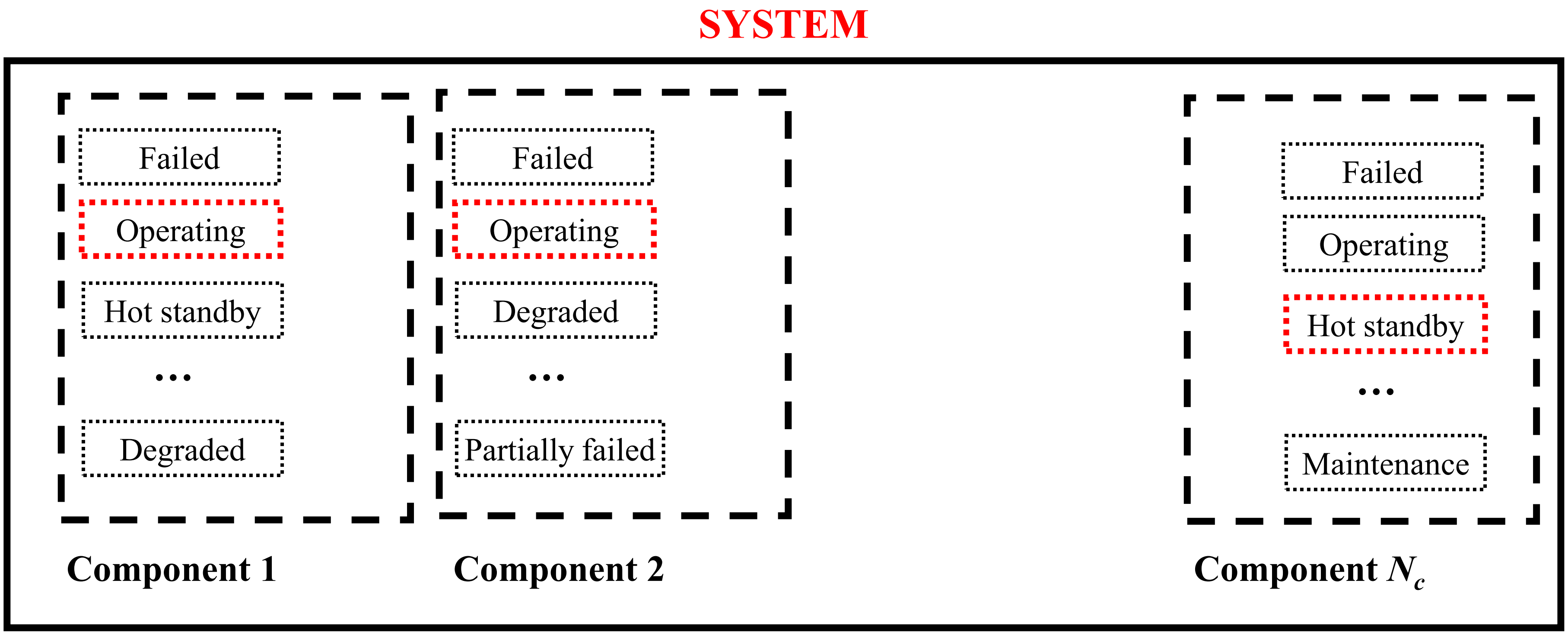
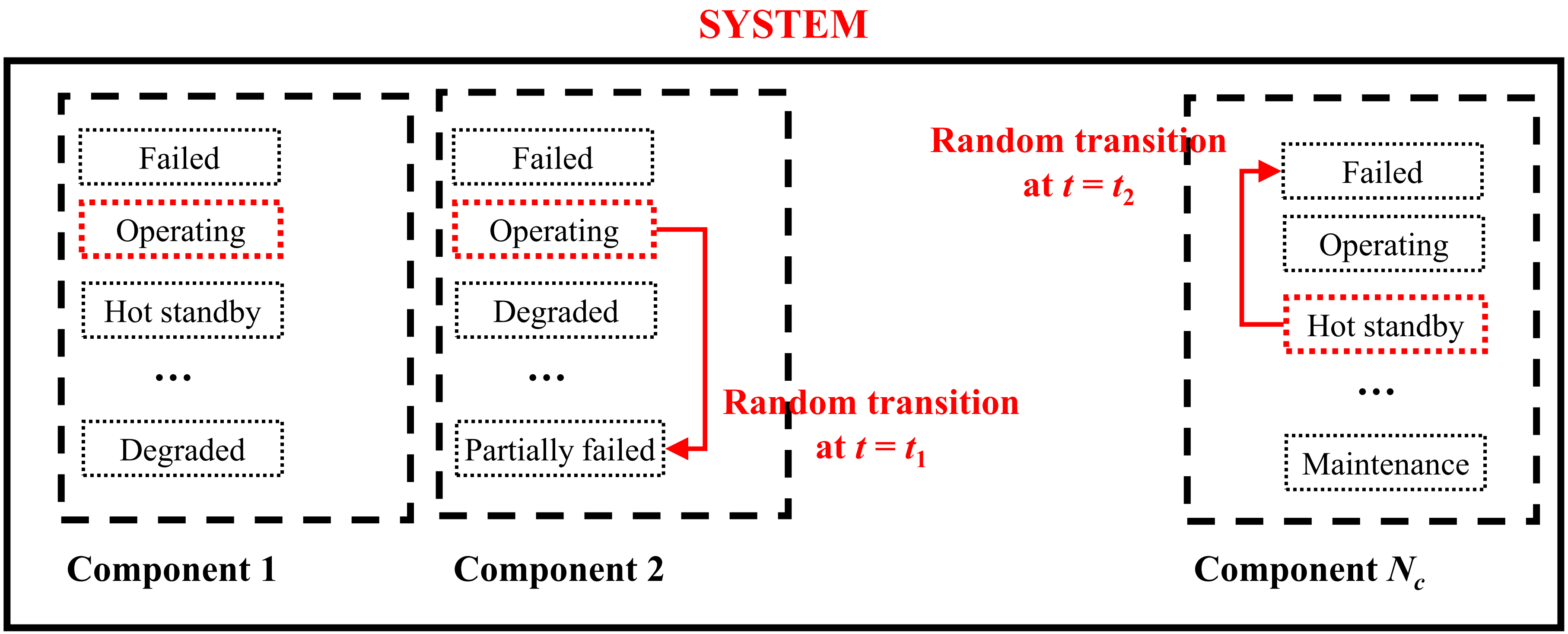
Markov Reliability and Availability Analysis

