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# Markov Reliability and Availability Analysis Part I: Discrete-Time Discrete State Markov Processes

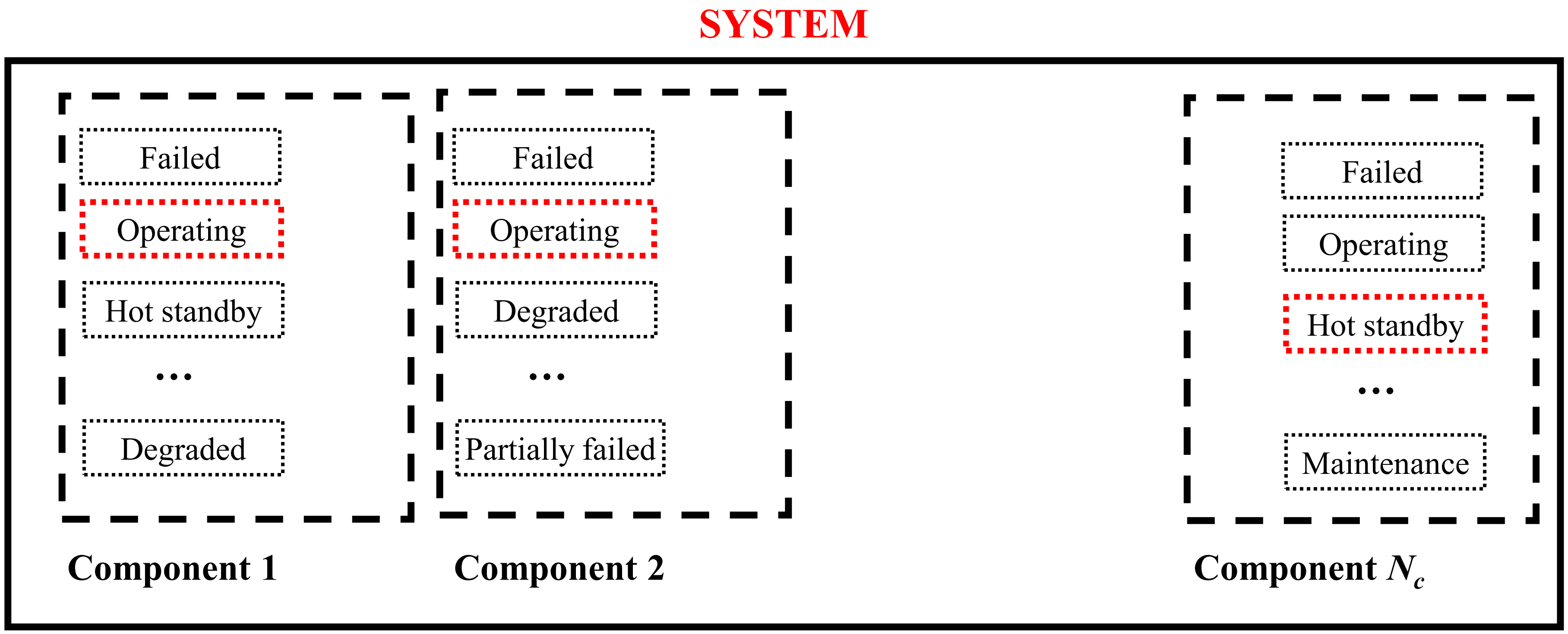
Ibrahim Ahmed

ibrahim.ahmed@polimi.it

# General Framework

# General Framework

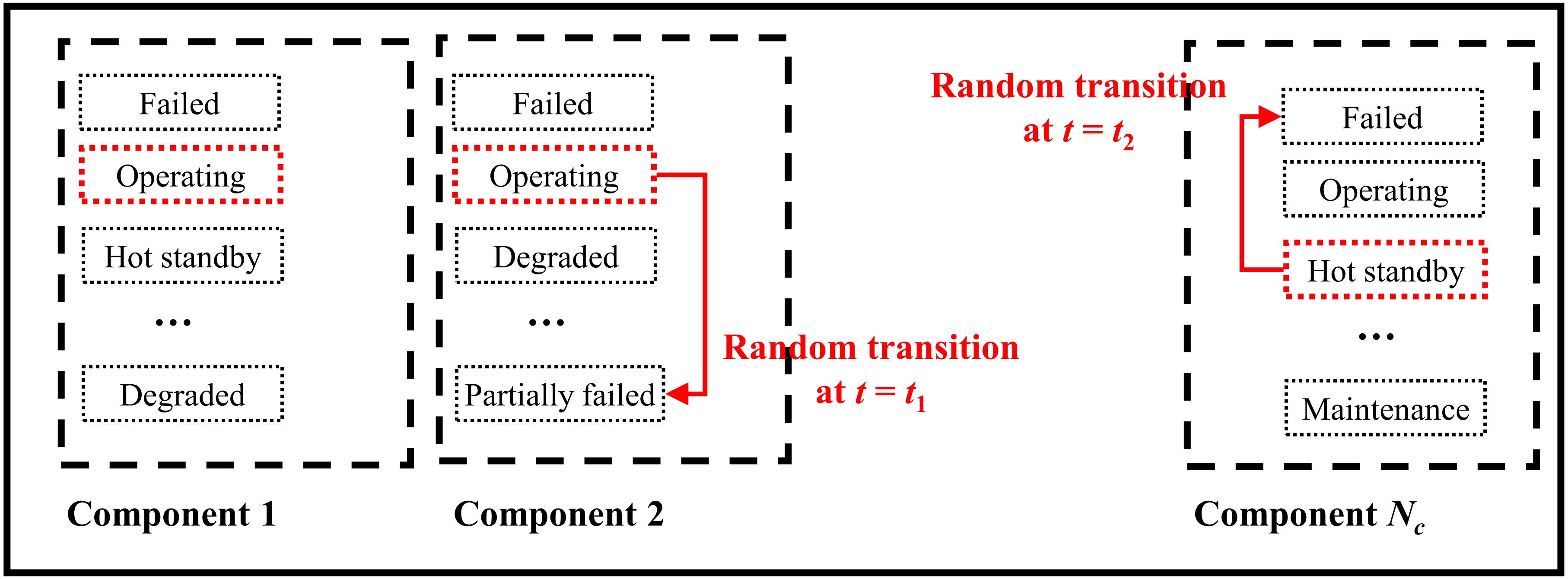




# General Framework



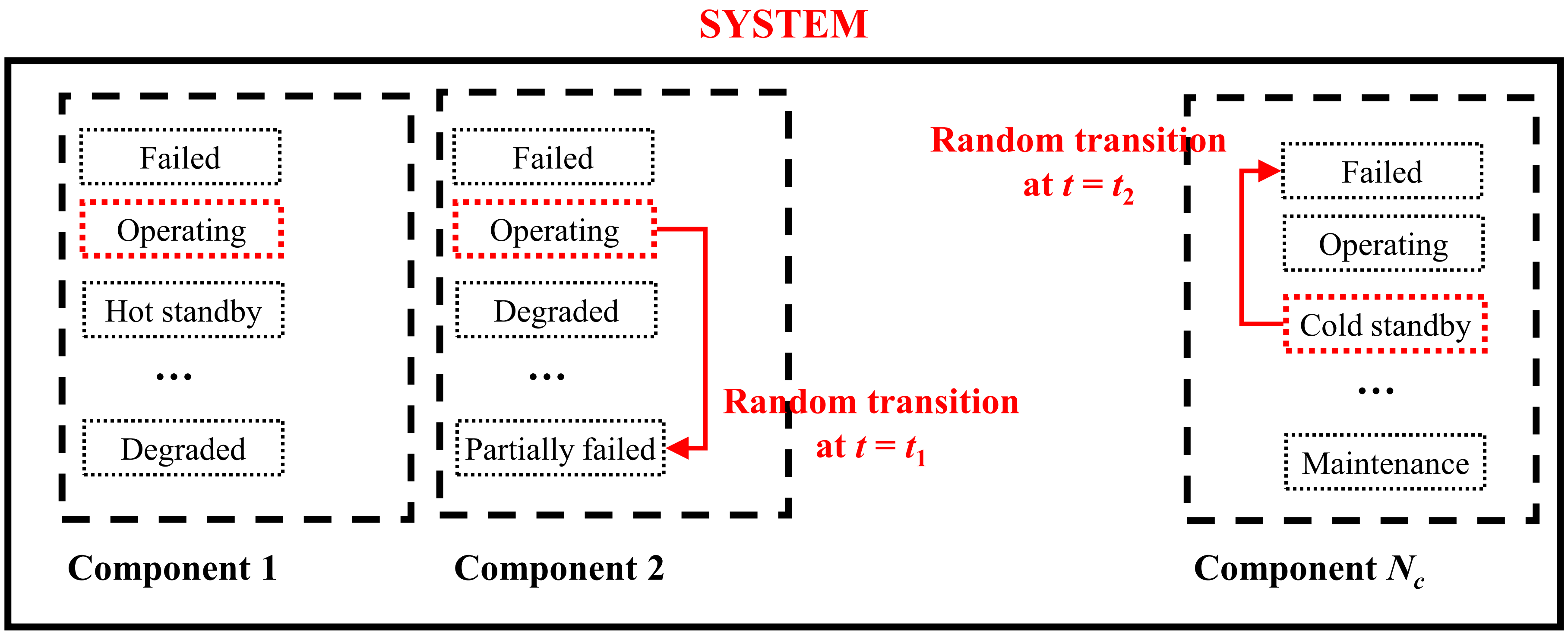
SYSTEM




**System evolution = Stochastic process**

# General Framework






Under specified conditions:

**System evolution = Stochastic process**

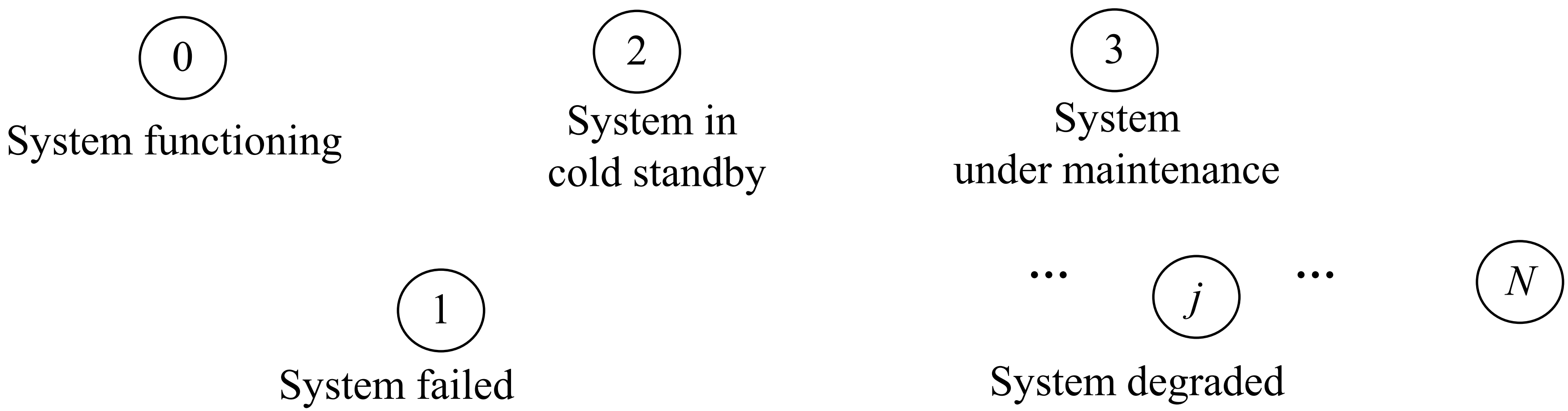
**=**

**MARKOV PROCESS**

# Markov Processes: Basic Elements

# Markov Processes: the System States (1)

- The **system** can occupy a **finite** or **countably infinite** number $N$ of states




**Set of possible states** $U = \{0, 1, 2, \ldots, N\}$

=

**State-space** of the random process

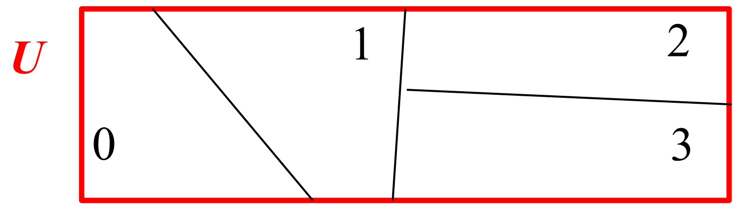
- The **States** are:
  - **Mutually Exclusive:** $P(\text{State} = i \cap \text{State} = j) = 0, \text{ if } i \neq j$

    (the system can be **only** in **one** state *at **each** time*)
  - **Exhaustive:** $P(U) = P\left(\bigcup_{i=1}^{N} \text{State} = i\right) = \sum_{i=1}^{N} P(\text{State} = i) = 1$

