln(1 - R^A_{t,3\to m_A}) \qquad k = 1,2 \qquad R^A_{t,3\to m_A} \sim U[0,1)$$

and placed at the proper position in the timeline of the succession of occurring transitions

- The simulation then proceeds to the successive times in the list, in correspondence of which a system transition occurs.
- After each transition, the timeline is updated with the times of the transitions that the component which has undergone the last transition can do from its new state.
- During the trial, each time the system enters a failed configuration, tallies are collected and in the end, after M trials, the unreliability and unavailability estimates are computed.

POLITECNICO DI MILANO

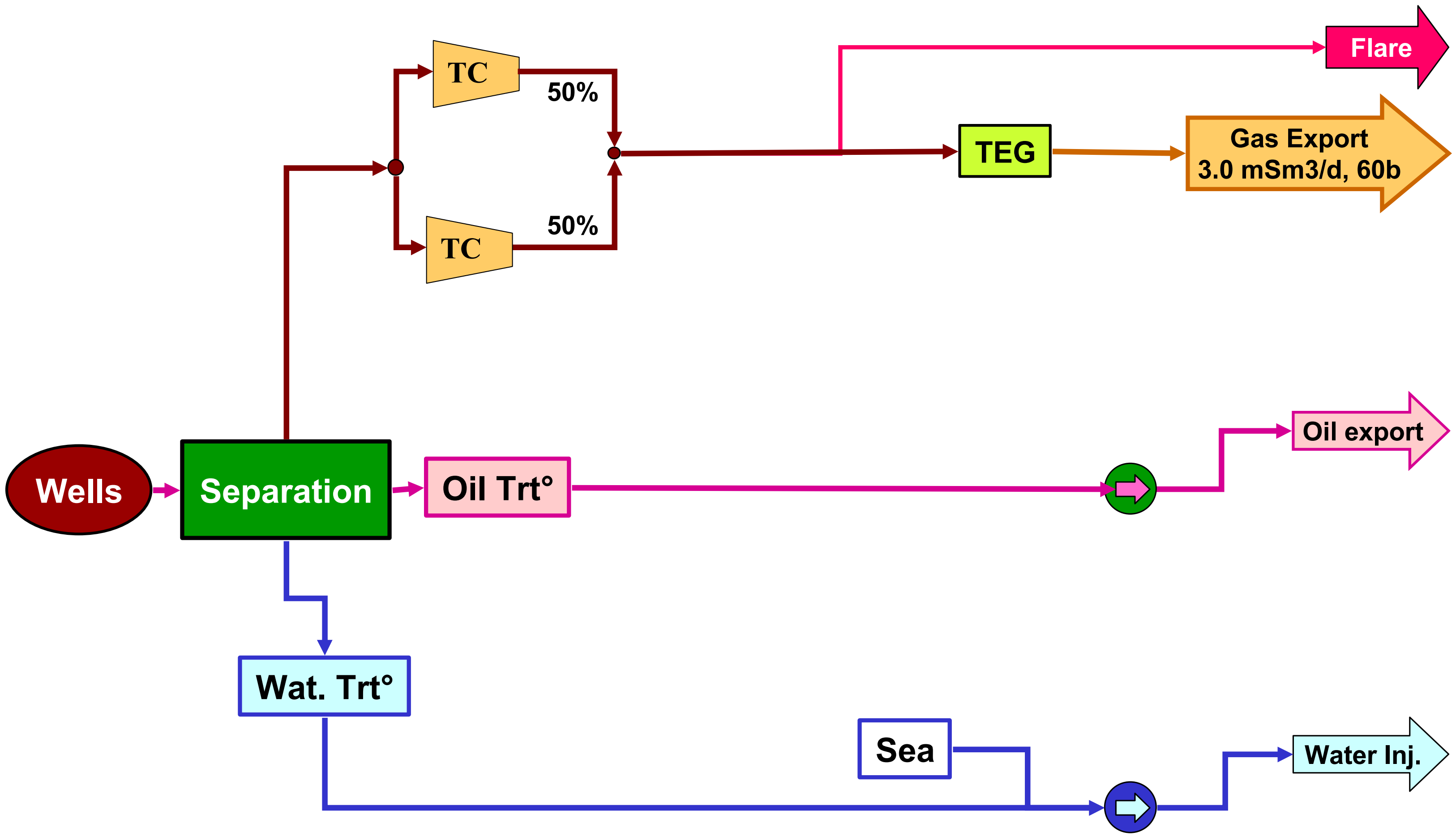
PRODUCTION AVAILABILITY EVALUATION OF AN OFFSHORE INSTALLATION

A real example of Indirect Simulation

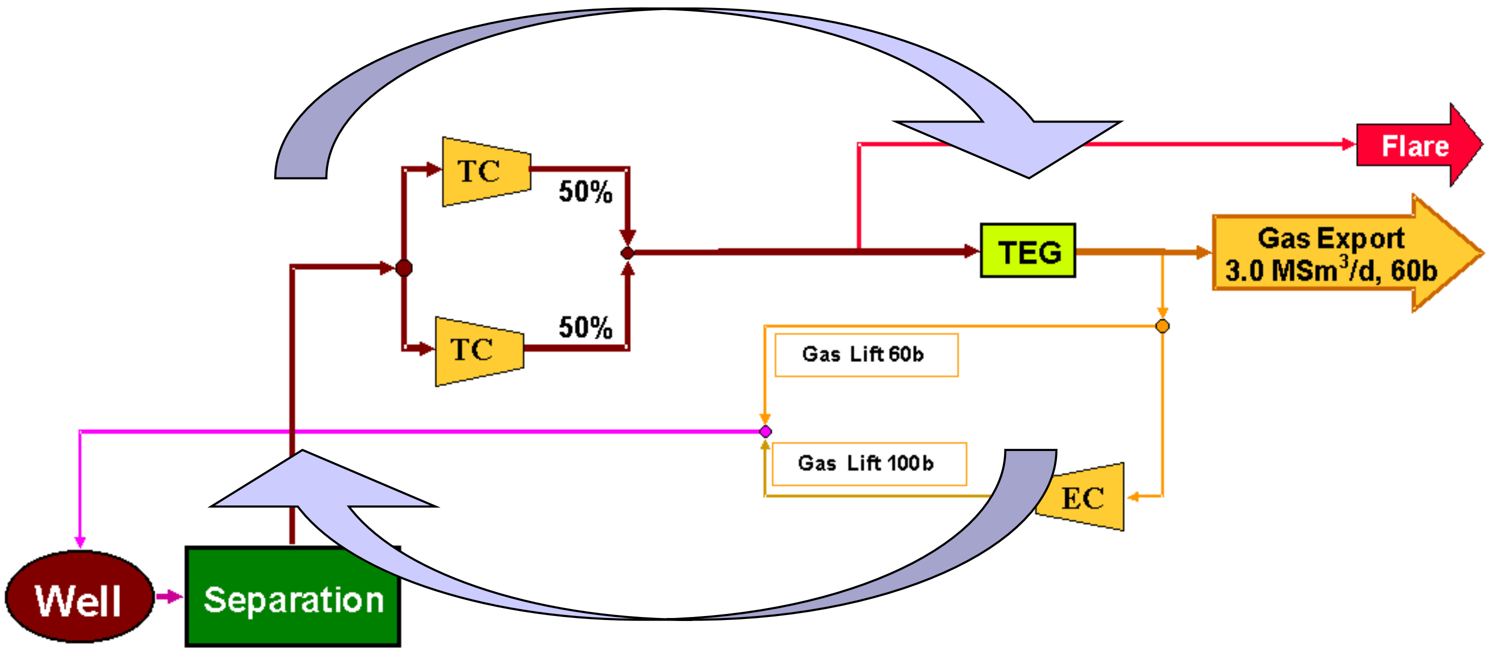
CENTRALE PARIS

Supélec

System description: basic scheme



System description: gas-lift



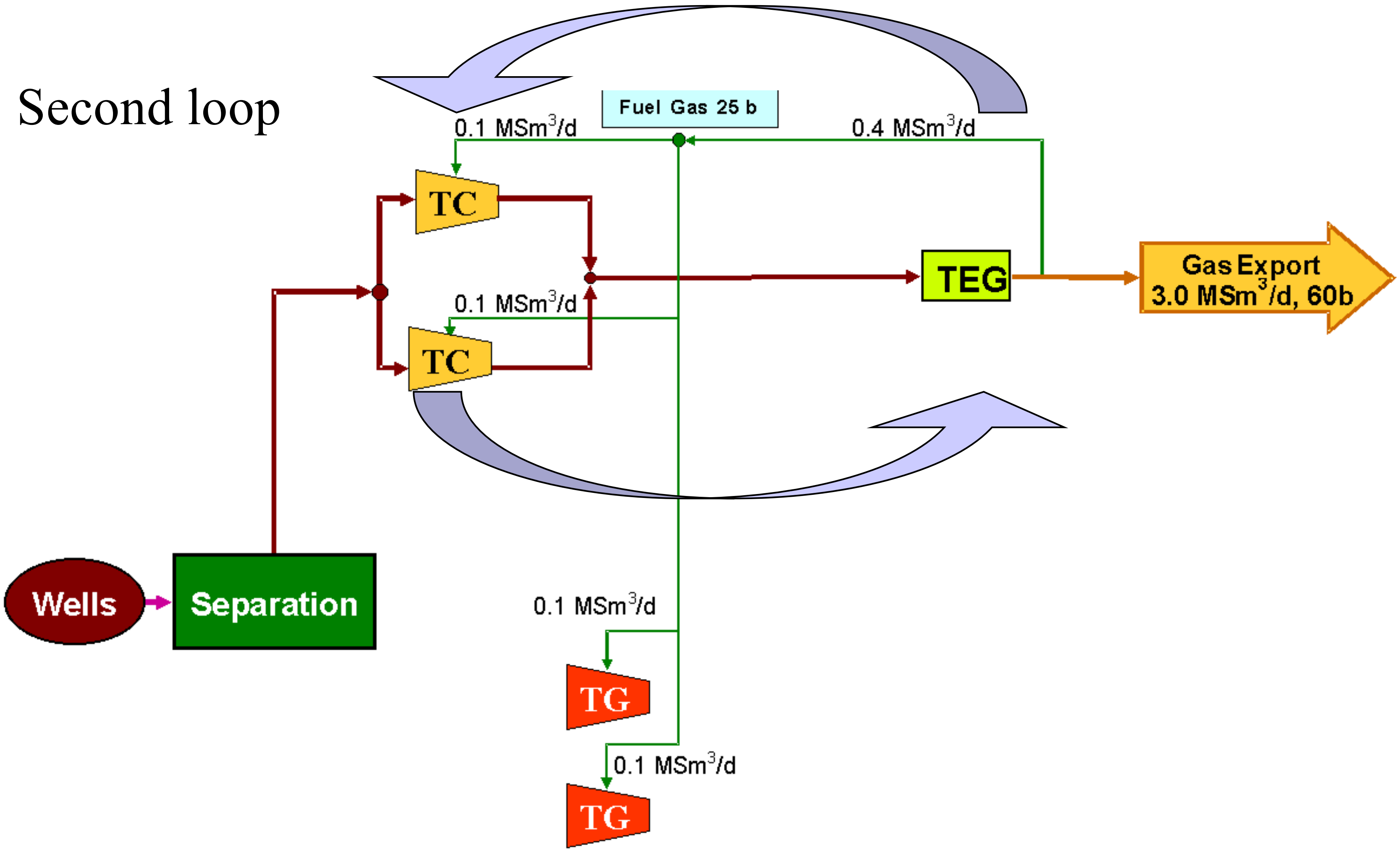
First loop

Gas-lift pressure	Production of the Well
100	100%
60	80%
0	60%

System description:

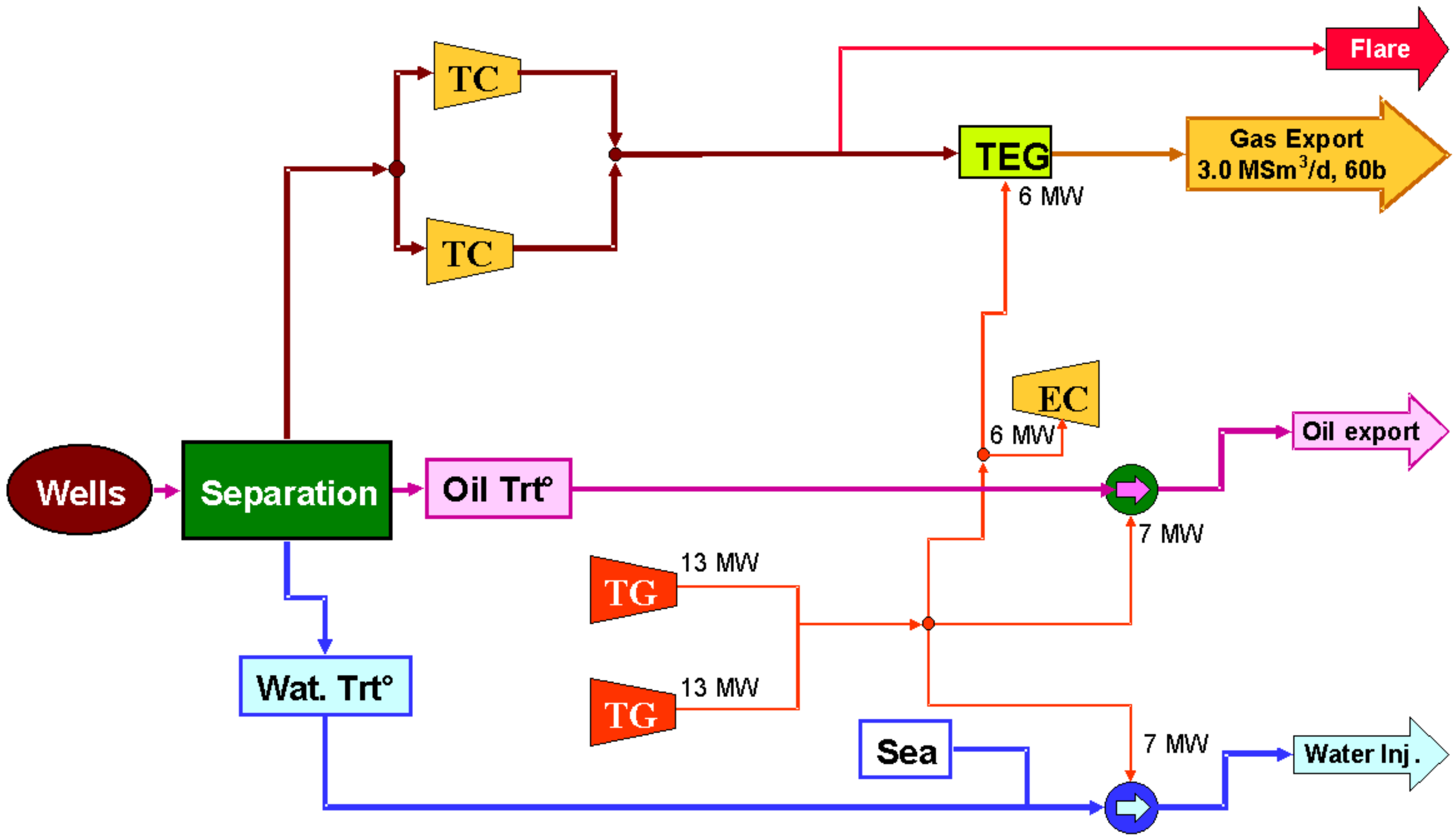
fuel gas generation and distribution

Second loop

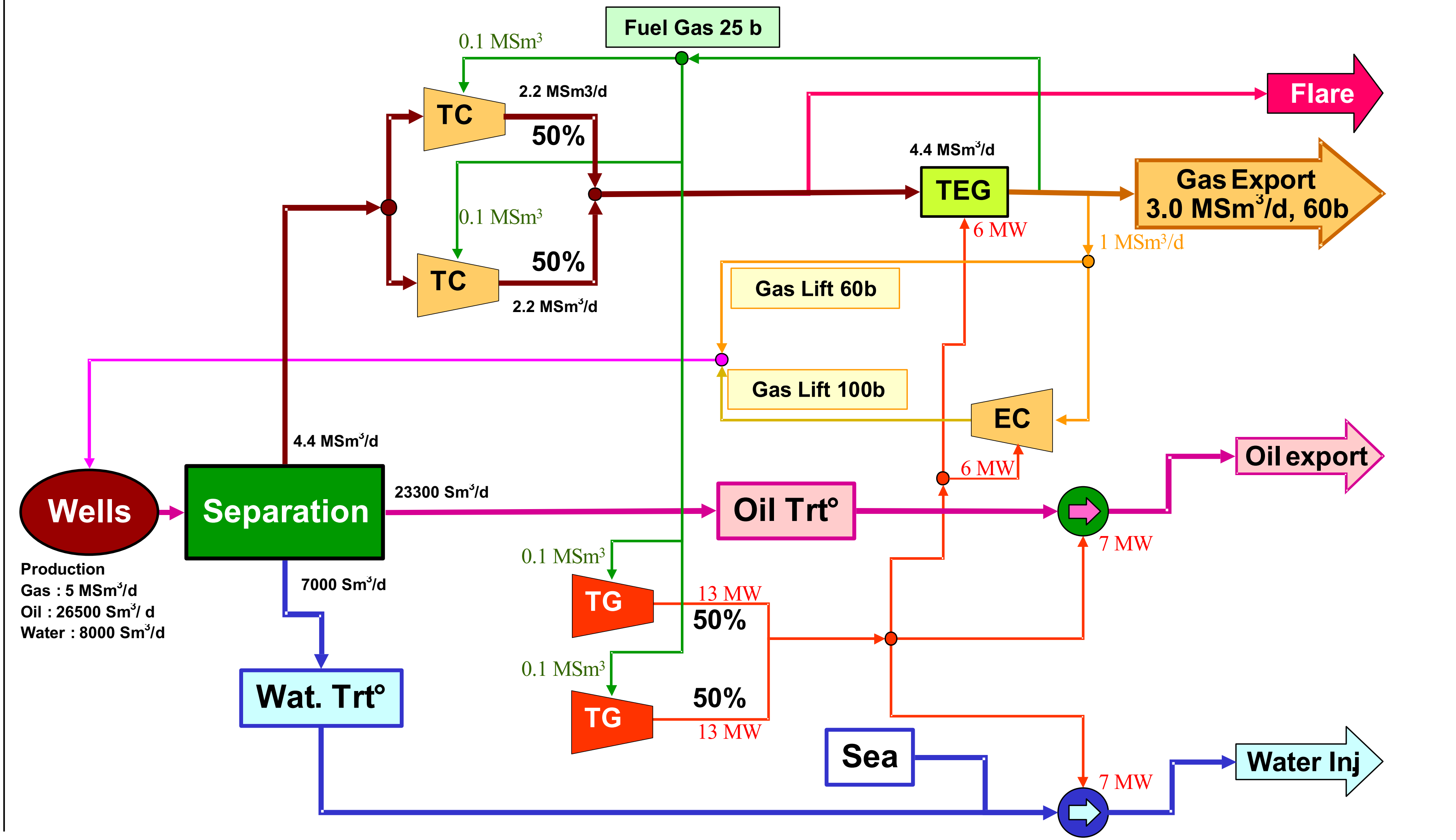


System description:

electricity power production and distribution

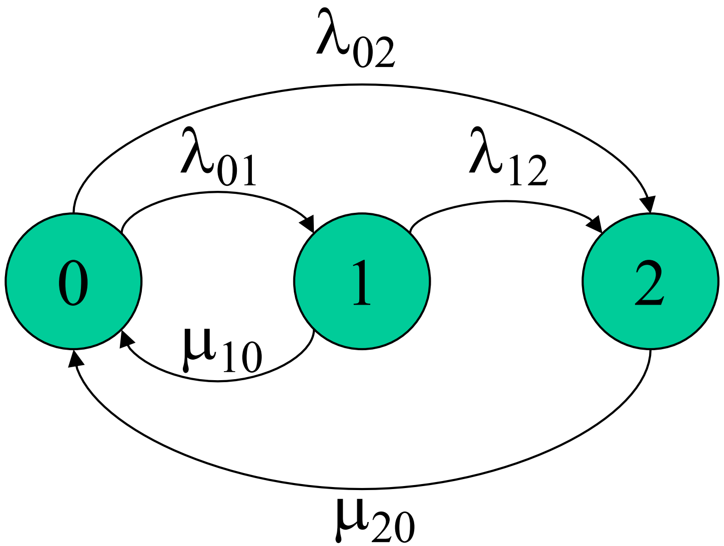


The offshore production plant



back

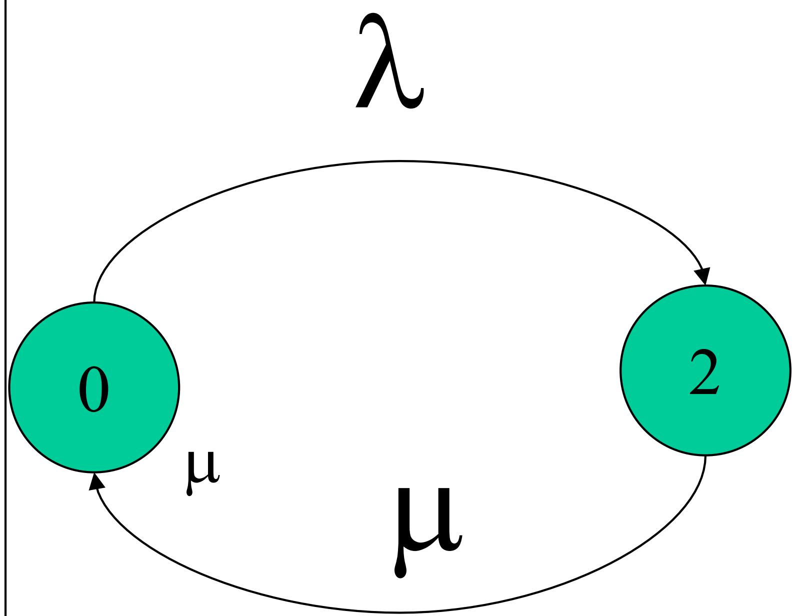
Component failures and repairs: TCs and TGs



