POLITECNICO MILANO 1863

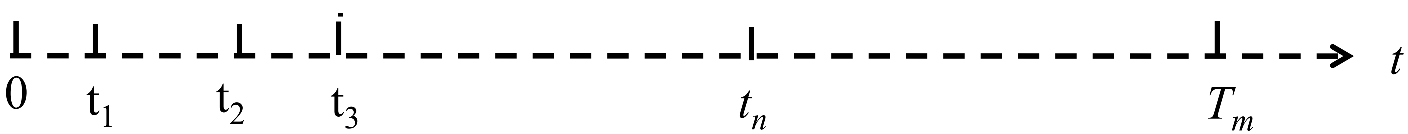
lasar
laboratory of signal and risk analysis

Markov Reliability and Availability Analysis Part II: Continuous Time Discrete State Markov Processes

- The **stochastic process** may be **observed** at:

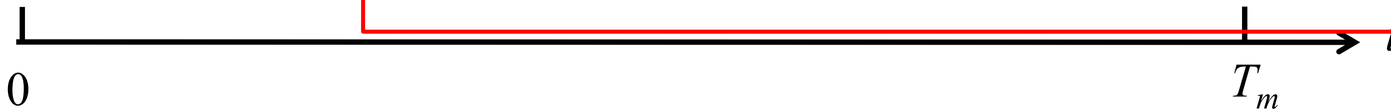
  - Discrete times

    → **DISCRETE-TIME DISCRETE-STATE MARKOV PROCESSES**



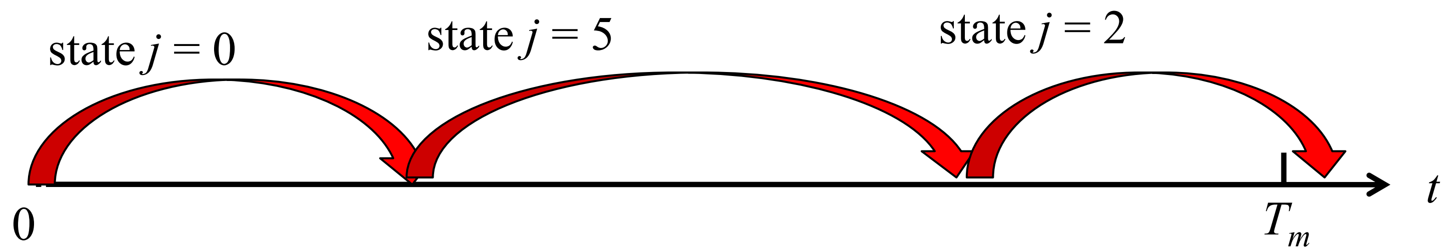
  - Continuously

    → **CONTINUOUS-TIME DISCRETE-STATE MARKOV PROCESS**



The conceptual model: Continuous-Time

- The stochastic process is **observed continuously** and **transitions** are assumed to **occur continuously in time**



The conceptual model: Finite State Space

- The random process of system transition between states in time is described by a **stochastic process** $\{X(t);\, t \geq 0\}$
- $X(t) :=$ system state at time t
    - $X(3.6) = 5$: the system is in state number 5 at time $t = 3.6$

OBJECTIVE:

Computing the <u>probability</u> that the system is in a <u>given state</u> as a <u>function of time</u>, for <u>all</u> possible states

$$P[X(t) = j],\, t \in [0, T_m],\, j = 0, 1, \ldots, N$$

Objective:

$$P[X(t) = j], t \in [0, T_m], j = 0,1, \dots, N$$

What do we need?

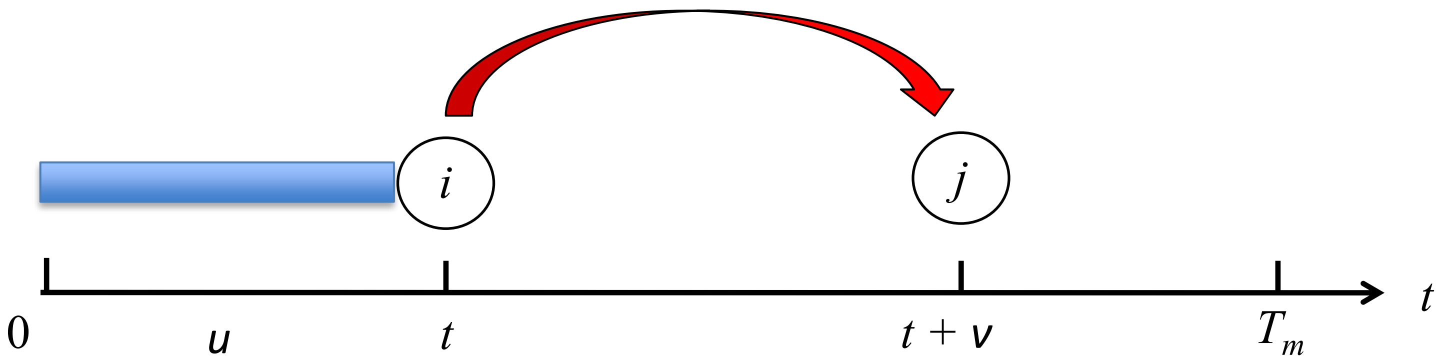
Transition Probabilities!