    (the system must be in **one** state *at **all** time*s

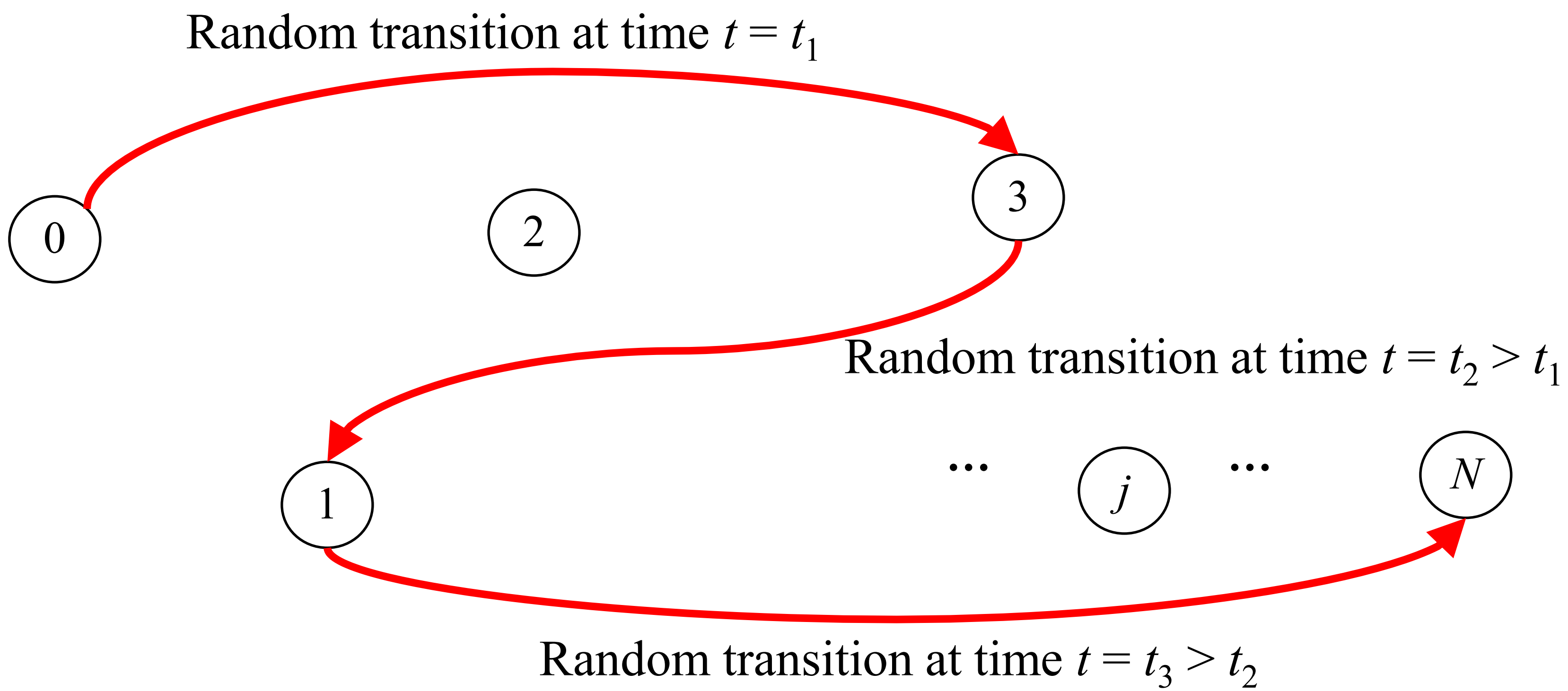
- **Example:**

**Set of possible states** $U = \{0, 1, 2, 3\}$




$$P(U) = P(\text{State} = 0 \cup \text{State} = 1 \cup \text{State} = 2 \cup \text{State} = 3)$$
$$= P(\text{State} = 0) + P(\text{State} = 1) + P(\text{State} = 2) + P(\text{State} = 3) = 1$$

- **Transitions** from one state to another occur **stochastically** (i.e., **randomly in time** and **in final transition state**)



# Markov Processes: Mathematical Representation

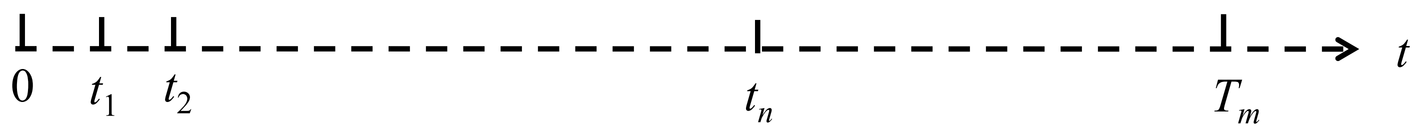


- The system state in **time** can be described by an **integer random variable** $X(t)$

  $X(t) = 5$ → the system occupies the **state** labelled by **number 5** at time $t$

- The **stochastic process** may be **observed** at:

  - Discrete times → **DISCRETE-TIME DISCRETE-STATE MARKOV CHAIN**



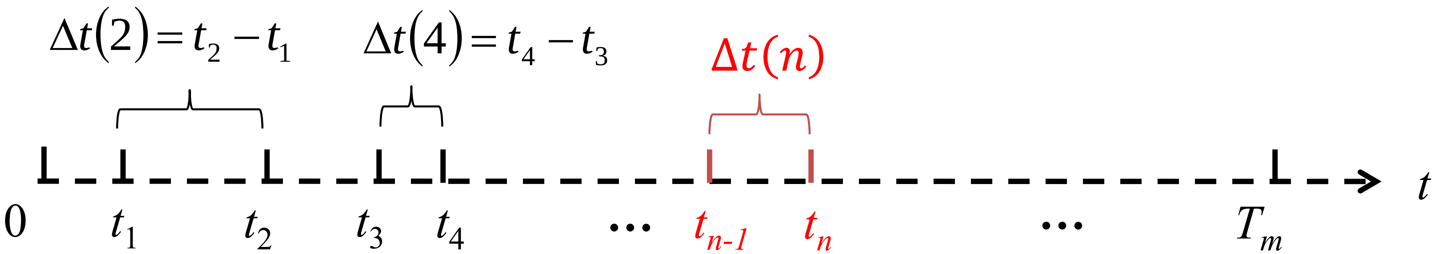

  - Continuously → **CONTINUOUS-TIME DISCRETE-STATE MARKOV PROCESS**

# Discrete-Time Markov Processes

# The Conceptual Model: Discrete Observation Times



- The stochastic process is **observed** at **discrete** times

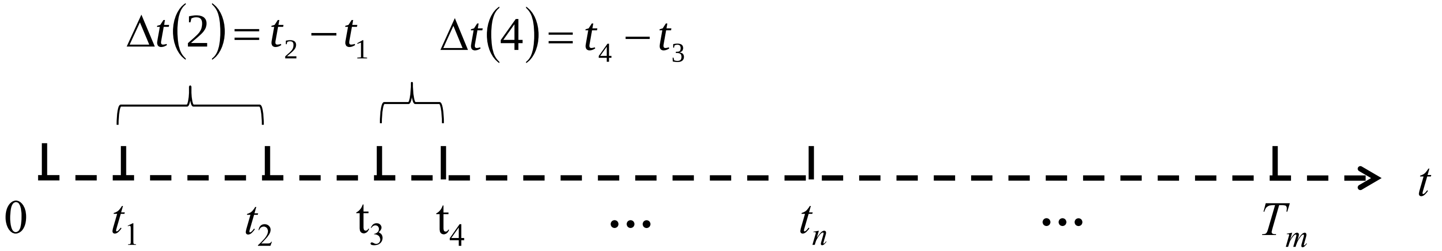



$$t_n = t_{n-1} + \Delta t(n)$$

# The Conceptual Model: Discrete Observation Times

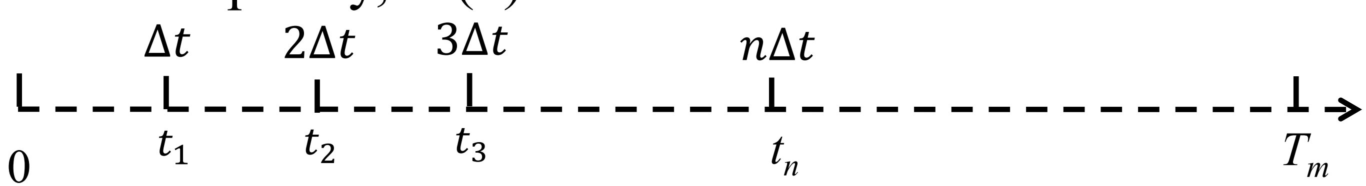


- The stochastic process is **observed** at **discrete** times




$$t_n = t_{n-1} + \Delta t(n)$$

- **Hypotheses**:
  - The time interval $\Delta t(n)$ is **small** enough such that **only one** event (i.e., stochastic transition) can occur within it
  - For simplicity, $\Delta t(n) = \Delta t =$ constant



# The Conceptual Model: Mathematical Representation



- The random process of system transition in time is described by an **integer random variable** $X(\cdot)$

- $X(n) \coloneqq$ **system state** at time $t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time $t_3$

# The Conceptual Model: Objective

- The random process of system transition in time is described by an **integer random variable** $X(\cdot)$
- $X(n) \coloneqq$ **system state** at time $t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time $t_3$

