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# Monte Carlo Simulation for Reliability and Availability Analysis

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## Some Examples:

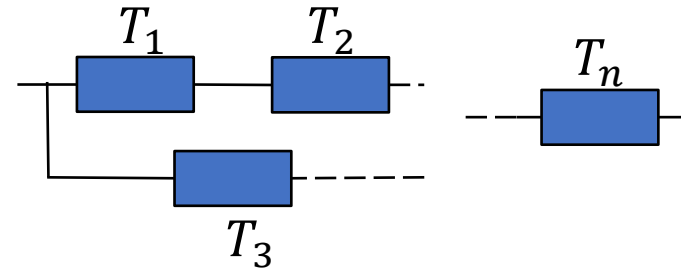
### 1) Unreliability @mission time $t_M$ :

$$F_T(t_{miss}) = P\{T \leq t_{miss}\} = P\{q(T_1, \dots, T_n) \leq t_{miss}\} = \\ = P\{q(T_1, \dots, T_n) \leq t_{miss}\} = \int_{(t_1, \dots, t_n): q(t_1, \dots, t_n) \leq t_{miss}} f_{T_1, \dots, T_n}(t_1, \dots, t_n) dt_1 \dots dt_n$$

- $T_i$  = Failure time of component  $i$
- $T = q(T_1, \dots, T_n)$  = System failure time

#### Examples for $q$ :

- Series system  $\rightarrow q(T_1, \dots, T_n) = \min(T_1, \dots, T_n)$
- Parallel system  $\rightarrow q(T_1, \dots, T_n) = \max(T_1, \dots, T_n)$



### 2) $MTTF = \int_0^{+\infty} t f_T(t) dt$

# MC Estimation of Definite Integrals

$$G = \int_a^b h(x) dx = \int_a^b g(x) f_X(x) dx$$



- $X$  is a random variable with pdf  $f_X(x)$ :
- $g(x)$  is a random variable

$$\begin{cases} f_X(x) \geq 0 \\ \int_a^b f_X(x) dx = 1 \end{cases}$$



$$E[g(x)] = \int_a^b g(x) f_X(x) dx = G$$

**Problem** → Estimate  $E[g(x)]$

**Solution** → Dart Game

- 1) for  $i = 1, 2, \dots, N$ 
  - Sample  $X_i$  from  $f_X(x)$
  - Compute  $g(X_i)$End

- the probability that a shot  $X_i$  hits  $[x, x + dx]$  is  $f(x)dx$
- the award is  $g(X_i)$

2) Compute

$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \bar{g}$$

**Random variable!**

$\swarrow \searrow$   
 $\text{Var}[G_N] \quad E[G_N]$



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**Random variable!**

$\swarrow$   
 $\searrow$   
 $Var[G_N]$   $E[G_N]$

Is  $G_N$  a good estimator of  $E[g(x)]$  ?

## Characteristics of a good estimator:

- Unbiased  $\rightarrow E[G_N] = G$
- Consistent  $\rightarrow \lim_{N \rightarrow +\infty} E[(G_N - G)^2] = 0$

# MC Evaluation of Definite Integrals (1D): Why $G_N$ is a good estimator of $G$ ?

$$G_N = \frac{1}{N} \sum_{i=1}^N g(x_i)$$



$G_N$  is a random variable with:

$$E[G_N] = E\left[\frac{1}{N} \sum_{i=1}^N g(X_i)\right] = \frac{1}{N} \sum_{i=1}^N E[g(X_i)] = \frac{1}{N} \sum_{i=1}^N E[g(x)] = G$$

$$\text{Var}[G_N] = \text{Var}\left[\frac{1}{N} \sum_{i=1}^N g(X_i)\right] = \frac{1}{N^2} \sum_{i=1}^N \text{Var}[g(X_i)] = \frac{1}{N} \text{Var}[g(x)]$$



$G_N$  is an unbiased estimator of  $G$ :

$$E[G_N] = G$$

$G_N$  is a consistent estimator of  $G$ :

$$\lim_{N \rightarrow \infty} \text{Var}[G_N] = 0$$



$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi} = 0.6366$$

How can we write the integral for  
MC estimation?