	TC	TG
λ_{01}	$0.89 \cdot 10^{-3}\ h^{-1}$	$0.67 \cdot 10^{-3}\ h^{-1}$
λ_{02}	$0.77 \cdot 10^{-3}\ h^{-1}$	$0.74 \cdot 10^{-3}\ h^{-1}$
λ_{12}	$1.86 \cdot 10^{-3}\ h^{-1}$	$2.12 \cdot 10^{-3}\ h^{-1}$
μ_{10}	$0.033\ h^{-1}$	$0.032\ h^{-1}$
μ_{20}	$0.048\ h^{-1}$	$0.038\ h^{-1}$

State 0 = as good as new
State 1 = degraded (no function lost, greater failure rate value)
State 2 = critical (function is lost)

Component failures and repairs: EC and TEG



	EC	TEG
λ	$0.17 \cdot 10^{-3}\ h^{-1}$	$5.7 \cdot 10^{-5}\ h^{-1}$
μ	$0.032\ h^{-1}$	$0.333\ h^{-1}$

State 0 = as good as new

State 2 = critical (function is lost)

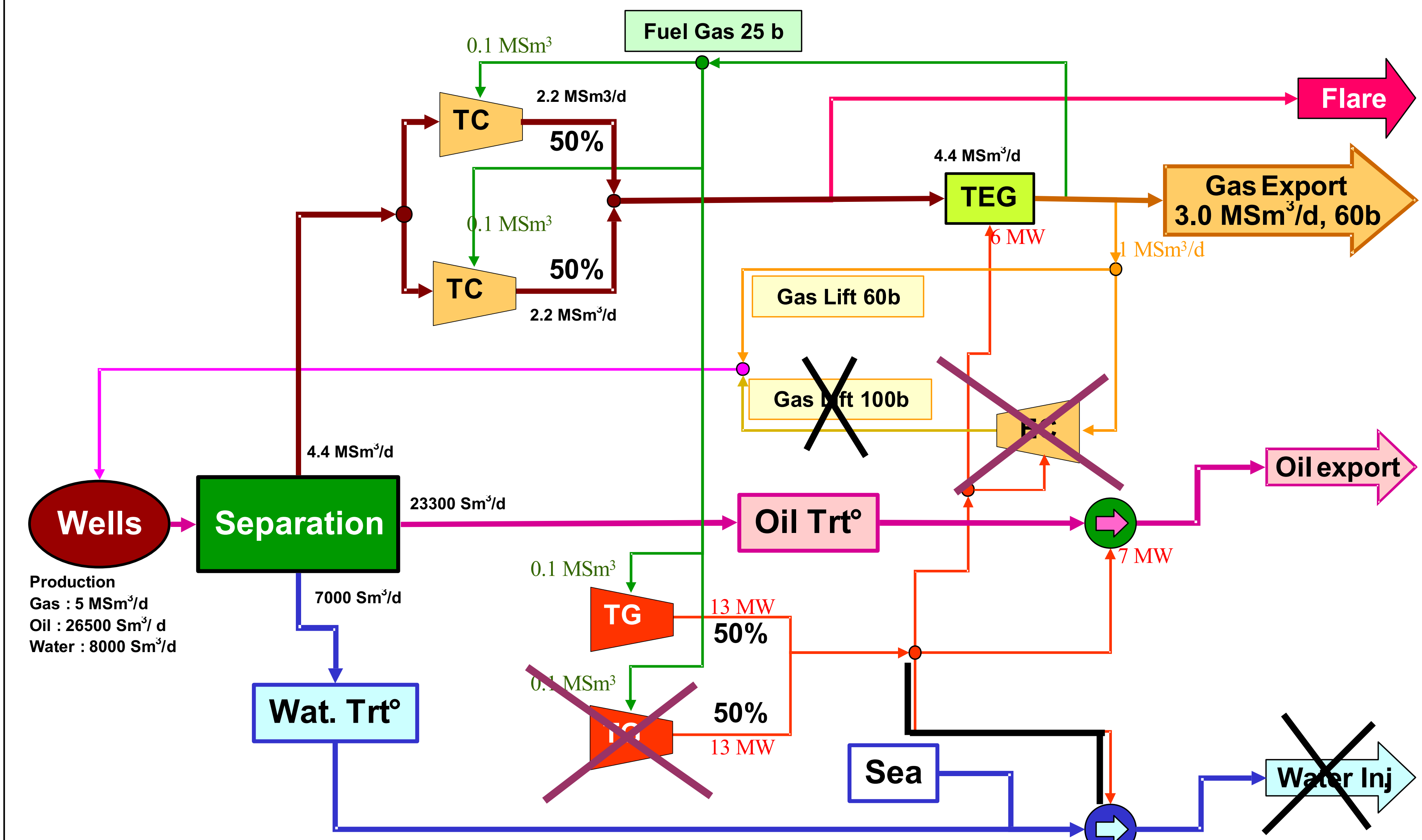
Production priority

When a failure occurs, the system is reconfigured to minimise (in order):

- the impact on the export oil production
- the impact on export gas production

➢ **The impact on water injection does not matter**

Production priority: example



Maintenance policy: reparation

Only 1 repair team

Priority levels of failures:

1. Failures leading to total loss of export oil (both TG's or both TC's or TEG)
2. Failures leading to partial loss of export oil (single TG or EC)
3. Failures leading to no loss of export oil (single TC failure)

Maintenance policy: preventive maintenance

- Only 1 preventive maintenance team
- The preventive maintenance takes place only if the system is in perfect state of operation

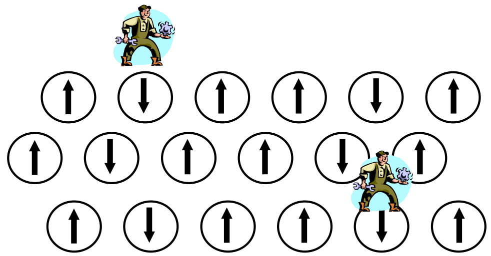
	Type of maintenance	Frequency [hours]	Duration [hours]
Turbo-Generator and Turbo-Compressors	Type 1	2160 (90 days)	4
	Type 2	8760 (1 year)	120 (5 days)
	Type 3	43800 (5 years)	672 (4 weeks)
Electro Compressor	Type 4	2666	113

MARKOV APPROACH

Number of components = 6
Number of states for component = 2 or 3 ⇨ $2^2 \cdot 3^4 = \mathbf{324}$ plant states