The Conceptual Model: Transition Probabilities

- **Transition probability** that the system will be in state j at time $t + v$ given that it is in state i at **current** time t and given **the previous system history**

$$P\left[X(t+v)=j \mid X(t)=i, X(u)=x(u), 0 \le u < t\right]$$

$(i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)$



The Conceptual Model: Markov Assumption

- **IN GENERAL STOCHASTIC PROCESSES:**

the **probability** of a **future** state of the system usually depends on its **entire life history**

$$P[X(t+v) = j \mid X(t) = i, X(u) = x(u), 0 \le u < t]$$

$$(i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)$$

- **IN MARKOV PROCESSES:**

the **probability** of a **future** state of the system **only** depends on its **present state**

$$P[X(t+v) = j \mid X(t) = i, \cancel{X(u) = x(u), 0 \le u < t}]$$

$$=$$

$$P[X(t+v) = j \mid X(t) = i]$$

$$(i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)$$

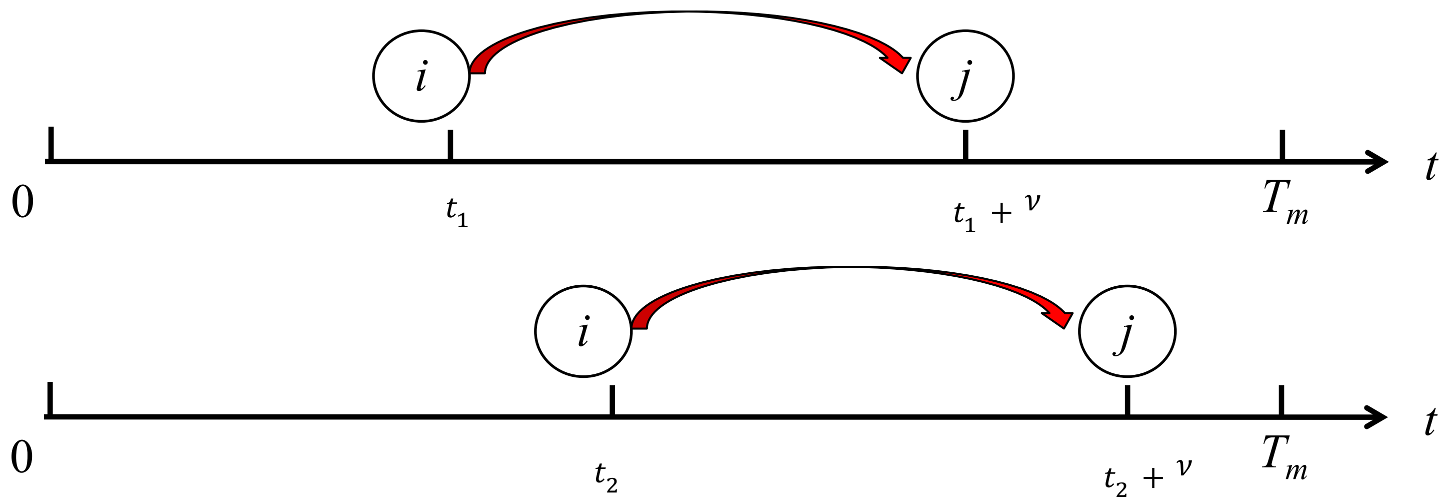
THE PROCESS HAS "NO MEMORY"

The Conceptual Model: homogeneous Markov process

If the **transition probability** depends on the **interval** ***ν*** and **not** on the **individual times** t and $t + \nu$

- the transition probabilities are **stationary**
- the Markov process is **homogeneous** in time

$$p_{ij}(t, t+\nu) = P[X(t+\nu) = j \mid X(t) = i] = p_{ij}(\nu)$$



- Homogeneus process without memory

Transition time → Exponential distribution

HYPOTHESIS:

- The time interval $\nu = dt$ is **small** such that **only one** event (i.e., one **stochastic transition**) can occur within it

$$p_{ij}(dt) = P[X(t+dt) = j | X(t) = i] = 1 - e^{-\alpha_{ij} \cdot dt}$$

α_{ij} = **transition rate** from state *i* to state *j*

HYPOTHESIS:

- The time interval $\nu = dt$ is **small** such that **only one** event (i.e., one **stochastic transition**) can occur within it

$$p_{ij}(dt) = P[X(t+dt) = j | X(t) = i] = 1 - e^{-\alpha_{ij} \cdot dt}$$

$=$ (Taylor 1^{st} order expansion)

$$\alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0$$

Analogy with the Discrete Time Markov Chains

Discrete-time	**Continuous-time**
$P[X(n+1)=j \mid X(n)=i]=p_{ij}$	$P[X(t+dt)=j \mid X(t)=i]=\alpha_{ij}\,dt$
$\underline{P}(n+1)=\underline{P}(n)\cdot\underline{\underline{A}}$	$\underline{P}(t+dt)=\underline{P}(t)\cdot\underline{\underline{A}}$

$$\underline{\underline{A}}=\begin{pmatrix} p_{00} & p_{01} & \dots & p_{0N} \\ p_{10} & p_{11} & \dots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \dots & p_{NN} \end{pmatrix}$$

$$\underline{\underline{A}}=\begin{pmatrix} 1-dt\cdot\sum_{j=1}^{N}\alpha_{0j} & \alpha_{01}\cdot dt & \dots & \alpha_{0N}\cdot dt \\ \alpha_{10}\cdot dt & 1-dt\cdot\sum_{\substack{j=0\\ j\neq 1}}^{N}\alpha_{1j} & \dots & \alpha_{1N}\cdot dt \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

The conceptual model: the fundamental matrix equation (1)

$$\left[P_0(t+dt)P_1(t+dt)\ldots P_N(t+dt)\right] = \left[P_0(t)P_1(t)\ldots P_N(t)\right]\cdot \begin{pmatrix} 1-dt\cdot\sum_{j=1}^{N}\alpha_{0j} & \alpha_{01}\cdot dt & \ldots & \alpha_{0N}\cdot dt \\ \alpha_{10}\cdot dt & 1-dt\cdot\sum_{\substack{j=0\\ j\neq 1}}^{N}\alpha_{1j} & \ldots & \alpha_{1N}\cdot dt \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

- First-equation:

$$P_0(t+dt) = \left[1-dt\sum_{j=1}^{N}\alpha_{0j}\right]P_0(t) + \alpha_{10}P_1(t)\cdot dt + \ldots + \alpha_{N0}P_N(t)dt$$

The conceptual model: the fundamental matrix equation (2)

$$P_0(t+dt) = \left[1 - dt\sum_{j=1}^{N}\alpha_{0j}\right]P_0(t) + \alpha_{10}P_1(t)\cdot dt + \ldots + \alpha_{N0}P_N(t)\,dt$$

subtract $P_0(t)$ on both sides

$$P_0(t+dt) - P_0(t) = P_0(t) - P_0(t) - \sum_{j=1}^{N}\alpha_{0j}P_0(t)dt + \alpha_{10}P_1(t)dt + \ldots + \alpha_{N0}P_N(t)dt$$

divide by dt

$$\frac{P_0(t+dt) - P_0(t)}{dt} = -\sum_{j=1}^{N}\alpha_{0j}P_0(t) + \alpha_{10}P_1(t) + \ldots + \alpha_{N0}P_N(t)$$

let $dt \to 0$

$$\lim_{dt\to 0}\frac{P_0(t+dt) - P_0(t)}{dt} = \frac{dP_0}{dt} = -\sum_{j=1}^{N}\alpha_{0j}\cdot P_0(t) + \alpha_{10}\cdot P_1(t) + \ldots + \alpha_{N0}\cdot P_N(t)$$

The conceptual model: The Transition Probability Matrix

$$\frac{dP_0}{dt} = -\sum_{j=1}^{N} \alpha_{0j} P_0(t) + \alpha_{10} P_1(t) + \cdots + \alpha_{N0} P_N(t) = [\ P_0(t), P_1(t), \ldots, P_N(t)] \cdot \begin{bmatrix} -\sum_{j=1}^{N} \alpha_{0j} \\ \alpha_{10} \\ \ldots \\ \alpha_{N0} \end{bmatrix}$$

- Extending to the other equations:

$$\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{\underline{A}}^*, \quad \underline{\underline{A}}^* = \begin{pmatrix} \underbrace{-\sum_{j=1}^{N} \alpha_{0j}}_{\alpha_{00}} & \alpha_{01} & \ldots & \alpha_{0N} \\ \alpha_{10} & \underbrace{-\sum_{\substack{j=0 \\ j \neq 1}}^{N} \alpha_{1j}}_{\alpha_{11}} & \ldots & \alpha_{1N} \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