$$f(x) = ?$$

$$g(x) = ?$$

# MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx = \frac{2}{\pi} = 0.6366$$

By setting:

$$f(x) = 1 \quad \text{for } x \in [0,1]$$

$$g(x) = \cos\left(\frac{\pi}{2}x\right)$$

We perform  $N=10^4$  trials:

$$\begin{array}{l} X_i \rightarrow U[0,1] \\ g(x_i) = \cos\left(\frac{\pi}{2}X_i\right) \end{array} \quad \Rightarrow \quad G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$

# MC Evaluation of Definite Integrals (1D) Example

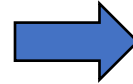
$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$$

We perform  $N=10^4$  trials:

$$x_i \rightarrow U[0,1)$$

$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$

$N = 10^4$



$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$



$$\text{Var}[G_N] = \frac{1}{N} \text{Var}[g(x)] = \frac{1}{N} \left( E[g^2(x)] - (E[g(x)])^2 \right) = \frac{1}{N} \left( E[g^2(x)] - G^2 \right)$$

Unknown in a practical case!

# MC Evaluation of Definite Integrals (1D) Example

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$$x_i \rightarrow U[0,1]$$
$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$
$$N = 10^4$$

→

$$G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0,6342$$



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$$G = E[g(x)] = \frac{2}{\pi}$$

$$E[g^2(x)] = \int_0^1 \cos^2\left(\frac{\pi}{2}x\right) dx = \frac{1}{2}$$

Unknown in a practical case!

$$\text{Var}[G_N] = \frac{1}{10^4} \left( \frac{1}{2} - \left(\frac{2}{\pi}\right)^2 \right) = 9.47 \cdot 10^{-6}$$

# MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right) dx$$

We perform  $N=10^4$  trials:

$$x_i \rightarrow U[0,1] \quad \begin{matrix} g(x_i) = \cos\left(\frac{\pi}{2}X_i\right) \\ N = 10^4 \end{matrix} \quad \longrightarrow \quad G_N = \frac{1}{N} \sum_{i=1}^N g(X_i) = \frac{1}{N} \sum_{i=1}^N \cos\left(\frac{\pi}{2}X_i\right) = 0.6342$$



$$\text{Var}[G_N] = \frac{1}{N} \text{Var}[g(x)] = \frac{1}{N} \left( E[g^2(x)] - (E[g(x)])^2 \right) = \frac{1}{N} \left( E[g^2(x)] - G^2 \right)$$



$$G \approx G_N = 0.6342$$

$$E[g^2(x)] \approx \frac{1}{N} \sum_{i=1}^N g^2(X_i)$$

They can be computed during the MC simulation!