+

1 repair team ⇨ 129 new plant states



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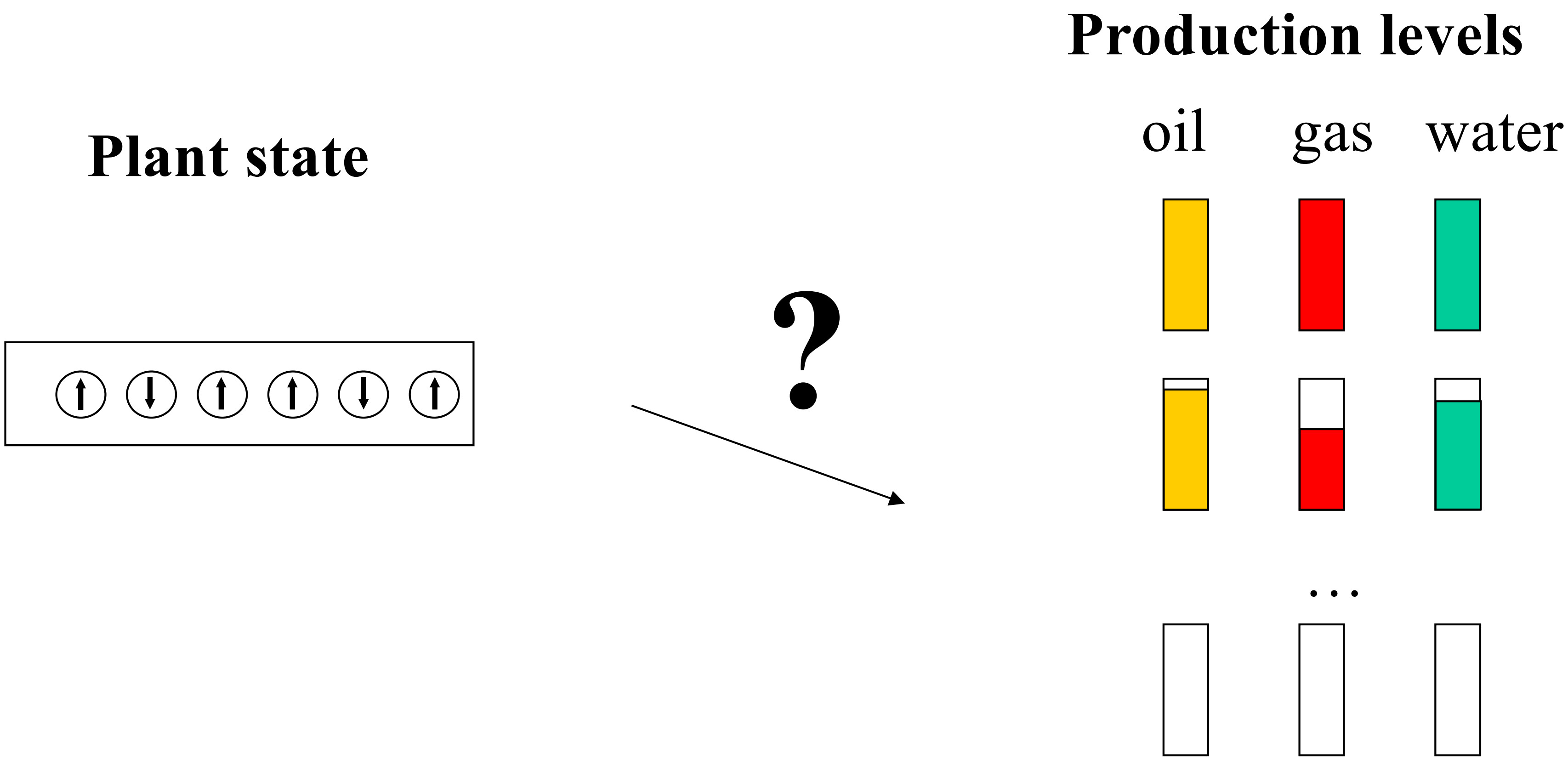
1 maintenance team ⇨ Non homogeneous Markov chain

Markov approach too complex

MONTE CARLO APPROACH

MONTE CARLO APPROACH



Associate a production level to each of the 453 plant states → too long, error prone

A systematic procedure

7 different production levels

6 different system faults

6 fault trees

6 families of mcs

Production Level	Gas [kSm^3/d]	Oil [k m^3/d]	Water [m^3/d]	mcs	MCS
0=(100%)	3000	23.3	7000		
1	900	23.3	7000	X5, X6	X5,X6
2	2700	21.2	0	X3, X4	X2X3,X2X4
3	1000	21.2	0	X3X5, X3X6, X4X5, X4X6	X2X3X5, X2X3X6, X2X4X5, X2X4X6
4	2600	21.2	6400	X2	X2
5	900	21.2	6400	X2X5, X2X6	X2X5, X2X6
6	0	0	0	X1, X3X4, X5X6	X1X2X3X4X5X6

Numerical results

Case A: corrective maintenance and no preventive maintenance (T_{miss}= $1 \cdot 10^3$ hours, trials=10^6)

CPU time ≈ 15 min

Case B: perfect system (no failures) and preventive maintenance (T_{miss}= 10^4 hours, trials=10^5)

CPU time ≈ 12 min

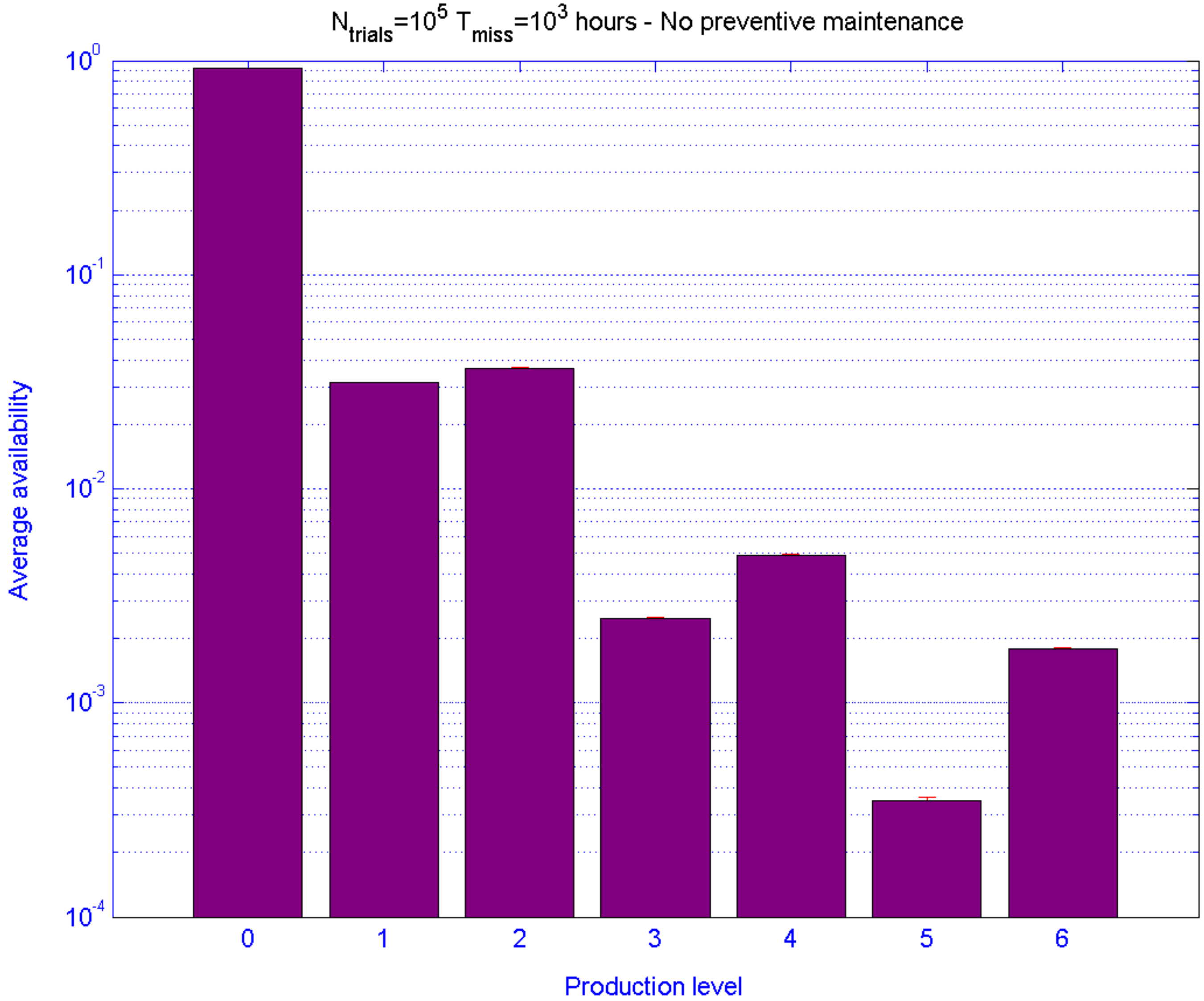
Case C: corrective and preventive maintenance

