Monte Carlo Simulation for Reliability and Availability Analysis

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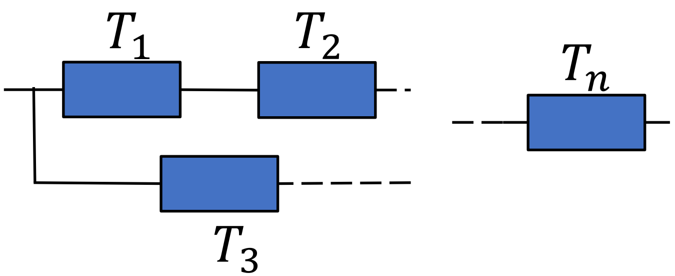
April 19th 2024

RAM quantities of interest are definite integrals



Some Examples:

1) Unreliability @mission time t_M:

$$F_T(t_{miss}) = P\{T \leq t_{miss}) = P\{q(T_1, \dots, T_n) \leq t_{miss}\} =$$

$$= P\{q(T_1, \dots, T_n) \leq t_{miss}\} = \int_{(t_1,\dots,t_n):q(t_1,\dots,t_n)\leq t_{miss}} f_{T_1,\dots,T_n}(t_1, \dots, t_n)\, dt_1 \dots dt_n$$

- T_i=Failure time of component i
- $\mathrm{T} = q(T_1, \dots, T_n)$ = System failure time

  Examples for q:
  - Series system → $q(T_1, \dots, T_n) = \min(T_1, \dots, T_n)$
  - Parallel system → $q(T_1, \dots, T_n) = \max(T_1, \dots, T_n)$

2) $MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$

MC Estimation of Definite Integrals

MC Evaluation of Definite Integrals (1D)

$$G = \int_a^b h(x)dx = \int_a^b g(x)\, f_X(x)dx$$

- X is a random variable with pdf $f_X(x)$:
- $g(x)$ is a random variable

$$\begin{cases} f_X(x) \geq 0 \\ \int_a^b f_X(x)\,dx = 1 \end{cases}$$

$$E[g(x)] = \int_a^b g(x)\, f_X(x)dx = G$$

MC Evaluation of Definite Integrals (1D)

Problem → Estimate $E[g(x)]$

Solution → Dart Game

1) for $i = 1,2, \dots, N$
   - Sample X_i from $f_X(x)$
   - Compute $g(X_i)$

   End

- the probability that a shot X_i hits $[x, x + dx]$ is *f*(*x*)dx
- the award is $g(X_i)$

2) Compute

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \bar{g}$$

Random variable!

$\boldsymbol{Var[G_N]}$ $\boldsymbol{E[G_N]}$

MC Evaluation of Definite Integrals (1D)

Problem → Estimate $E[g(x)]$

Solution → Dart Game

1) for $i = 1,2, \dots, N$
   - Sample X_i from $f_X(x)$
   - Compute $g(X_i)$

   End

- the probability that a shot X_i hits $[x, x + dx]$ is $f(x)dx$
- the award is $g(X_i)$

2) Compute

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \bar{g}$$

Random variable!

$\boldsymbol{Var[G_N]}$ $\boldsymbol{E[G_N]}$

Is G_N a good estimator of $E[g(x)]$?

MC Evaluation of Definite Integrals (1D)

Characteristics of a good estimator:

- Unbiased → $E[G_N] = G$
- Consistent → $\lim_{N \to +\infty} \mathrm{E}[(G_N - G)^2] = 0$

MC Evaluation of Definite Integrals (1D): Why G_N is a good estimator of G?

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(x_i)$$

⇓

G_N is a random variable with:

$$E[G_N] = E\left[\frac{1}{N}\sum_{i=1}^{N} g(X_i)\right] = \frac{1}{N}\sum_{i=1}^{N} E[g(X_i)] = \frac{1}{N}\sum_{i=1}^{N} E[g(x)] = G$$

$$Var[G_N] = Var\left[\frac{1}{N}\sum_{i=1}^{N} g(X_i)\right] = \frac{1}{N^2}\sum_{i=1}^{N} Var[g(X_i)] = \frac{1}{N} Var[g(x)]$$

⇓

G_N is an unbiased estimator of *G:* $\quad E[G_N] = G$

G_N is a consistent estimator of *G:* $\quad \lim_{N\to\infty} Var[G_N] = 0$

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx = \frac{2}{\pi} = 0.6366$$

How can we write the integral for MC estimation?