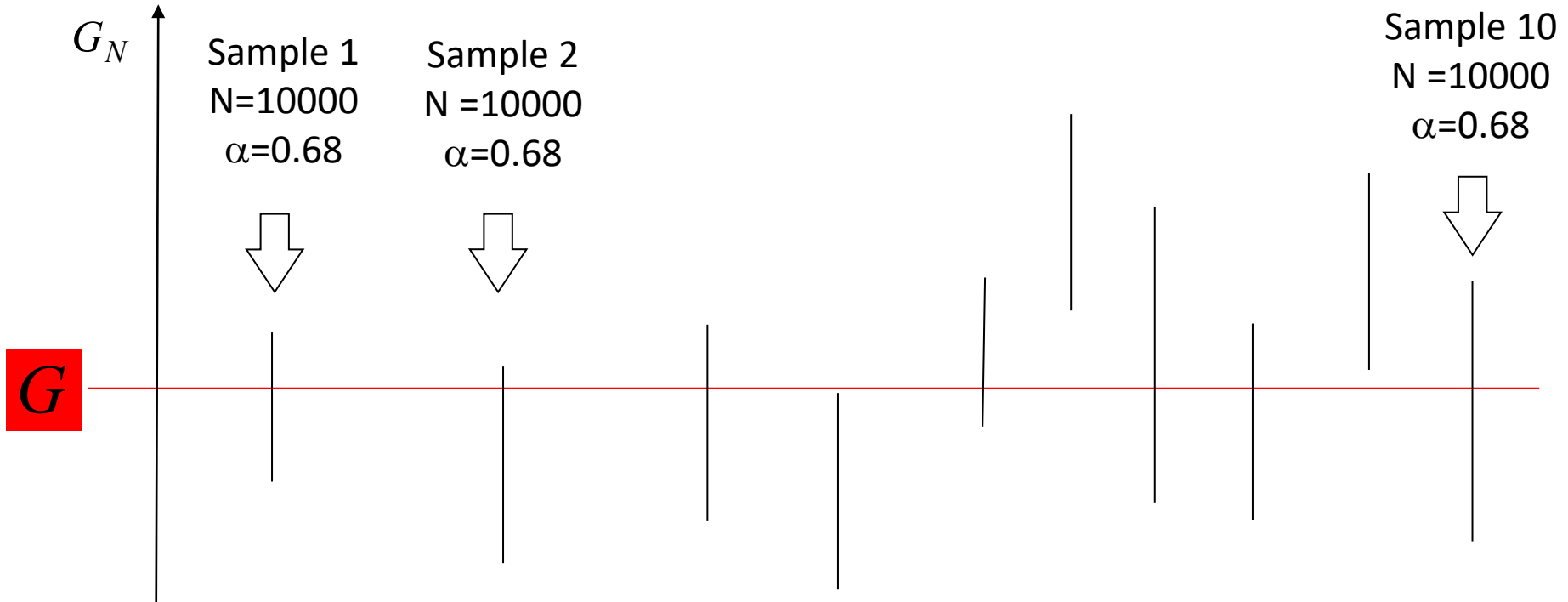
$$\text{Var}[G_N] \approx \frac{1}{N} \left( \overline{g^2} - G_N^2 \right) = 9.6 \cdot 10^{-6}$$

Estimated Variance!



$$G = 0.6342 \pm \sqrt{9.6 \cdot 10^{-6}} = 0.6342 \pm 0.0031$$

True value is 0.6366



unknown

$$P(G_N \in [G - \sigma_{G_N}; G + \sigma_{G_N}]) \approx 0.68$$

$$P(G \in [G_N - \sigma_{G_N}; G_N + \sigma_{G_N}]) \approx 0.68$$

- Why Monte Carlo instead of deterministic numerical integration?

Because the latter suffers from two major issues when dealing with highly multidimensional problems:

1. The number of function evaluations (grid) increases combinatorially with the number of dimensions
2. The boundaries of the multidimensional integration domain  $D$  become intractable

- The estimate  $G_N$  becomes more precise (less uncertain) as the estimator variance  $Var[G_N]$  decreases!
- How can we achieve lower  $Var[G_N] = \frac{1}{N} Var[g(x)]$ ?
  1. Increasing the number  $N$  of MC trials  $\Rightarrow$  “brute force”
  2. Decreasing  $Var[g(x)] \Rightarrow$  variance reduction techniques

$$G = \int_D \left[ \frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx \equiv \int_D g_1(x) f_1(x) dx$$

Forced (biased) MC simulation



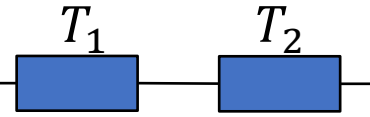
# Some Examples of MC Estimation of RAM quantities of Interest

$T = \text{System failure time}$

$T_i \approx f_{T_i}(t_i) = \text{Component failure time}$



$F_T(t) ???$



$T_1 \approx f_{T_1}(t_1)$

$T_2 \approx f_{T_2}(t_2)$

$$F_T(t_{miss}) = P\{T \leq t_{miss}\} = \int_0^{t_{miss}} f_T(t) dt = \int_0^{+\infty} I_g(t) f_T(t) dt$$

with

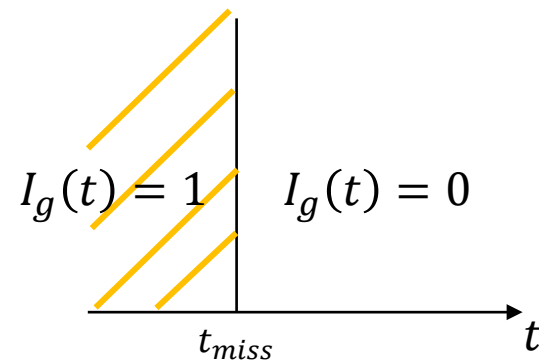
$$I_g(t) = \begin{cases} 1 & \text{if } t \leq t_{miss} \\ 0 & \text{otherwise} \end{cases}$$