**OBJECTIVE:**

**Compute the probability that the system is in a given state at a given time, for all possible states and times**

$$P[X(n) = j], n = 1, 2, \ldots, N_{time}, \; j = 0, 1, \ldots, N$$

**Objective:**

$$P\left[X(n)=j\right], n=1,2,\ldots,N_{time}, \; j=0,1,\ldots,N$$

**What do we need?**

**Objective:**

$$P\left[X\left(n\right)=j\right], n=1,2,\ldots,N_{time}, j=0,1,\ldots,N$$

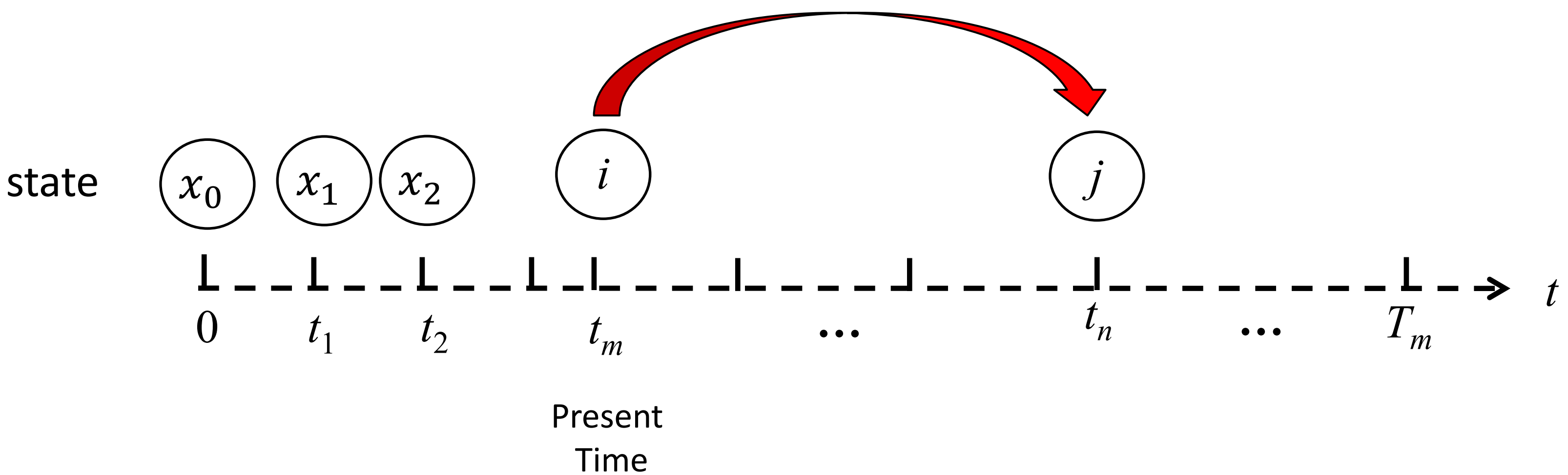
**What do we need?**

Transition Probabilities!

# The Conceptual Model: the Transition Probabilities

- **Transition probability:** conditional probability that the system moves to state $j$ at time $t_n$ given that it is in state $i$ at current time $t_m$ and given the previous system history

$$P[X(n) = j | X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots, X(m) = x_m = i]$$
$$\forall j = 0,1,\dots,N$$



# The Conceptual Model: the Markov Assumption

## In general for stochastic processes:

- the **probability** of a transition to a **future** state depends on its **entire life history**

$$P[X(n) = j | X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots, X(m) = x_m = i]$$

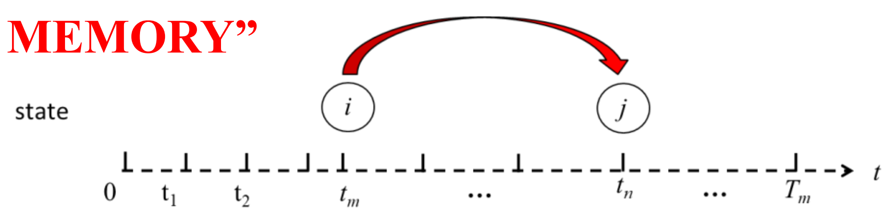
## In Markov Processes:

- the **probability** of a transition to a **future** state **only** depends on its **present state**

$$P[X(n) = j | \cancel{X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots}, X_m = x_m = i]$$

$$=$$

**THE PROCESS HAS "NO MEMORY"**



# The Conceptual Model: the Markov Assumption - Notation



$$p_{ij}(m,n) = P[X(n) = j|X(m) = i] \qquad n > m \geq 0$$

1. Transition probabilities $p_{ij}(m, n)$ are **larger than or equal to 0**

$$p_{ij}(m,n) \geq 0, \quad n > m \geq 0$$

$i = 0,1, 2, \ldots, N, j = 0, 1, 2, \ldots, N$

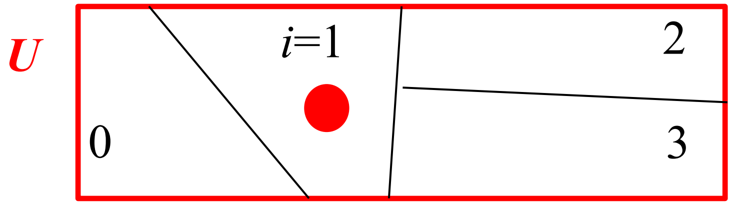
(**definition** of **probability**)

2. Transition probabilities **must sum to 1**

$$\sum_{all\ j} p_{ij}(m,n) = \sum_{j=0}^{N} p_{ij}(m,n) = 1, n > m \geq 0$$

$i = 0,1, 2, \ldots, N$

(the **set of states** is **exhaustive**)




$$\sum_{j=0}^{3} p_{1j}(m,n) = 1, n > m \geq 0$$

Starting from $i = 1$, the system either **remains in $i = 1$** or it goes **somewhere else, i.e.,** to **$j = 0$ or 2 or 3**

3. $$p_{ij}(m,n)=\sum_k p_{ik}(m,r)p_{kj}(r,n) \quad i=0,1,2,\ldots,N,\ j=0,1,2,\ldots,N$$

$$p[X(n)=j, X(m)=i]=\sum_k p[X(n)=j, X(r)=k, X(m)=i]$$ **(theorem of total probability)**

↓ **conditional probability**

$$=\sum_k p[X(n)=j \mid X(r)=k, X(m)=i]P[X(r)=k, X(m)=i]$$

↓ **Markov assumption**

$$=\sum_k p[X(n)=j \mid X(r)=k]P[X(r)=k, X(m)=i]$$

$$p_{ij}(m,n)=P[X(n)=j \mid X(m)=i]=\frac{P[X(n)=j, X(m)=i]}{P[X(m)=i]}$$ **(conditional probability)**

↓ **formula above**

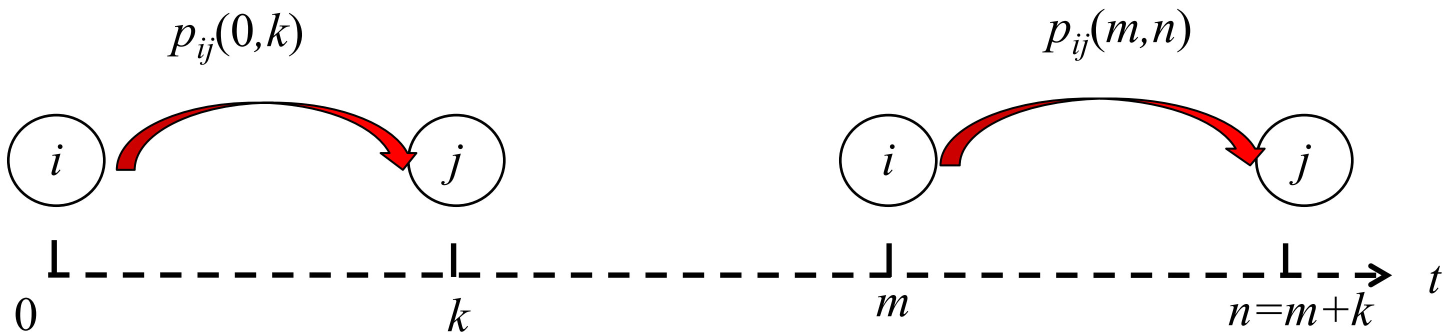
$$=\sum_k p[X(n)=j \mid X(r)=k]\frac{P[X(r)=k, X(m)=i]}{P[X(m)=i]}$$