(T_{miss}=$5 \cdot 10^5$ hours, trials=10^5)

CPU time ≈ 20 h

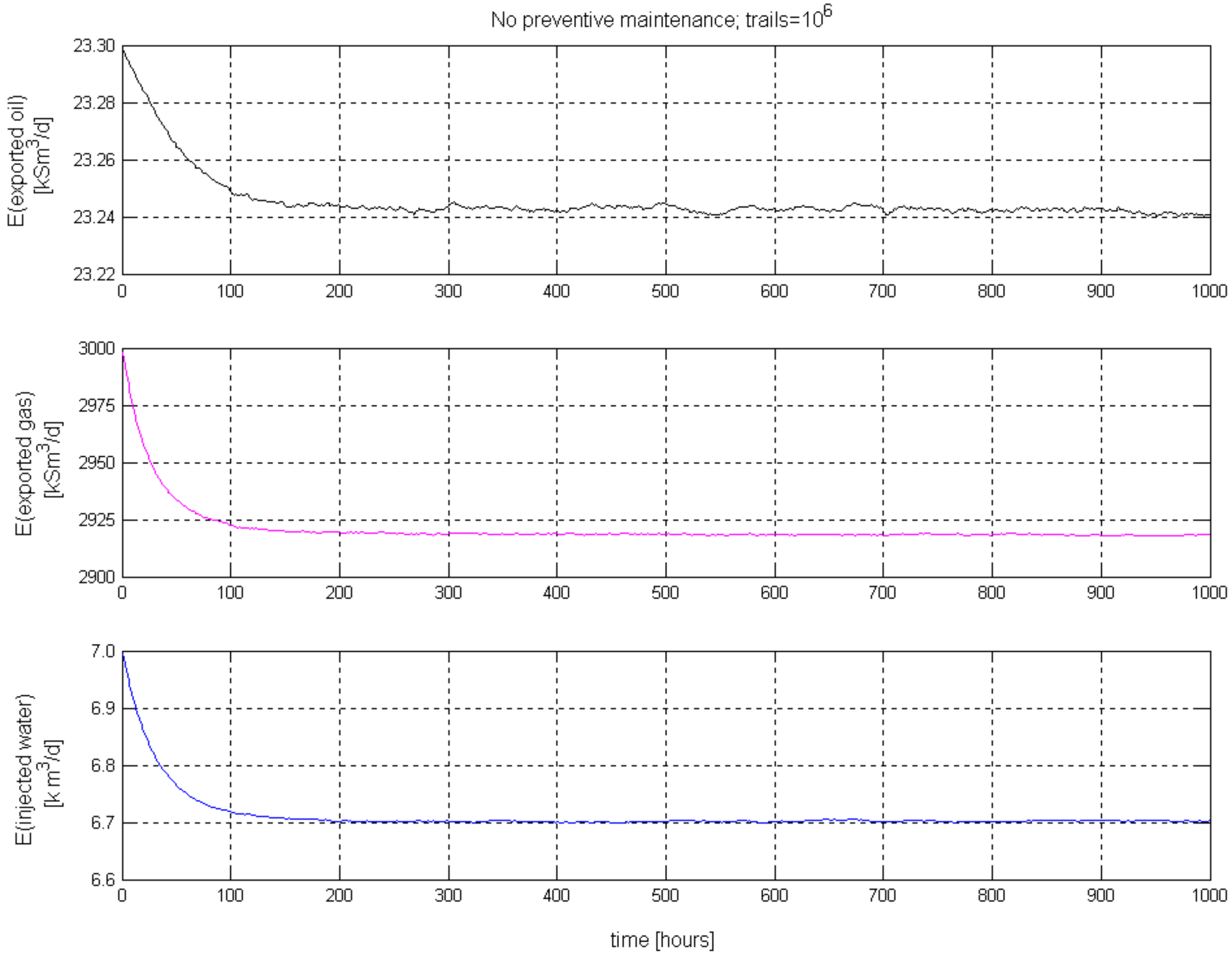
Case A: no preventive maintenances

Production level	Average availability
0	9.23E-1
1	3.13E-2
2	3.67E-2
3	2.47E-3
4	4.88E-3
5	3.50E-4
6	1.79E-3