TRANSITION RATE MATRIX

It will be indicated as A

System of N+1 **linear**, **first-order differential equations** in the unknown state probabilities

$$P_j(t),\ j = 0,1,2,\ldots,N,\ t \geq 0$$

Exercise 1

Consider a system made by one component that can be in two states: working or failed. Assume constant failure rate λ and constant repair rate μ. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix, A

Exercise 2

Consider a system made by $\boldsymbol{N}$ identical components in parallel which can be in two states: working or failed. Assume constant failure rate λ, that $\boldsymbol{N}$ repairman are available and that the single component repair rate is constant and equal to μ. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix, *A*

Exercise 3: system with N identical components and 1 repairman available

Consider a system made by $\boldsymbol{N}$ identical components which can be in two states: working or failed. Assume constant failure rate λ, that 1 repairman is available and that the component repair rate is constant and equal to μ. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix

Solution to the Fundamental Equation

Solution to the fundamental equation of the Markov process continuous in time: problem setting

$$\begin{cases} \dfrac{d\underline{P}}{dt} = \underline{P}(t)\cdot \underline{\underline{A}} \\ \underline{P}(0) = \underline{C} \end{cases} \quad \text{where} \quad \underline{\underline{A}} = \begin{pmatrix} -\sum_{j=1}^{N}\alpha_{0j} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j\neq 1}}^{N}\alpha_{1j} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

System of N**+1 linear**, **first-order differential equations** in the unknown state probabilities

$$P_j(t),\ j = 0,1,2,\dots,N,\ t \geq 0$$

Solution to the fundamental equation of the Markov process continuous in time

$$\begin{cases} \dfrac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{\underline{A}} \\ \underline{P}(0) = \underline{C} \end{cases}$$

where

$$\underline{\underline{A}} = \begin{pmatrix} -\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \dots & \alpha_{0N} \\ \alpha_{10} & -\sum_{\substack{j=0 \\ j\neq 1}}^{N} \alpha_{1j} & \dots & \alpha_{1N} \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

System of ***N*+1 linear**, **first-order differential equations** in the unknown state probabilities

$$P_j(t),\ j = 0,1,2,\dots,N,\ t \geq 0$$

USE LAPLACE TRANSFORM

- Laplace Transform: $\tilde{P}_j(s) = L\left[P_j(t)\right] = \int_0^{\infty} e^{-st} P_j(t)\,dt, \; j = 0,1,\ldots,N$
- First derivative: $L\left(\frac{dP_j(t)}{dt}\right) = s \cdot \tilde{P}_j(s) - P_j(0), \quad j = 0,1,\ldots,N$
- Apply the Laplace operator to $\frac{d\underline{P}}{dt} = \underline{P}(t) \cdot \underline{\underline{A}}$

$$L\left[\frac{d\underline{P}(t)}{dt}\right] = L\left[\underline{P}(t) \cdot \underline{\underline{A}}\right]$$

First derivative ← $\boxed{s\underline{\tilde{P}}(s) - \underline{C}} = \boxed{\underline{\tilde{P}}(s) \cdot \underline{\underline{A}}}$ → Linearity

$$\underline{\tilde{P}}(s) = \underline{C} \cdot \left[s \cdot \underline{\underline{I}} - \underline{\underline{A}}\right]^{-1}$$

$\underline{P}(t)$ = **inverse transform of** $\underline{\tilde{P}}(s)$

Solution to the Fundamental Equation: Steady State Probabilities

- At steady state $\underline{P}(t) = \underline{\Pi}$

$$\frac{d\underline{P}(t)}{dt} = 0 \quad \Rightarrow \quad \frac{d\underline{P}(t)}{dt} = \underline{P}(t)\cdot \underline{\underline{A}} = \underline{\Pi}\cdot \underline{\underline{A}} = 0$$

- Solve the (linear) system: $\begin{cases} \underline{\Pi}\cdot \underline{\underline{A}} = 0 \\ \sum_{j=0}^{N} \Pi_j = 1 \end{cases}$

Solution to the Fundamental Equation: Steady State Probabilities

- At steady state: $\underline{P}(t) = \underline{\Pi}$

$$\frac{d\underline{P}(t)}{dt} = 0 \quad \Rightarrow \quad \frac{d\underline{P}(t)}{dt} = \underline{P}(t) \cdot \underline{\underline{A}} = \underline{\Pi} \cdot \underline{\underline{A}} = 0$$