↓ **conditional probability**

$$=\sum_k P[X(n)=j \mid X(r)=k]P[X(r)=k \mid X(m)=i]=\sum_k p_{kj}(r,n)p_{ik}(m,r)$$

# The Conceptual Model: Stationary Transition Probabilities





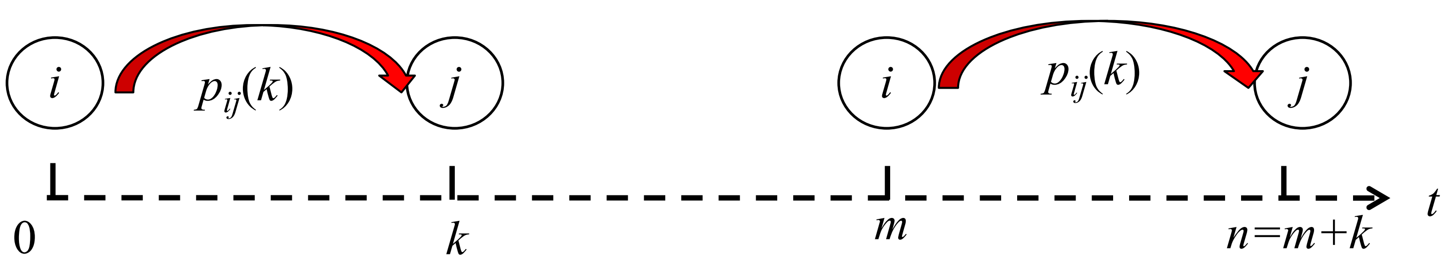

- If the **transition probability** $p_{ij}(m, n)$ depends on the **interval** $(t_n - t_m)$ and **not** on the **individual times** $t_m$ and $t_n$, then
  - the **transition probabilities** are **stationary**
  - the **Markov process** is **homogeneous in time**

# The Conceptual Model: Stationary Transition Probabilities



- If the **transition probability** $p_{ij}(m, n)$ depends on the **interval** $(t_n - t_m)$ and **not** on the **individual time** $t_m$ then:
  - the **transition probabilities** are **stationary**
  - the **Markov process** is **homogeneous in time**

$$p_{ij}(m,n) = p_{ij}\big(m, m + \overbrace{(n-m)}^{\textbf{\textit{k} time steps}}\big) = p_{ij}(m, m+k) = P\left[X(m+k) = j \mid X(m) = i\right]$$
$$= P\left[X(k) = j \mid X(0) = i\right]$$
$$= p_{ij}(k), \;\; k \geq 0 \quad i = 0, 1, 2, \ldots, N, \; j = 0, 1, 2, \ldots, N$$



- We know:
  - The one-step transition probabilities: $p_{ij}(1) = \boxed{p_{ij}}$

    $(i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N)$
  - The state probabilities at time $n = 0$ (initial condition):

    $$\boxed{c_j} = P[X(0) = j]$$

- Objective:
  - Compute the probability that the system is in a given state $j$ at a given time $t_n$, for all possible states and times

$$P[X(n) = j] = \boxed{P_j(n)},\ n = 1, 2, \ldots, N_{time},\ j = 0, 1, \ldots, N$$

# The Conceptual Model: Notation - the Transition Probability Matrix



$$
\underline{\underline{A}} = \begin{array}{c} i/j \\ 0 \\ 1 \\ \dots \\ N \end{array}
\begin{array}{c}
\begin{array}{cccc} 0 & 1 & \dots & N \end{array} \\
\begin{pmatrix}
p_{00} & p_{01} & \dots & p_{0N} \\
p_{10} & p_{11} & \dots & p_{1N} \\
\dots & \dots & \dots & \dots \\
p_{N0} & p_{N1} & \dots & p_{NN}
\end{pmatrix}
\end{array}
$$

**Properties**:

- $\dim\left(\underline{\underline{A}}\right) = (N+1)\times(N+1)$
- $0 \le p_{ij} \le 1, \forall i, j \in \{0,1,2,\dots,N\}$

(all elements are **probabilities**)

$$\underline{\underline{A}} = \begin{array}{c} i/j \\ 0 \; \Sigma \\ 1 \\ \cdots \\ N \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \ldots & N \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & \cdots & p_{0N} \\ p_{10} & p_{11} & \cdots & p_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ p_{N0} & p_{N1} & \cdots & p_{NN} \end{pmatrix} \end{array} \begin{array}{c} \\ =1 \\ \\ \\ \\ \end{array}$$

**Properties:**

- $\dim(\underline{\underline{A}}) = (N+1)\times(N+1)$
- $0 \leq p_{ij} \leq 1, \forall i, j \in \{0,1,2,\ldots,N\}$

(all elements are **probabilities**)

**only ($N$+1)x$N$ elements need to be known**

- $\sum_{j=0}^{N} p_{ij} = 1, i = 0,1,2,\ldots,N$

(the **set of states** is **exhaustive**)

$\underline{\underline{A}}$ **is a Stochastic Matrix**

# The Conceptual Model: Notation - Unconditional State Probabilities



- Introduce the row vector:

$\underline{P}(n) = \left[P_0(n) P_1(n) \ldots P_j(n) \ldots P_N(n)\right]$ = probabilities of the system being in state 0, 1, 2, …, $N$ at the $n$-th time step

- Initialize the vector $\underline{P}(n)$ at time step $n = 0$:

$$\underline{P}(0) = \underline{C} = \left[C_0\; C_1 \ldots C_j \ldots C_N\right]$$

## Computation of the Unconditional State Probabilities (1)



$$P_j(1) = P\left[X(1) = j\right]$$

↓ theorem of total probability

$$= \boxed{\sum_{i=0}^{N} \boxed{P\left[X(1) = j \middle| X(0) = i\right]} \cdot P\left[X(0) = i\right]}$$

↓ homogeneous process

$$= \sum_{i=0}^{N} \boxed{p_{ij}} C_i = p_{0j} \cdot C_0 + p_{1j} \cdot C_1 + p_{2j} \cdot C_2 + \ldots + p_{Nj} \cdot C_N,$$

$$with \ \ j = 0, 1, 2, \ldots, N$$

**Using Matrix Notation:**

$$\underline{P}(1) = \underline{C} \cdot \underline{\underline{A}}$$

# Computation of the Unconditional State Probabilities (2)

- At the second time step $n = 2$:

$$P_j(2) = P\left[X(2) = j\right]$$

↓ theorem of total probability + Markov assumption

$$= \sum_{k=0}^{N} P\left[X(2) = j \middle| X(1) = k\right] \cdot P\left[X(1) = k\right]$$

↓ homogeneous process

$$= \sum_{k=0}^{N} p_{kj} \cdot P_k(1)$$

$$= P_0(1) \cdot p_{0j} + P_1(1) \cdot p_{1j} + P_2(1) \cdot p_{2j} + \ldots + P_N(1) \cdot p_{Nj},$$

$$with \;\; j = 0, 1, 2, \ldots, N$$

$$\underline{P}(2) = \underline{P}(1) \cdot \underline{\underline{A}} = \left(\underline{C}\,\underline{\underline{A}}\right)\underline{\underline{A}} = \underline{C}\,\underline{\underline{A}}^2$$