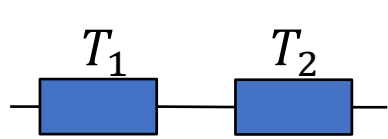
$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

$$G = F_T(t_{miss}) \quad g(x) = I_g(t) \quad f(x) = f_T(t)$$

$T$  is evaluated by means of MC simulation

$$F_T(t_{miss}) = P\{T \leq t_{miss}\} = \int_0^{t_{miss}} f_T(t) dt = \int_0^{+\infty} I_g(t) f_T(t) dt$$

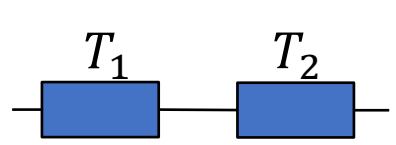




$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$
$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

$$t_{miss} = 8760 \text{ h}$$
$$\lambda_1 = 2 \cdot 10^{-4} \text{ h}^{-1}$$
$$\lambda_2 = 5 \cdot 10^{-3} \text{ h}^{-1}$$

OBJECTIVE:  
MC Estimate System Unreliability  
at the Mission Time



$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$

$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

$$t_{miss} = 8760 \text{ h}$$

$$\lambda_1 = 2 \cdot 10^{-4} \text{ h}^{-1}$$

$$\lambda_2 = 5 \cdot 10^{-3} \text{ h}^{-1}$$

OBJECTIVE:  
MC Estimate System Unreliability  
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$$N = 10000 \rightarrow \lambda_{sys} = \lambda_1 + \lambda_2 \rightarrow \text{for } i = 1, \dots, N \rightarrow t_i = -\frac{1}{\lambda_{sys}} \ln(1 - r_i), r_i \rightarrow U[0,1)$$

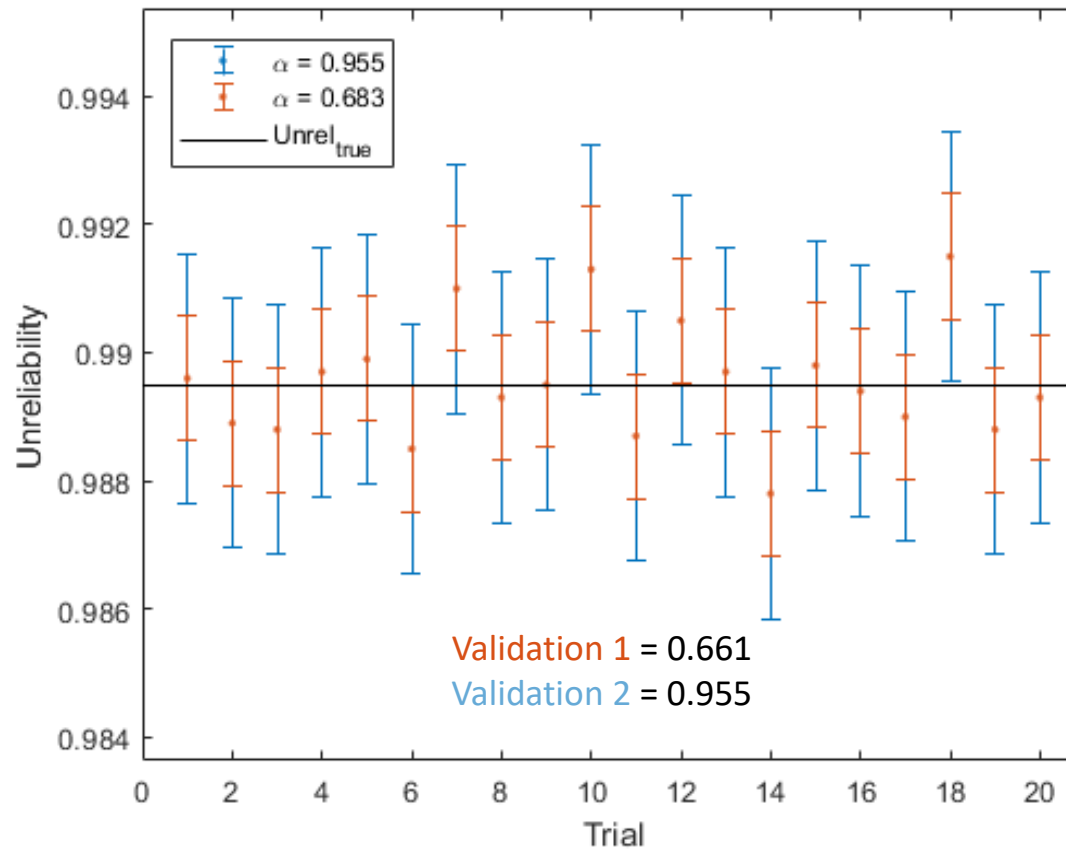
$$F_N(t_{miss}) = \frac{1}{N} \sum_{i=1}^N I_g(t_i) = 0,9891 \text{ where } I_g(t_i) = \begin{cases} 1 & \text{if } t_i < t_{miss} \\ 0 & \text{otherwise} \end{cases}$$