- Solve the (linear) system:

$$\begin{cases} \underline{\Pi} \cdot \underline{\underline{A}} = 0 \\ \sum_{j=0}^{N} \Pi_j = 1 \end{cases}$$

- It can be shown that:

$$\Pi_j = \frac{D_j}{\sum_{i=0}^{N} D_i} \qquad j = 0, 1, 2, \ldots, N$$

D_j = determinant of the **square matrix** obtained from $\underline{\underline{A}}$ by **deleting** the *j*-th row and column

Exercise 4: one component/one repairman – Solution to the fundamental equation (1)

Consider a system made by one component which can be in two states: working ('0') or Failed ('1'). Assume constant failure rate λ and constant repair rate μ and that the component is working at $t = 0$.

You are required to find:

- the component steady state availability
- the component instantaneous availability

Exercise 4: one component/one repairman – Solution to the fundamental equation (4)

- **TIME-DEPENDENT STATE PROBABILITIES**

$$P_0(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\lambda+\mu)t}$$ (system instantaneous **availability**)

$$P_1(t) = \frac{\lambda}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\lambda+\mu)t}$$ (system instantaneous **unavailability**)

- **STEADY STATE PROBABILITIES**

$$\Pi_0 = \lim_{t\to\infty} P_0(t) = \frac{\mu}{\mu+\lambda} = \frac{1/\lambda}{1/\mu+1/\lambda} = \frac{MTBF}{MTTR+MTBF}$$

= average fraction of time the system is functioning

$$\Pi_1 = \lim_{t\to\infty} P_1(t) = \frac{\lambda}{\mu+\lambda} = \frac{1/\mu}{1/\mu+1/\lambda} = \frac{MTTR}{MTTR+MTBF}$$

= average fraction of time the system is down (i.e., under repair)

Quantity of Interest

Frequency of departure from a state

- **Unconditional** probability of **arriving in state *j*** in the next dt **departing from state *i*** at time t: $P[X(t+dt)=j, X(t)=i]$

$$P[X(t+dt)=j, X(t)=i] = P[X(t+dt)=j \mid X(t)=i] \cdot P[X(t)=i] = p_{ij}(dt)P_i(t)$$

- **Frequency of departure from state *i* to state *j***:

$$v_{ij}^{dep}(t) = \lim_{dt \to 0} \frac{P[X(t+dt)=j, X(t)=i]}{dt} = \lim_{dt \to 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t)$$

- **Total frequency of <u>departure</u> from state *i* to any other state *j***:

$$v_i^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^{N} v_{ij}^{dep}(t) = \sum_{\substack{j=0 \\ j \neq i}}^{N} \alpha_{ij} \cdot P_i(t) = P_i(t) \sum_{\substack{j=0 \\ j \neq i}}^{N} \alpha_{ij} = -\alpha_{ii} \cdot P_i(t)$$

Frequency of departure from a state at steady state

- **Unconditional** probability of **arriving in state *j*** in the next dt **departing from state *i*** at time *t*: $P[X(t+dt)=j, X(t)=i]$

$$P[X(t+dt)=j, X(t)=i]=P[X(t+dt)=j|X(t)=i]\cdot P[X(t)=i]=p_{ij}(dt)P_i(t)$$

- **Frequency of departure from state *i* to state *j***:

$$v_{ij}^{dep}(t)=\lim_{dt\to 0}\frac{P[X(t+dt)=j, X(t)=i]}{dt}=\lim_{dt\to 0}\frac{p_{ij}(dt)P_i(t)}{dt}=\alpha_{ij}P_i(t)$$ (at steady state) $= \boxed{v_{ij}^{dep}=\alpha_{ij}\cdot\Pi_i}$

- **Total frequency of <u>departure</u> from state *i* to any other state *j***:

$$v_i^{dep}(t)=\sum_{\substack{j=0\\ j\neq i}}^{N}v_{ij}^{dep}(t)=\sum_{\substack{j=0\\ j\neq i}}^{N}\alpha_{ij}\cdot P_i(t)=P_i(t)\sum_{\substack{j=0\\ j\neq i}}^{N}\alpha_{ij}=-\alpha_{ii}\cdot P_i(t)$$ (at steady state) $\boxed{v_i^{dep}=-\alpha_{ii}\cdot\Pi_i}$

Frequency of arrival to a state

- **Unconditional** probability of **arriving in state *j*** in the next dt **departing from state *i*** at time *t*: $P[X(t+dt)=j, X(t)=i]$

$$P[X(t+dt)=j, X(t)=i] = P[X(t+dt)=j \mid X(t)=i] \cdot P[X(t)=i] = p_{ij}(dt) P_i(t)$$