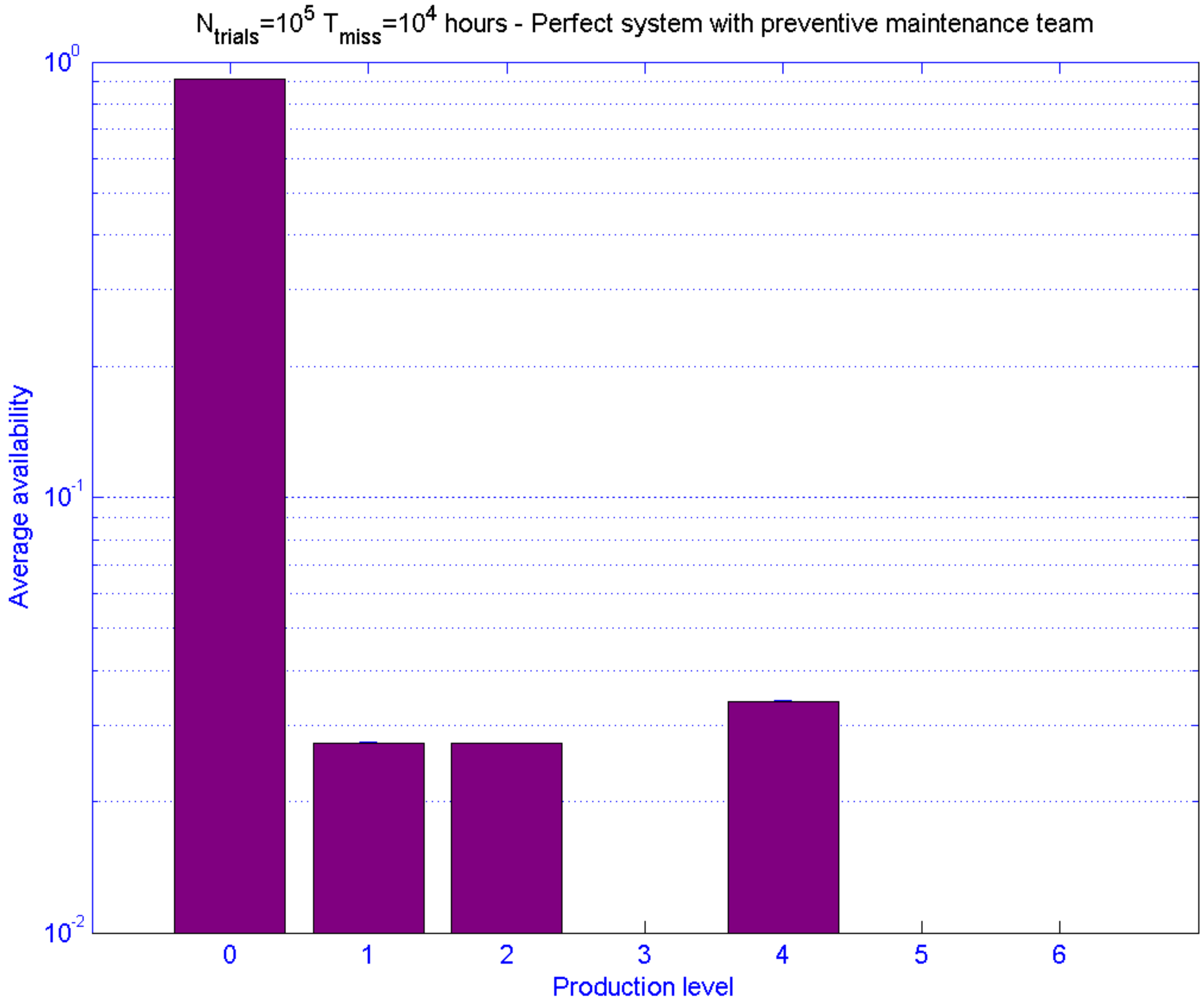
Case A: no preventive maintenances

	Asymptotic values
Oil [k m³/d]	23.24
Gas [k Sm³/d]	2918
Water [k m³/d]	6.703



Case B: perfect system and preventive maintenances

Production level	Average availability
0	9.12E-1
1	2.73E-2