$f(x) = ?$

$g(x) = ?$

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx = \frac{2}{\pi} = 0.6366$$

By setting:

$f(x) = 1 \quad for \quad x \in [0,1]$ Continuous uniform distribution

$$g(x) = \cos\left(\frac{\pi}{2}x\right)$$

We perform N=10^4 trials:

$$X_i \to U[0,1)$$

$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$

$$\Rightarrow \quad G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \frac{1}{N}\sum_{i=1}^{N} cos\left(\frac{\pi}{2}X_i\right) = 0{,}6342$$

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2} x\right) dx$$

We perform N=10[4] trials:

$$x_i \to U[0,1)$$
$$g(x_i) = \cos\left(\frac{\pi}{2} X_i\right)$$
$$N = 10^4$$

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \frac{1}{N}\sum_{i=1}^{N} cos\left(\frac{\pi}{2} X_i\right) = 0{,}6342$$

$$Var[G_N] = \frac{1}{N} Var[g(x)] = \frac{1}{N}\left(E[g^2(x)] - \left(E[g(x)]\right)^2\right) = \frac{1}{N}\left(E[g^2(x)] - G^2\right)$$

Unknown in a practical case!

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

We perform N=10[4] trials:

$$x_i \rightarrow U[0,1)$$
$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$
$$N = 10^4$$

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \frac{1}{N}\sum_{i=1}^{N} cos\left(\frac{\pi}{2}X_i\right) = 0{,}6342$$

$$Var[G_N] = \frac{1}{N}Var[g(x)] = \frac{1}{N}\left(E[g^2(x)] - (E[g(x)])^2\right) = \frac{1}{N}\left(E[g^2(x)] - G^2\right)$$

Unknown in a practical case!

$$G = E[g(x)] = \frac{2}{\pi}$$

$$E[g^2(x)] = \int_0^1 \cos^2\left(\frac{\pi}{2}x\right)dx = \frac{1}{2}$$

$$Var[G_N] = \frac{1}{10^4}\left(\frac{1}{2} - \left(\frac{2}{\pi}\right)^2\right) = 9.47 \cdot 10^{-6}$$

MC Evaluation of Definite Integrals (1D) Example

$$G = \int_0^1 \cos\left(\frac{\pi}{2}x\right)dx$$

We perform $N=10^4$ trials:

$$x_i \to U[0,1)$$
$$g(x_i) = \cos\left(\frac{\pi}{2}X_i\right)$$
$$N = 10^4$$

$$G_N = \frac{1}{N}\sum_{i=1}^{N} g(X_i) = \frac{1}{N}\sum_{i=1}^{N} \cos\left(\frac{\pi}{2}X_i\right) = 0.6342$$

$$Var[G_N] = \frac{1}{N}Var[g(x)] = \frac{1}{N}\left(E[g^2(x)] - (E[g(x)])^2\right) = \frac{1}{N}\left(E[g^2(x)] - G^2\right)$$

$$G \approx G_N = 0.6342$$

$$E[g^2(x)] \approx \frac{1}{N}\sum_{i=1}^{N} g^2(X_i)$$

They can be computed during the MC simulation!

$$Var[G_N] \approx \frac{1}{N}\left(\overline{g^2} - G_N{}^2\right) = 9.6 \cdot 10^{-6}$$

Estimated Variance!

$$G = 0.6342 \pm \sqrt{9.6 \cdot 10^{-6}} = 0.6342 \pm 0.0031$$

True value is 0.6366

Definite Integral – Monte Carlo Vs Deterministic Numerical Integration

- Why Monte Carlo instead of deterministic numerical integration?

  Because the latter suffers from two major issues when dealing with highly multidimensional problems:

  1. The number of function evaluations (grid) increases combinatorially with the number of dimensions

  2. The boundaries of the multidimensional integration domain *D* become intractable

Estimation error – variance reduction

- The estimate G_N becomes more precise (less uncertain) as the estimator variance $Var[G_N]$ decreases!

- How can we achieve lower $Var[G_N] = \frac{1}{N} Var[g(x)]$?

  1. Increasing the number *N* of MC trials ⇒ "brute force"
  2. Decreasing $Var[g(x)]$⇒ variance reduction techniques

$$G = \int_D \left[\frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx \equiv \int_D g_1(x) f_1(x) dx$$

Forced (biased) MC simulation

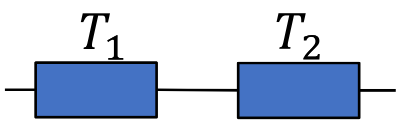
Some Examples of MC Estimation of RAM quantities of Interest

Unreliability estimation example

T = System failure time

$T_i \approx f_{T_i}(t_i)$ = Component failure time

$F_T(t)$???