- **Frequency of <u>arrival from</u> state *i* to state *j***:

$$v_{ij}^{arr}(t) = \lim_{dt \to 0} \frac{P[X(t+dt)=j, X(t)=i]}{dt} = \lim_{dt \to 0} \frac{p_{ij}(dt) P_i(t)}{dt} = \alpha_{ij} P_i(t)$$

- **Total frequency of <u>arrival</u> to state *j***:

$$v_j^{arr}(t) = \sum_{\substack{i=0 \\ i \neq j}}^{N} v_{ij}^{arr}(t) = \sum_{\substack{i=0 \\ i \neq j}}^{N} \alpha_{ij} \cdot P_i(t)$$

Frequency of arrival to a state at steady state

- **Unconditional** probability of **arriving in state *j*** in the next dt **departing from state *i*** at time *t*: $P[X(t+dt)=j, X(t)=i]$

$$P[X(t+dt)=j, X(t)=i] = P[X(t+dt)=j | X(t)=i] \cdot P[X(t)=i] = p_{ij}(dt)P_i(t)$$

- **Frequency of <u>arrival from</u> state *i* to state *j***:

$$v_{ij}^{arr}(t) = \lim_{dt\to 0} \frac{P[X(t+dt)=j, X(t)=i]}{dt} = \lim_{dt\to 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t)$$ (at steady state) $v_{ij}^{arr} = \alpha_{ij}\Pi_i$

- **Total frequency of <u>arrival</u> to state *j***:

$$v_j^{arr}(t) = \sum_{\substack{i=0\\ i\neq j}}^{N} v_{ij}^{arr}(t) = \sum_{\substack{i=0\\ i\neq j}}^{N} \alpha_{ij} \cdot P_i(t)$$ (at steady state) $v_j^{arr} = \sum_{\substack{i=0\\ i\neq j}}^{N} \alpha_{ij} \cdot \Pi_i = \ldots$

$$\nu_i^{arr}(t) = \sum_{\substack{k=0 \\ k \neq i}}^{N} \alpha_{ki} \cdot P_k(t)$$

$$\nu_i^{arr} = \sum_{\substack{k=0 \\ k \neq i}}^{N} \alpha_{ki} \cdot \Pi_k \quad \text{(at steady state)}$$

$$\underline{\Pi} \cdot \underline{\underline{A}} = 0 \quad \Rightarrow \quad \sum_{k=0}^{N} \alpha_{ki} \cdot \Pi_k = 0 \quad (i = 0, 1, 2, \ldots, N)$$

$$-\alpha_{ii} \cdot \Pi_i = \sum_{\substack{k=0 \\ k \neq i}}^{N} \alpha_{ki} \cdot \Pi_k \quad (i = 0, 1, 2, \ldots, N)$$

AT STEADY STATE:

frequency of departures from state *i* = frequency of arrivals to state *i*

- **SYSTEM FAILURE INTENSITY W_f:**
  - **Rate** at which **system failures** occur

1 failure state [*f*], *N* operating states

- **SYSTEM FAILURE INTENSITY W_f:**
  - **Rate** at which **system failures** occur

$$W_f(t) = v_f^{arr}(t) = \sum_{\substack{i=0 \\ i \neq f}}^{N} v_{if}^{arr}(t) = \sum_{\substack{i=0 \\ i \neq f}}^{N} \alpha_{if} \cdot P_i(t)$$

S =Set of the success states; F = Set of the operating states

- **SYSTEM FAILURE INTENSITY W_f:**
  - **Rate** at which **system failures** occur

$$W_F(t) = \sum_{j \in F} v_f^{arr}(t) = \sum_{j \in F} \sum_{i \in S} v_{ij}^{arr}(t) =$$

$$= \sum_{j \in F} \sum_{i \in S} \alpha_{ij} P_i(t) = \sum_{i \in S} P_i(t) \sum_{j \in F} \alpha_{ij} = \sum_{i \in S} P_i(t) \lambda_{i \to F}$$

$\lambda_{i \to F}$ = conditional (transition) probability of leaving success state i towards failure states

- **SYSTEM REPAIR INTENSIT W_r:**
  - **Rate** at which **system repairs** occur

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \to S}$$

S = set of success states of the system

F = set of failure states of the system

$P_j(t)$ = probability of the system being in the failure state j at time t

$\mu_{j \to S}$ = conditional (transition) probability of leaving failure state j towards success states

Exercise 5: one component/one repairman

Consider a system made by one component which can be in two states: working ('0') or Failed ('1'). Assume constant failure rate λ and constant repair rate μ and that the component is working at $t = 0$:

- You are required to find the failure and repair intensities

Sojourn Time in a state (1)

- **Time of occupance of state *i* (sojourn time) = $\boldsymbol{T_i}$**

- Is the time (t) that the system has already been in state i influencing the time (s) the system will remain in state i?

$$\mathrm{P}(T_i > t + s | T_i > t) = P(X(t+u) = i, 0 \leq u \leq s |\, X(\tau) = i, 0 \leq \tau \leq t) =$$

$$= \mathrm{P}(X(t+u) = i, 0 \leq u \leq s | X(t) = i) \text{ (by Markov property)}$$

$$= \mathrm{P}(X(u) = i, 0 \leq u \leq s | X(0) = i) \text{ (by homogeneity)}$$

$$= P(T_i > s)$$ **Memoryless Property**

- The only distribution satisfying the memoryless property is the **Exponential distribution**

$$T_i \sim Exp$$

Sojourn Time in a state (2)

- System departure **rate from state *i*** (at steady state): $-\alpha_{ii}$

$$T_i \sim Exp(-\alpha_{ii})$$

- **Expected sojourn time $\boldsymbol{l_i}$**: average time of occupancy of state i

$$l_i = \mathbb{E}\{T_i\} = \frac{1}{-\alpha_{ii}}$$

Sojourn Time in a state (3)

- Total frequency of departure at steady state: $\boxed{\nu_i^{dep} = -\alpha_{ii} \cdot \Pi_i}$
- Average time of occupancy of state: $\boxed{l_i = \dfrac{1}{-\alpha_{ii}}}$

$$\nu_i^{dep} = -\alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i}$$

$$\boxed{v_i^{dep} = v_i^{arr}}$$

$$\Pi_i = v_i^{arr} \cdot l_i$$

The **mean** proportion of time Π_i that the system spends in state *i* is equal to the **total frequency of <u>arrivals</u> to state *i*** multiplied by the mean duration of one visit in state *i*

System Availability

- **System instantaneous availability** at time *t*

  = **sum** of the **probabilities** of being in a **success** state at time *t*

$$p(t) = \sum_{i \in S} P_i(t) = 1 - q(t) = 1 - \sum_{j \in F} P_j(t)$$

In the Laplace domain

$$\tilde{p}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)$$

S = set of success states of the system

F = set of failure states of the system

- **TWO CASES:**

1) Non-Reparaible Systems
→ No repairs allowed

2) Reparaible Systems
→ Repairs allowed

- No repairs allowed ⇒ **Reliability = Availability** $R(t) \equiv p(t) = 1 - q(t)$
- In the Laplace Domain: $\tilde{R}(s) = \sum_{i \in S} \tilde{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)$
- **Mean Time to Failure (MTTF):**

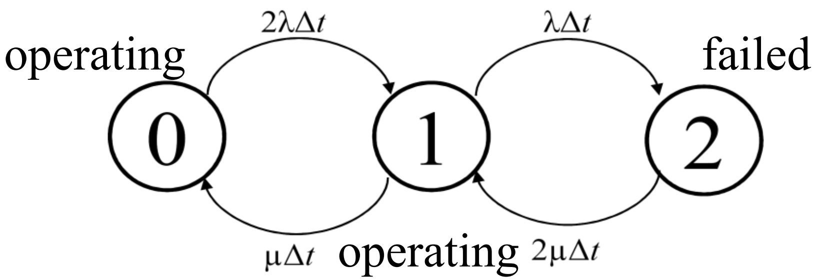
$$MTTF = \int_0^\infty R(t)dt = \left[\int_0^\infty R(t)e^{-st}dt\right]_{s=0} = \tilde{R}(0) = \sum_{i \in S} \tilde{P}_i(0) = \left[\frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s)\right]_{s=0}$$

- **TWO CASES:**

1) **Non-reparaible systems**
   → No repairs allowed

2) **Reparaible systems**
   → Repairs allowed

Parallel System of two identical components



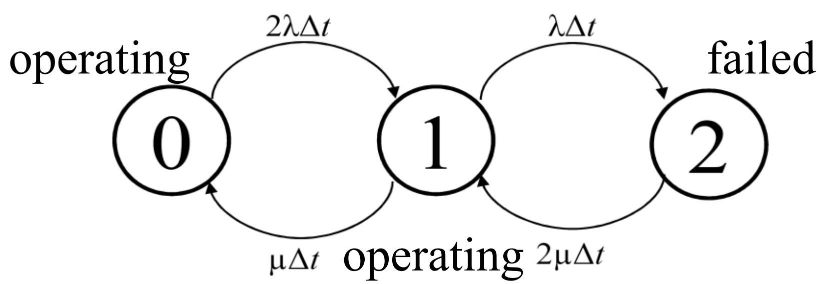
$$S = \{0,1\}$$
$$F = \{2\}$$

$$R(t) = P(T > t) = P\{[X(\tau) = 0 \ \ or \ X(\tau) = 1], \forall \tau \in [0, t)\} = P_0^*(t) + P_1^*(t)$$

Can I trasnsform the Markov Diagram in such a way that $R(t) = P_0^*(t) + P_1^*(t)$?