2	2.72E-2
3	0.00
4	3.40E-2
5	0.00
6	0.00



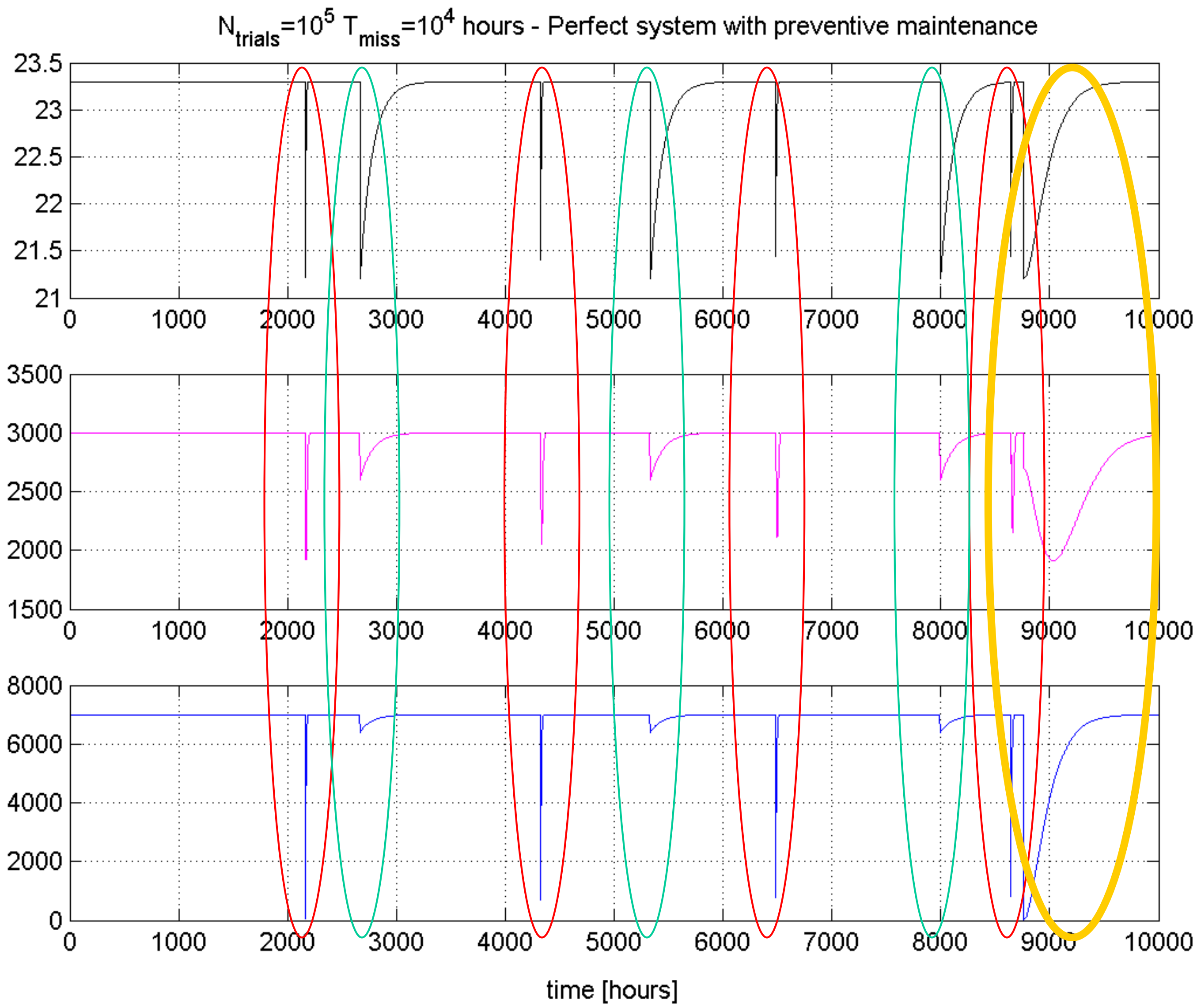
Case B: perfect system and preventive maintenances

P.Maintenance Type 1 (TC,TG)

P.Maintenance Type 2 (EC)

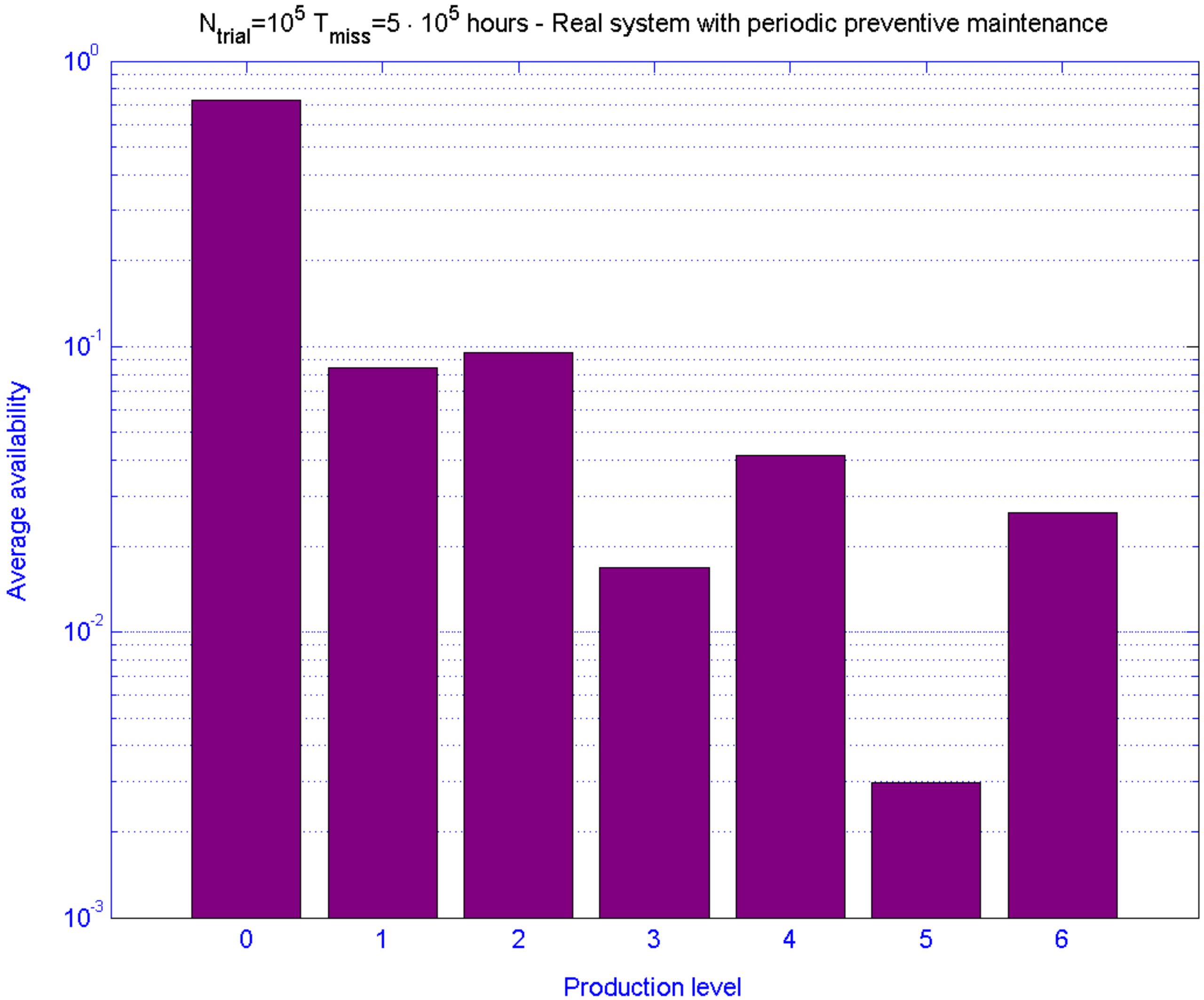
P.Maintenance Type 3 (TC,TG)