$T_1 \approx f_{T_1}(t_1)$
$T_2 \approx f_{T_2}(t_2)$

$$F_T(t_{miss}) = P\{T \le t_{miss}) = $$
$$= \int_0^{t_{miss}} f_T(t)dt = \int_0^{+\infty} I_g(t) f_T(t) dt$$

with

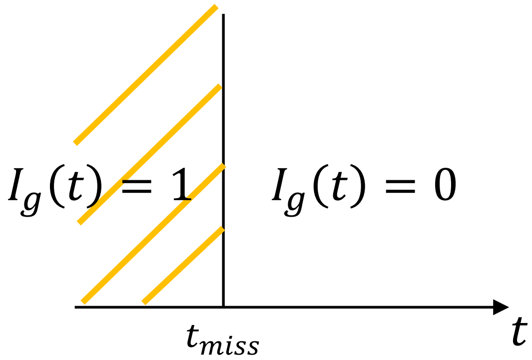
$$I_g(t) = \begin{cases} 1 & if\ t \le t_{miss} \\ 0 & otherwise \end{cases}$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)]$$

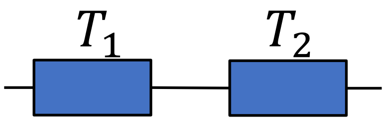
$G = F_T(t_{miss})$ $g(x) = I_g(t)$ $f(x) = f_T(t)$

T is evaluated by means of MC simulation

$$F_T(t_{miss}) = P\{T \le t_{miss}) = $$
$$= \int_0^{t_{miss}} f_T(t)dt = \int_0^{+\infty} I_g(t) f_T(t) dt$$



Unreliability estimation example

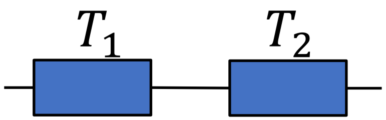


$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$
$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

t_{miss} = 8760 h
λ_1 = 2 10^{-4} h^{-1}
λ_2 = 5 10^{-3} h^{-1}

OBJECTIVE:
MC Estimate System Unreliability at the Mission Time

Unreliability estimation example



$$f_{T_1}(t) = \lambda_1 e^{-\lambda_1 t}$$
$$f_{T_2}(t) = \lambda_2 e^{-\lambda_2 t}$$

t_{miss} = 8760 h
λ_1 = 2 10^{-4}h^{-1}
λ_2 = 5 10^{-3} h^{-1}

OBJECTIVE:
MC Estimate System Unreliability at the Mission Time

N = 10000 → $\lambda_{sys} = \lambda_1 + \lambda_2 \rightarrow for\ i = 1, \ldots, N \rightarrow t_i = -\frac{1}{\lambda_{sys}} \ln(1 - r_i), r_i \rightarrow U[0,1)$

$$F_N(t_{miss}) = \frac{1}{N}\sum_{i=1}^{N} I_g(t_i) = 0{,}9891 \quad where \quad I_g(t_i) = \begin{cases} 1 & if\ t_i < t_{miss} \\ 0 & otherwise \end{cases}$$