System Reliability: Reparaible Systems (1)

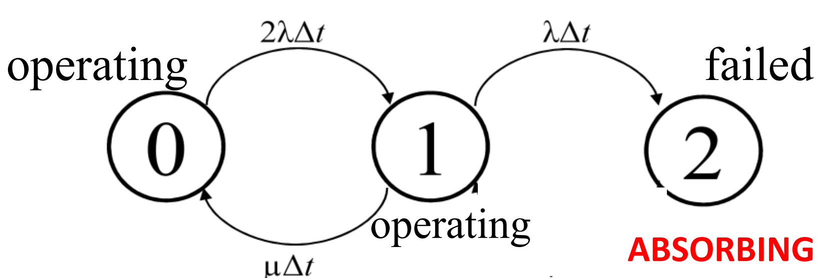
Parallel System of two identical components



$$S = \{0,1\}$$
$$F = \{2\}$$

$$R(t) = P(T > t) = P\{X(\tau) = 0 \;\; or \; X(\tau) = 1, \forall\tau \in [0,t)\} = P_0^*(t) + P_1^*(t)$$

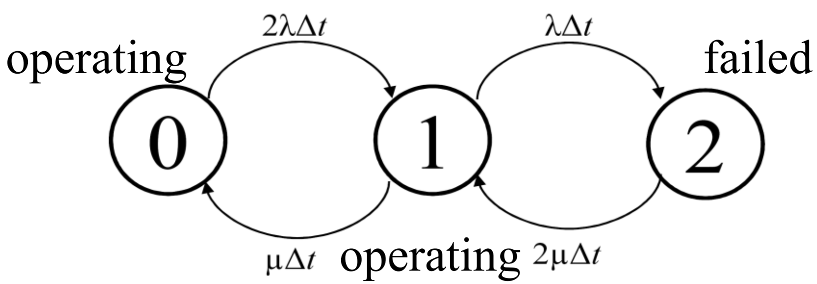
Can I trasnsform the Markov Diagram in such a way that $R(t) = P_0^*(t) + P_1^*(t)$?



ABSORBING STATE!

System Reliability: Reparaible Systems (1)

1. Transform the failed states $j \in F$ into absorbing states (the system cannot be repaired → it is not possible to escape from a failed state)



$$S = \{0,1\}$$
$$F = \{2\}$$

$$\underline{\underline{A}} = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \;\Rightarrow\; \underline{\underline{A}}^* = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 \end{bmatrix} \;\Rightarrow\; \underline{\underline{A}}^* = \begin{bmatrix} -2\lambda & 2\lambda \\ \mu & -(\mu+\lambda) \end{bmatrix}$$

The new matrix $\underline{\underline{A}}^*$ contains the transition rates for transitions **only among the success states** $i \in S$

(the “reduced” system is virtually functioning continuously with no interruptions)

2. Solve the **reduced problem** of $\underline{\underline{A}}^*$ for the probabilities $P_i^*(t),\ i \in S$ of being in these **(transient) safe states**

$$\frac{d\underline{P}^*(t)}{dt} = \underline{P}^*(t) \cdot \underline{\underline{A}}^*$$

Reliability

$$R(t) = \sum_{i \in S} P_i^*(t)$$

Mean Time To Failure (MTTF)

$$MTTF = \int_0^{\infty} R(t)\,dt = \sum_{i \in S} \tilde{P}_i^*(0) = \tilde{R}(0)$$

NOTICE: in the reduced problem we have only transient states $\Rightarrow \Pi_i^* = P_i^*(\infty) = 0$

Exercise 6

Consider a system made by 2 identical components in series. Each component can be in two states: working or failed. Assume constant failure rate λ, that 2 repairman are available and that the single component repair rate is constant and equal to μ. You are required to:

- find the system reliability
- find the system MTTF

Exercise 7

Consider a system made by 2 identical components in parallel. Each component can be in two states: working or failed. Assume constant failure rate λ, that 2 repairman are available and that the single component repair rate is constant and equal to μ. You are required to:

- find the system reliability
- find the system MTTF