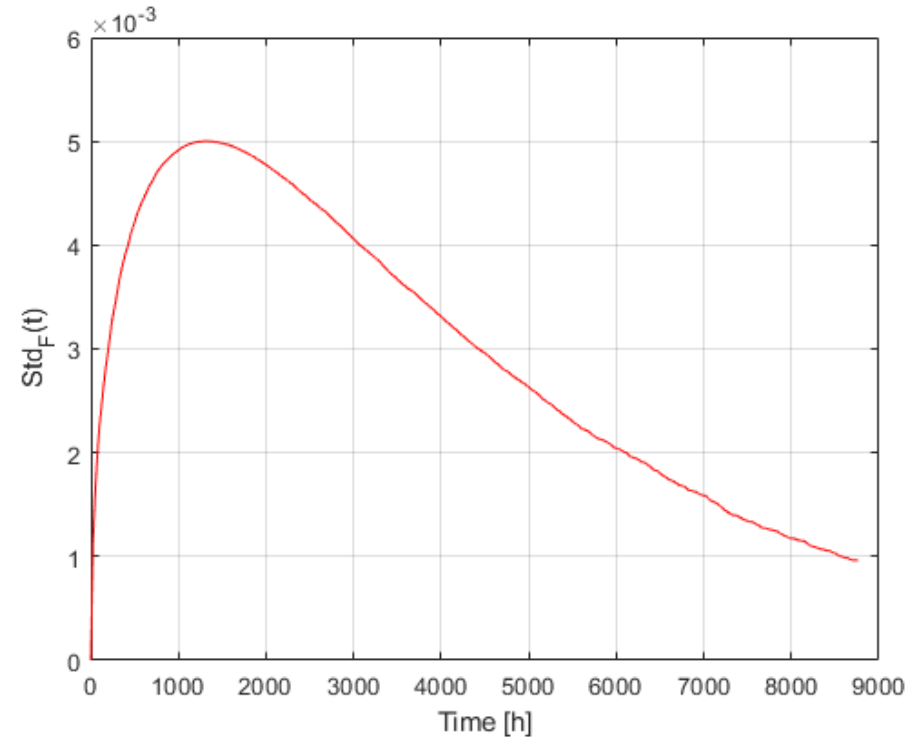
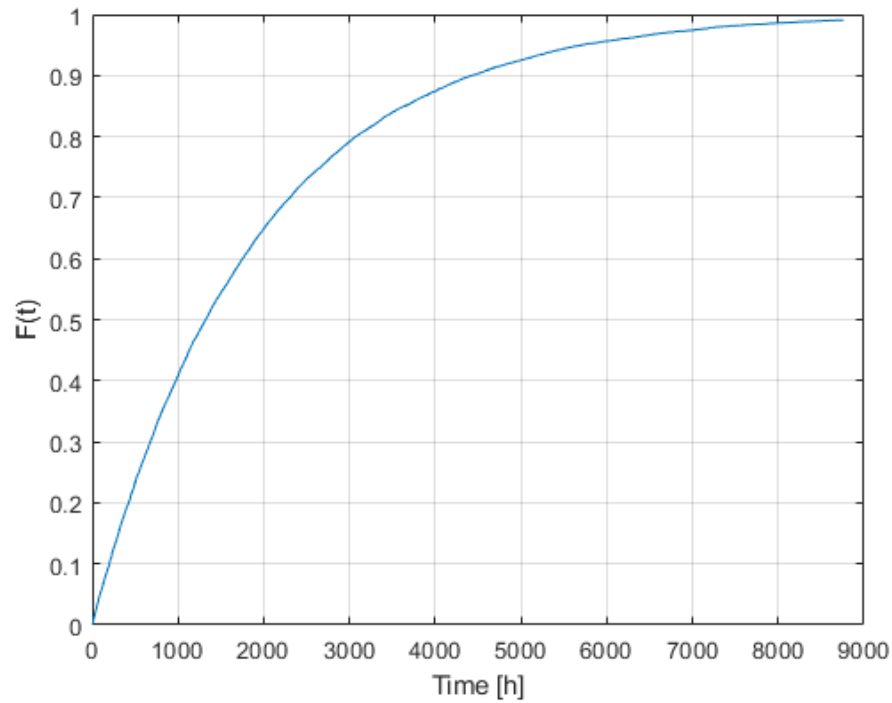
$$Var[F_N(t_{miss})] \approx \frac{1}{N} \left( \left( \frac{1}{N} \sum_{i=1}^N (I_g(t_i))^2 \right) - F_N^2(t_{miss}) \right) \approx \frac{1}{N} (F_N(t_{miss}) - F_N^2(t_{miss})) = 1,08 \cdot 10^{-6}$$