**Proceeding in the same recursive way…**

**FUNDAMENTAL EQUATION OF THE HOMOGENEOUS DISCRETE-TIME DISCRETE-STATE MARKOV PROCESS**

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$$

# Problem Setting & Found Solution

- We know:
  - The one-step transition probabilities: $p_{ij}$
  - The initial condition $c_j = P[X(0) = j]$
- Objective:
  - Compute the probability that the system is in a given state $j$ at a given time $t_n$, for all possible states and times: $\underline{P}(n)$
- Solution:

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{A}^n = \underline{C} \cdot \underline{A}^n}$$

**FUNDAMENTAL EQUATION**

# Multi-step Transition Probabilities: Interpretation



**FUNDAMENTAL EQUATION** $\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$

$$\underline{\underline{A}}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \dots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \dots & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & \dots & p_{NN}(n) \end{pmatrix}$$

***n*-th step**
**transition probability matrix**

$$p_{ij}(n) = P[X(n) = j | X(0) = i]$$

**probability of arriving in state *j* after *n* steps**
**given that the initial state was *i***

**EXAMPLE WITH $N = 2$ STATES AND $n = 2$ time steps**

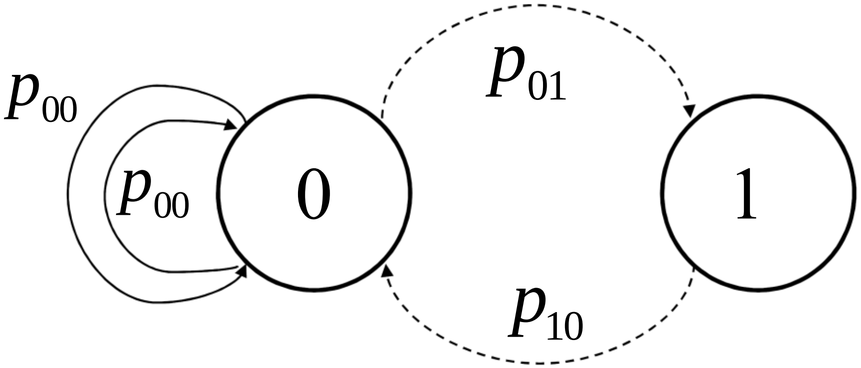
$$\underline{\underline{A}} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad (i = 0,1, j = 0, 1)$$

$$\underline{\underline{A}}^2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \cdot \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \boxed{p_{00} \cdot p_{00} + p_{01} \cdot p_{10}} & \boxed{p_{00} \cdot p_{01} + p_{01} \cdot p_{11}} \\ p_{10} \cdot p_{00} + p_{11} \cdot p_{10} & p_{10} \cdot p_{01} + p_{11} \cdot p_{11} \end{pmatrix}$$

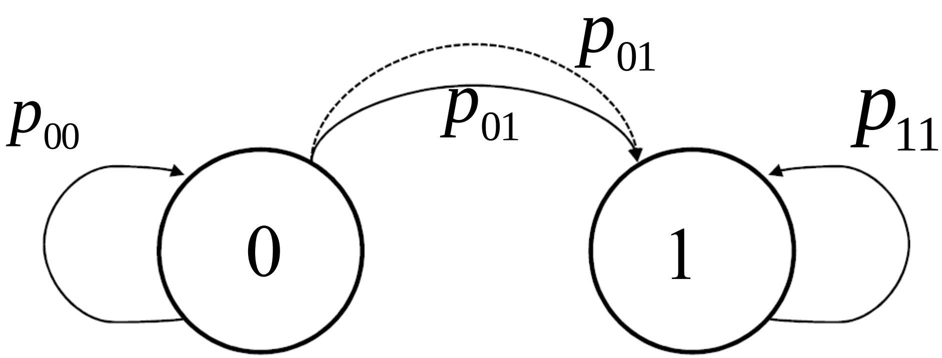
**WHAT IS THE "PHYSICAL" MEANING?**

# Multi-step Transition Probabilities (3)






$$p_{00}(2) = p_{00} \cdot p_{00} + p_{01} \cdot p_{10}$$




$$p_{01}(2) = p_{00} \cdot p_{01} + p_{01} \cdot p_{11}$$

$p_{ij}(n) = P[X(n) = j \mid X(0) = i]$ **, $p_{ij}(n)$ is the sum of the probabilities of all trajectories with length $n$ which originate in state $i$ and end in state $j$**

## Example 1: wet and dry days in a town



- Stochastic process of raining in a town (transitions between wet and dry days)

**DISCRETE STATES**

State 1: dry day

State 2: wet day

**DISCRETE TIME**

Time step = 1 day

**TRANSITION MATRIX**

$$\underline{\underline{A}} = \begin{array}{c c} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

*You are required to:*

1) Draw the Markov diagram
2) If today the weather is dry, what is the probability that it will be **dry two days from now**?

# Open Problems



- We provided an analytical framework for computing the state probabilities
- Still open issues:
  1. Estimate the transition matrix $A$ → Problem of parameter identification from data or expert knowledge
  2. Solve for a generic time $n$, i.e. find $P_j(n)$ as a function of $n$, without the need of multiplying $n$ times the matrix $A$

# Solution to the fundamental equation

# Solution to the Fundamental Equation (1)



$$\begin{cases} \underline{P}(n) = \underline{P}(0)\underline{\underline{A}}^n \\ \underline{P}(0) = \underline{C} \end{cases}$$

**SOLVE THE EIGENVALUE PROBLEM ASSOCIATED TO MATRIX A**

i) Set the **eigenvalue problem** $\underline{V} \cdot \underline{\underline{A}} = \omega \cdot \underline{V}$

ii) Write the **homogeneous form** $\underline{V} \cdot \left( \underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$

iii) Find **non-trivial solutions** by setting $\det\left( \underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$

iv) From $\det\left( \underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$ compute the **eigenvalues** $\omega_j,\ j = 0,1,\dots,N$

v) Set the ***N+1* eigenvalue problems** $\underline{V_j} \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V_j}$ $\quad j = 0,1,\dots,N$

vi) From $\underline{V_j} \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V_j}$ compute the **eigenvectors** $\underline{V_j},\ j = 0,1,\dots,N$

# Eigenvalues of a Stocastic Matrix



- $A$ is a stocastic matrix
- The Markov process is regular and Ergodic

$$\omega_0 = 1 \text{ and } |\omega_j| < 1, j = 1,2,\ldots,N$$

# Solution to the fundamental equation (2)



The **eigenvectors** $\underline{V}_j$ span the $(N+1)$-dimensional space and can be used as a **basis** to write **any** $(N+1)$-dimensional vector as a **linear combination** of them

$$\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j} \qquad \text{AND} \qquad \underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$$

WE NEED TO FIND THE COEFFICIENTS $\alpha_j$ AND $c_j, j = 0, 1, \ldots, N$

# Solution to the fundamental equation (3)



**FIND THE COEFFICIENTS** $c_j, j = 0,1,\ldots,N$ **FOR** $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$

**SOLVE THE ASSOCIATED ADJOINT EIGENVALUE PROBLEM**

i) Set the **adjoint eigenvalue problem**

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+$$

ii) Since for **real valued** matrices $\underline{\underline{A}}^+ = \underline{\underline{A}}^T$ then:

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+ \Rightarrow \underline{V}^+ \cdot \underline{\underline{A}}^T = \omega^+ \cdot \underline{V}^+$$

iii) Since the eigenvalues $\omega_j^+, j = 0,1,\ldots,N$ depend **only** on $\det\left(\underline{\underline{A}}^T\right) = \det\left(\underline{\underline{A}}\right)$

$$\Rightarrow \omega_j^+ = \omega_j, j = 0,1,\ldots,N$$

# Solution to the fundamental equation (4)

iv) From $\underline{V}_j^+ \cdot \underline{\underline{A}}^+ = \omega_j \cdot \underline{V}_j^+,\ j = 0,1,\ldots,N$ compute the adjoint eigenvectors

$$\underline{V}_j^+ \ ,\ j = 0,1,\ldots,N$$

v) By **definition** of the adjoint problem and since $\underline{V}_j^+$ and $\underline{V}_j$ are **orthogonal**

$$\Rightarrow \quad < \underline{V}_j^+, \underline{V}_i > \equiv \underline{V}_j^+ \cdot \underline{V}_i^T = \begin{cases} 0 \ \ if\ i \neq j \\ k\, otherwise \end{cases}$$

# Solution of the fundamental equation (4)



iv) From $\underline{V}_j^+ \cdot \underline{\underline{A}}^+ = \omega_j \cdot \underline{V}_j^+,\ j = 0,1,\ldots,N$ compute the adjoint eigenvectors

$$\underline{V}_j^+ \ ,\ j = 0,1,\ldots,N$$

v) By **definition** of the adjoint problem and since $\underline{V}_j^+$ and $\underline{V}_j$ are **orthogonal**

$$\Rightarrow \quad < \underline{V}_j^+, \underline{V}_i > \equiv \underline{V}_j^+ \cdot \underline{V}_i^T = \begin{cases} 0 \ \ if\ i \neq j \\ k\, otherwise \end{cases}$$

vi) Multiply the left- and right-hand sides of $\underline{C} = \sum_{i=0}^{N} c_i \underline{V}_i$ by $\underline{V}_j^+$

$$\Downarrow$$

$$\left\langle \underline{V}_j^+, \underline{C} \right\rangle = \sum_{i=0}^{N} c_i \left\langle \underline{V}_j^+, \underline{V}_i \right\rangle = c_j \left\langle \underline{V}_j^+, \underline{V}_j \right\rangle \rightarrow c_j = \frac{\left\langle \underline{V}_j^+, \underline{C} \right\rangle}{\left\langle \underline{V}_j^+, \underline{V}_j \right\rangle}$$

(orthogonality)

# Solution to the fundamental equation (5)



**FIND THE COEFFICIENTS** $\alpha_j,\ j = 0,1,\dots,N$ **FOR** $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$

USE $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$ , $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j}$ AND $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$

# Solution to the fundamental equation (5)



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i) Substitute $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ into $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$ to obtain $P(n) = \left( \sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

ii) Set $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \underline{C} \cdot \underline{\underline{A}}^n = \left( \sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

iii) Multiply $\underline{V}_j \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V}_j$ by $\underline{\underline{A}}$ to obtain $\underline{V}_j \cdot \underline{\underline{A}} \cdot \underline{\underline{A}} = \omega_j \cdot \boxed{\underline{V}_j \cdot \underline{\underline{A}}}$

Since $\boxed{\underline{V}_j \cdot \underline{\underline{A}}} = \omega_j \cdot \underline{V}_j$ then $\underline{V}_j \cdot \underline{\underline{A}}^2 = \omega_j \cdot \boxed{\omega_j \cdot \underline{V}_j} = \omega_j^2 \cdot \underline{V}_j$

••• (proceeding in the same recursive way)

$$\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$$

iv) Substitute $\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$ into $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j \quad = \underline{C} \cdot \underline{\underline{A}}^n = \sum_{j=0}^{N} c_j \cdot \underline{V}_j \underline{\underline{A}}^n$

$$\Rightarrow \quad \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \sum_{j=0}^{N} c_j \cdot \omega_j^{\,n} \cdot \underline{V}_j$$

$$\alpha_j = c_j \cdot \omega_j^{\,n}$$

# Example 2: wet and dry days in a town – HOMEWORK send your solution by Friday before 8:00 Correct solution: 0.714



- Stochastic process of raining in a town (transitions between wet and dry days)

**DISCRETE STATES**

State 1: dry day

State 2: wet day

**DISCRETE TIME**

Time step = 1 day

**TRANSITION MATRIX**

$$\underline{\underline{A}} = \begin{array}{c c} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

Today the weather is dry

*You are required to:*

1) Estimate the probability that it will be dry ***n*** days from now?

# Quantity of Interest

A Markov process is called ergodic if it is possible to eventually get from every state to every other state with positive probability

$$A = \begin{pmatrix} 0.8 & 0.2 \\ 0.50 & 0.5 \end{pmatrix}$$

*Ergodic*

$$A = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 1 \end{pmatrix}$$

*Non Ergodic*

A Markov process is said to be regular if some power of the stochastic matrix $A$ has all positive entries (i.e. strictly greater than zero).

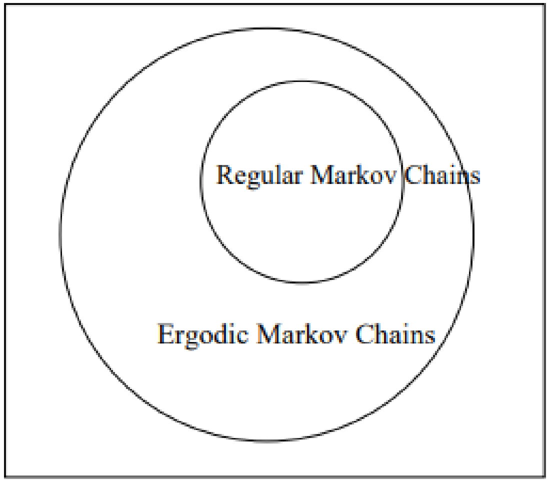
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = A^4 = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^5 = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*Ergodic – Non Regular*

Is it possible to make long-term predictions ($n \rightarrow +\infty$) of a Markov process?




It is possible to show that if the Markov process is regular then:

$$\lim_{n \rightarrow +\infty} \underline{P}(n) = \Pi$$

**Steady state probabilities**

- **Steady state probabilities $\pi_j$:** probability of the system being in state *j* **asymptotically**

- **TWO ALTERNATIVE APPROACHES:**

  1) Since $\omega_0 = 1$ and $|\omega_j| < 1,\ j = 1,2,\ldots,N$

  **AT STEADY STATE:** $\lim_{n\to\infty} \underline{P}(n) = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{\alpha_j} \cdot \underline{V_j} = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{c_j \cdot \omega_j^n} \cdot \underline{V_j} = c_0 \underline{V_0} = \underline{\Pi}$

# Steady state probabilities

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2) Use the recursive equation $\underline{P}(n) = \underline{P}(n-1) \cdot \underline{\underline{A}}$

**AT STEADY STATE:** $\underline{P}(n) = \underline{P}(n-1) = \underline{\Pi}$

**SOLVE** $\underline{\Pi} = \underline{\Pi} \cdot \underline{\underline{A}}$ subject to $\sum_{j=0}^{N} \Pi_j = 1$

# Example 3: wet and dry days in a town (continue)



$$\underline{\underline{A}} = \begin{array}{c} \\ dry \\ wet \end{array} \begin{array}{c} \begin{array}{cc} dry & wet \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array} \qquad \underline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- *Question*: what is the probability that **one year from now** the day will be **dry**?
  - ❑ Use the approximation based on the recursive equation