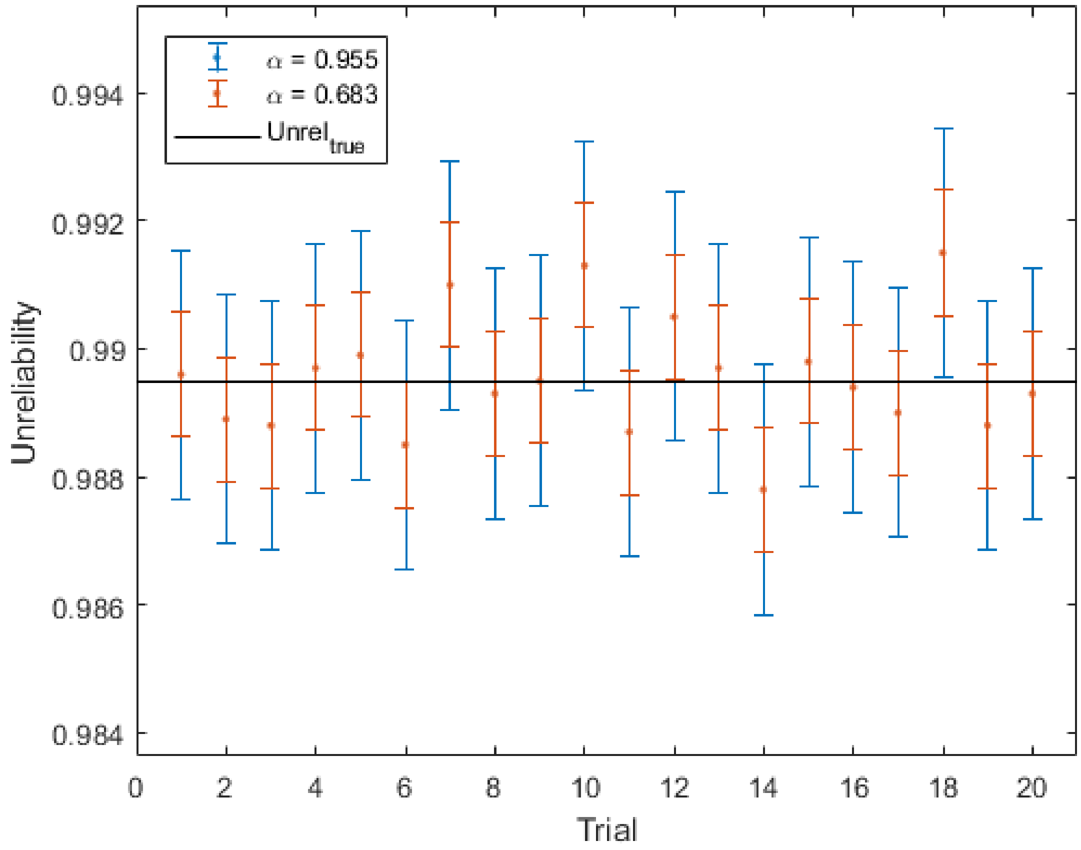
$$Var[F_N(t_{miss})] \approx \frac{1}{N}\left(\left(\frac{1}{N}\sum_{i=1}^{N}\left(I_g(t_i)\right)^2\right) - F_N^2(t_{miss})\right) \approx \frac{1}{N}\left(F_N(t_{miss}) - F_N^2(t_{miss})\right) = 1{,}08\ 10^{-6}$$

MC ESTIMATION OF SYSTEM UNRELIABILITY =$F_N(t_{miss}) \pm \sqrt{Var[F_N(t_{miss})]} = 0{,}9891 \pm 1{,}0\ 10^{-3}$

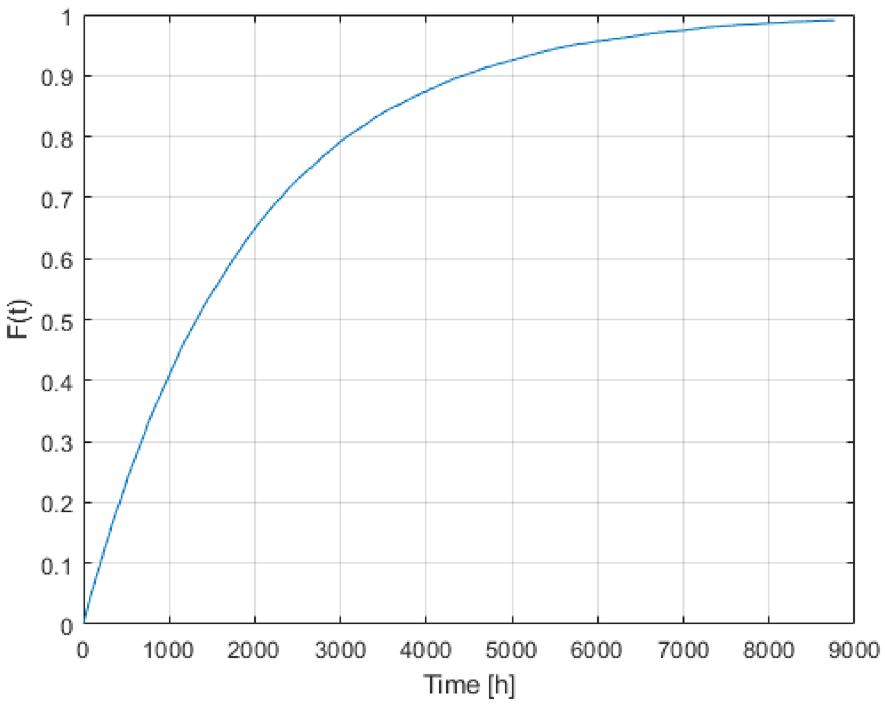
TRUE VALUE OF SYSTEM UNRELIABILITY = $1 - e^{-(\lambda_1+\lambda_2)t_{miss}} = 0{,}9895$