MC ESTIMATION OF SYSTEM UNRELIABILITY =  $F_N(t_{miss}) \pm \sqrt{Var[F_N(t_{miss})]}$  =  $0,9891 \pm 1,0 \cdot 10^{-3}$

TRUE VALUE OF SYSTEM UNRELIABILITY =  $1 - e^{-(\lambda_1 + \lambda_2)t_{miss}}$  = 0,9895

Repeating the system unreliability estimation 1000 times ...





$$MTTF = \int_0^{+\infty} t f_T(t) dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

$$G = MTTF \quad g(x) = t \quad f(x) = f_T(t)$$

Exponential failure time T



$$f_T(t) = \lambda e^{-\lambda t}$$

$$\lambda = 0,2 \text{ h}^{-1}$$

OBJECTIVE:  
MC Estimate System MTTF

$$MTTF = \int_0^{+\infty} t f_T(t) dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

$$G = MTTF \quad g(x) = t \quad f(x) = f_T(t)$$

Exponential failure time T



$$f_T(t) = \lambda e^{-\lambda t}$$

$$\lambda = 0,2 \text{ h}^{-1}$$

OBJECTIVE:  
MC Estimate System MTTF

Considering  $N = 10000$  trials:

$$MTTF_N = \frac{1}{N} \sum_{i=1}^N T_i = 4,98 \text{ h}$$

$$Var[MTTF_N] \approx \frac{1}{N} \left( \left( \frac{1}{N} \sum_{i=1}^N T_i^2 \right) - MTTF_N^2 \right) = 0,0024 \text{ h}$$

$$\text{MC ESTIMATION OF SYSTEM } MTTF = MTTF_N \pm \sqrt{Var[MTTF_N]} = 4,98 \pm 0,049$$

$$\text{TRUE VALUE OF SYSTEM } MTTF = \frac{1}{\lambda} = 5 \text{ h}$$



Repeating the system MTTF estimation 1000 times ...