	Mean	Std
Oil [k m³/d]	23.230	0.263
Gas [k Sm³/d]	2929	194.0
Water [k m³/d]	6.811	0.883



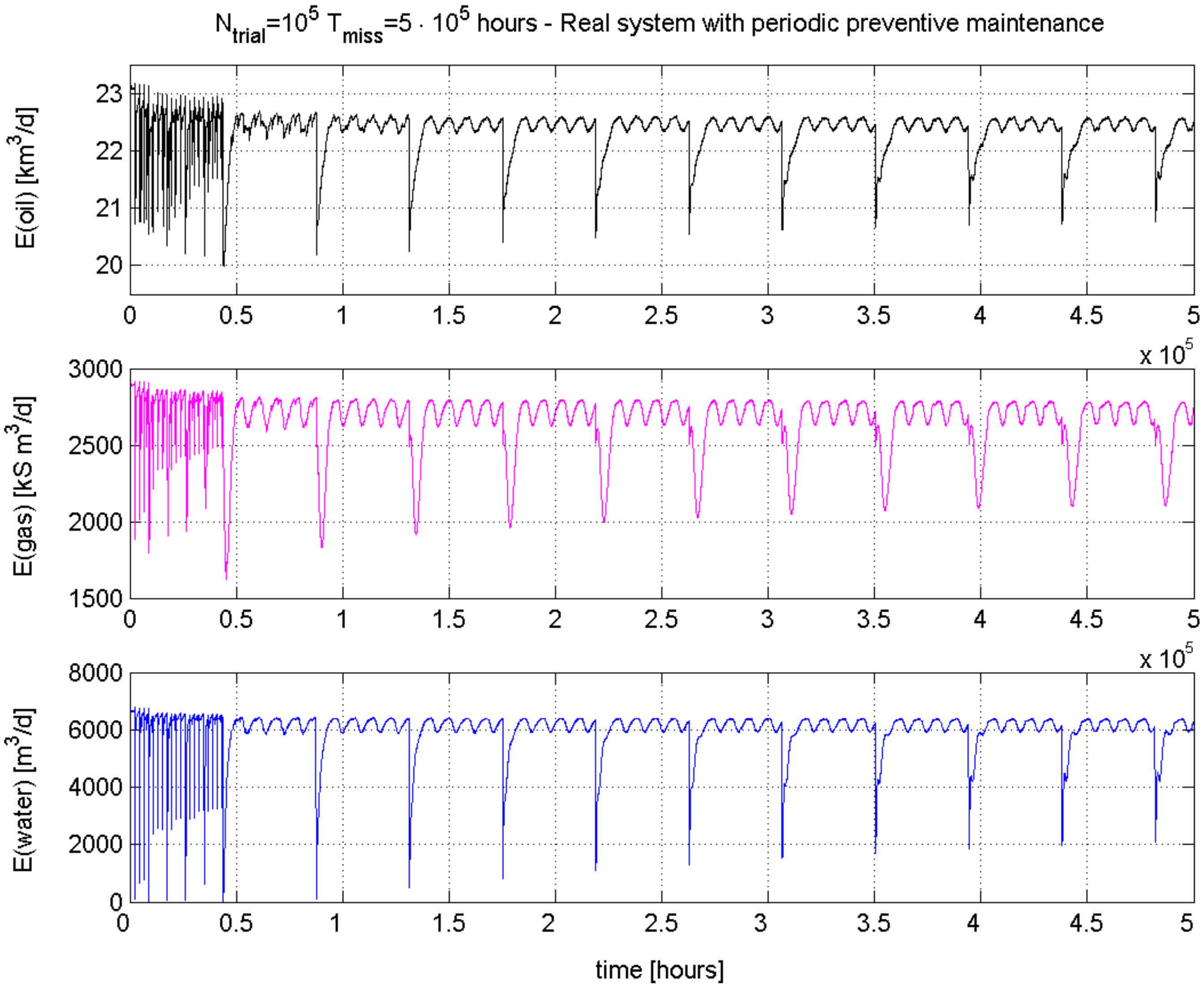
Case C: real system with preventive maintenances

Production level	Average availability
0	8.13E-1
1	5.68E-2
2	6.58E-2
3	1.19E-2
4	3.55E-2
5	2.34E-3
6	1.50E-2



Case C: real system with preventive maintenances

	Mean	Std
Oil [k m³/d]	22.60	0.42
Gas [k Sm³/d]	2687	194.3
Water [k m³/d]	6.04	0.76

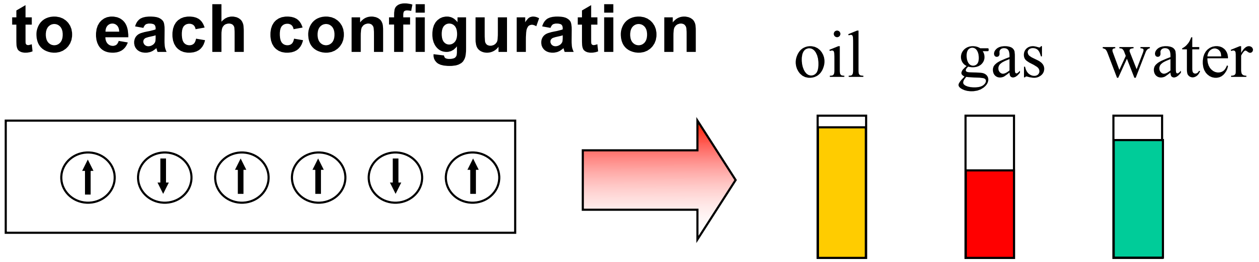


Conclusions

- Complex multi-state system with maintenance and operational loops

MC simulation

- Systematic procedure to assign a production level to each configuration



- Investigation of effects maintenance on production

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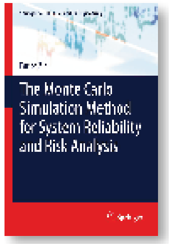

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E. Zio, Ecole Centrale Paris, Chatenay-Malabry, France

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