- **FIRST PASSAGE PROBABILITY AFTER *n* TIME STEPS:**

Probability that the system arrives **for the first time** in state *j* **after *n* steps**, given that it was in state *i* at the initial time 0

$$f_{ij}(n) = P\left[X(n) = j \;\; \textit{for the first time} \middle| X(0) = i\right]$$

=

$$f_{ij}(n) = P\left[X(n) = j, X(m) \neq j, 0 < m < n \mid X(0) = i\right]$$

**NOTICE:**

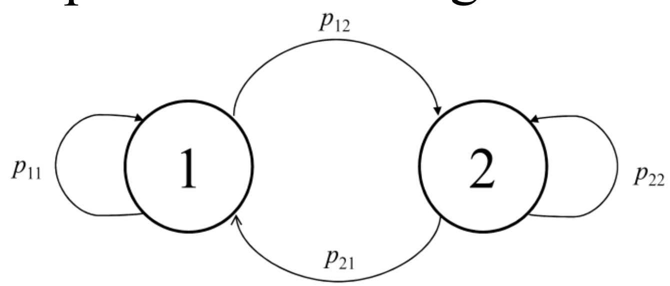
$$f_{ij}(n) \neq p_{ij}(n)$$

$p_{ij}(n)$ =probability that the system reaches state *j* **after *n* steps** starting from state *i*, but **not necessarily for the first time**

# Example 4: First Passage Probabilities

Compute for the markov process in the Figure below:

- $f_{11}(1)$
- $f_{11}(n)$
- $f_{12}(n)$




- Probability of going from state 1 to state 1 in 1 step for the first time

$$f_{11}(1) = ?$$

- Probability that the system, starting from state 1, will return to the same state 1 for the first time after *n* steps

$$f_{11}(n) = ?$$

- Probability that the system will arrive for the first time in state 2 after *n* steps

$$f_{12}(n) = ?$$

# First Passage Probabilities (4)



- RELATIONSHIP WITH TRANSITION PROBABILITIES

$$f_{ij}(1) = p_{ij}(1) = p_{ij}$$

$$f_{ij}(2) = p_{ij}(2) - f_{ij}(1) \cdot p_{jj}$$

Probability that the system reaches state $j$ at step 2, given that it was in $i$ at 0

Probability that the system reaches state $j$ for the first time at step 1 (starting from $i$ at 0) and that it remains in $j$ at the successive step

$$f_{ij}(3) = p_{ij}(3) - f_{ij}(1) \cdot p_{jj}(2) - f_{ij}(2) \cdot p_{jj}$$

…

$$f_{ij}(k) = p_{ij}(k) - \sum_{l=1}^{k-1} f_{ij}(k-l) p_{jj}(l)$$ (Renewal Equation)

# Recurrent, Transient and Absorbing States (1)



**DEFINITIONS:**

- First passage probability that the system goes to state $j$ **within $m$ steps** given that it was in $i$ at time 0:

$$q_{ij}(m) = \sum_{n=1}^{m} f_{ij}(n)$$

= sum of the probabilities of the **mutually exclusive events** of reaching $j$ for the first time after $n$ = 1, 2, 3, …, $m$ steps

- Probability that the system **eventually** reaches state $j$ from state $i$:

$$q_{ij}(\infty) = \lim_{m \to \infty} q_{ij}(m)$$

- Probability that the system **eventually** returns to the initial state:

$$f_{ii} = q_{ii}(\infty)$$

# Recurrent, transient and absorbing states (2)



- State $i$ is **recurrent** if the system starting at such state will **surely** return to it **sooner or later** (i.e., in finite time):

$$f_{ii} = q_{ii}(\infty) = 1$$

  - For recurrent states $\Pi_i \neq 0$

# Recurrent, transient and absorbing states (2)



- State *i* is **recurrent** if the system starting at such state will **surely** return to it **sooner or later** (i.e., in finite time):

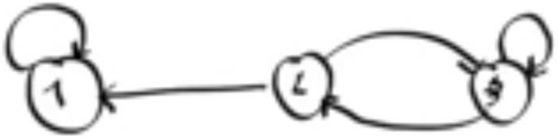
$$f_{ii} = q_{ii}(\infty) = 1$$

  - For recurrent states $\Pi_i \neq 0$

- State *i* is **transient** if the system starting at such state has a **finite probability** of **never** returning to it:

$$f_{ii} = q_{ii}(\infty) < 1$$

  - For these states, at steady state $\Pi_i = 0$



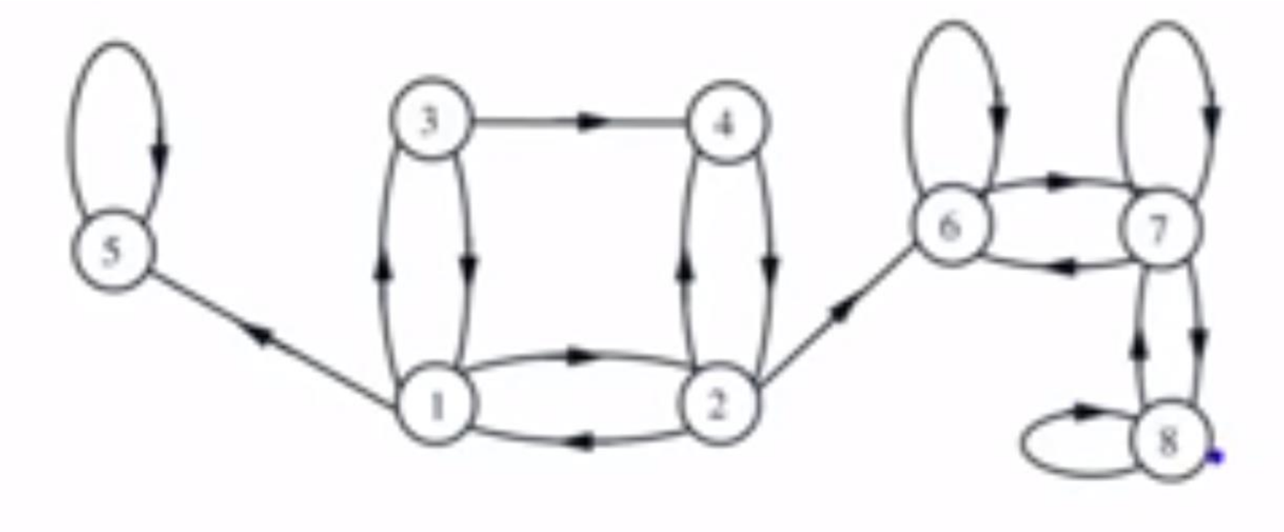
we **cannot** have a **finite Markov process** in which **all states** are **transients** because eventually it will leave them and **somewhere** it must go **at steady state**

- State *i* is **absorbing** if the system cannot leave it once it enters: $p_{ii} = 1$

# Example 5



Classify the states of the following Markov Chain



# Sojourn Time in a state (Average Occupation Time of a State)



$S_i$= number of consecutive time steps the system remains in state *i*

$\boldsymbol{E}[\boldsymbol{S_i}] = \boldsymbol{l_i}$= **Average occupation time** of state *i*

=

average number of time steps before the system exits state *i*

- Recalling that:

$p_{ii}$ = probability that the system "moves to" *i* in one step, given that it was in *i*

$1 - p_{ii}$ = probability that the system exits *i* in one step, given that it was in *i*

$$\mathrm{P}(S_i = n) = p_{ii}^n(1 - p_{ii})$$

$$S_i \sim \mathrm{Geom}(1 - p_{ii})$$

$$\boldsymbol{l_i} = \boldsymbol{E}[\boldsymbol{S_i}] = \frac{1}{1 - p_{ii}}$$