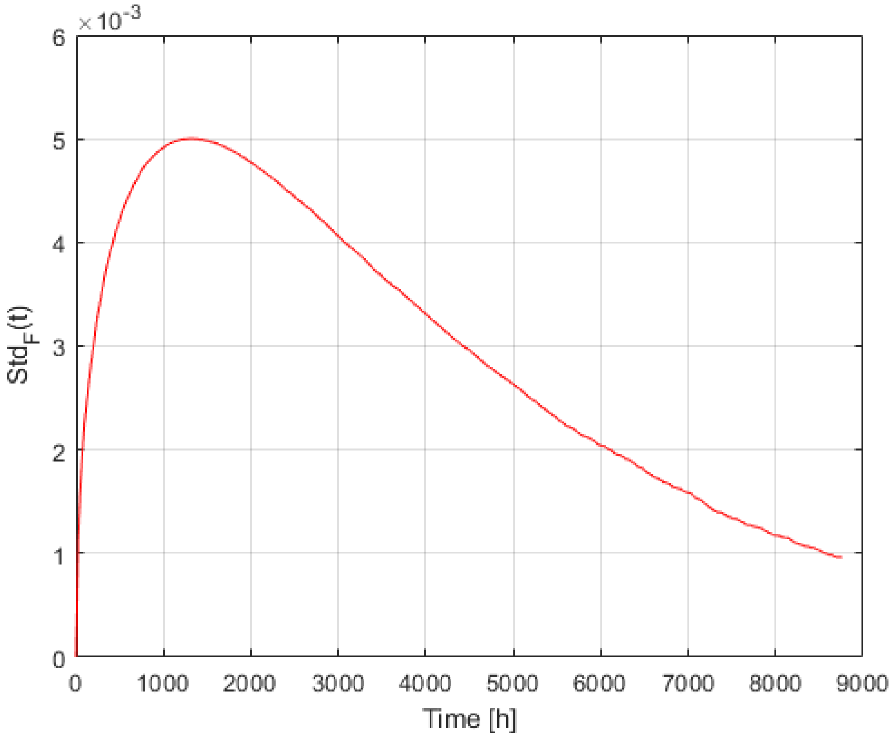
Unreliability estimation example

Repeating the system unreliability estimation 1000 times …



Unreliability estimation (time evolution)





MTTF estimation example

$$MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$$

$G = \int_a^b g(x) f(x) dx = E[g(x)]$ $\quad$ $G = MTTF$ $\quad$ $g(x) = t$ $\quad$ $f(x) = f_T(t)$

Exponential failure time T

$f_T(t) = \lambda e^{-\lambda t}$ $\quad$ λ = 0,2 h^{-1}

OBJECTIVE:
MC Estimate System MTTF

MTTF estimation example

$$MTTF = \int_0^{+\infty} t\, f_T(t)\, dt$$

$$G = \int_a^b g(x) f(x) dx = E[g(x)] \qquad G = MTTF \qquad g(x) = t \qquad f(x) = f_T(t)$$

Exponential failure time T

$f_T(t) = \lambda e^{-\lambda t}$ $\qquad$ λ = 0,2 h^{-1}

OBJECTIVE:
MC Estimate System MTTF

Considering $N = 10000$ trials:

$$MTTF_N = \frac{1}{N} \sum_{i=1}^{N} T_i = 4{,}98\ h$$

$$Var[MTTF_N] \approx \frac{1}{N} \left(\left(\frac{1}{N} \sum_{i=1}^{N} T_i^2 \right) - MTTF_N^2 \right) = 0{,}0024\ h$$

MC ESTIMATION OF SYSTEM $MTTF = MTTF_N \pm \sqrt{Var[MTTF_N]} = 4{,}98 \pm 0{,}049$

TRUE VALUE OF SYSTEM $MTTF = \frac{1}{\lambda} = 5\ h$

MTTF estimation example

Repeating the system MTTF estimation 1000 times …