General Framework

General Framework

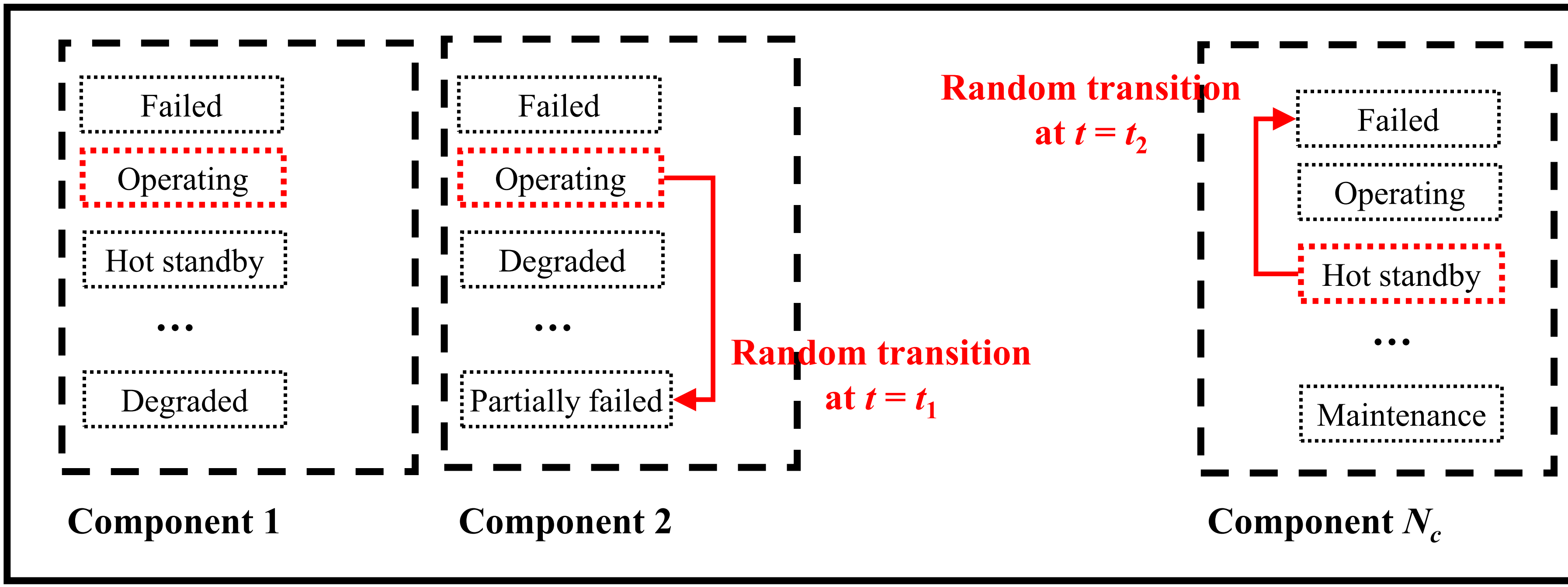


General Framework



System evolution = Stochastic process

SYSTEM



Under specified conditions:

System evolution = Stochastic process

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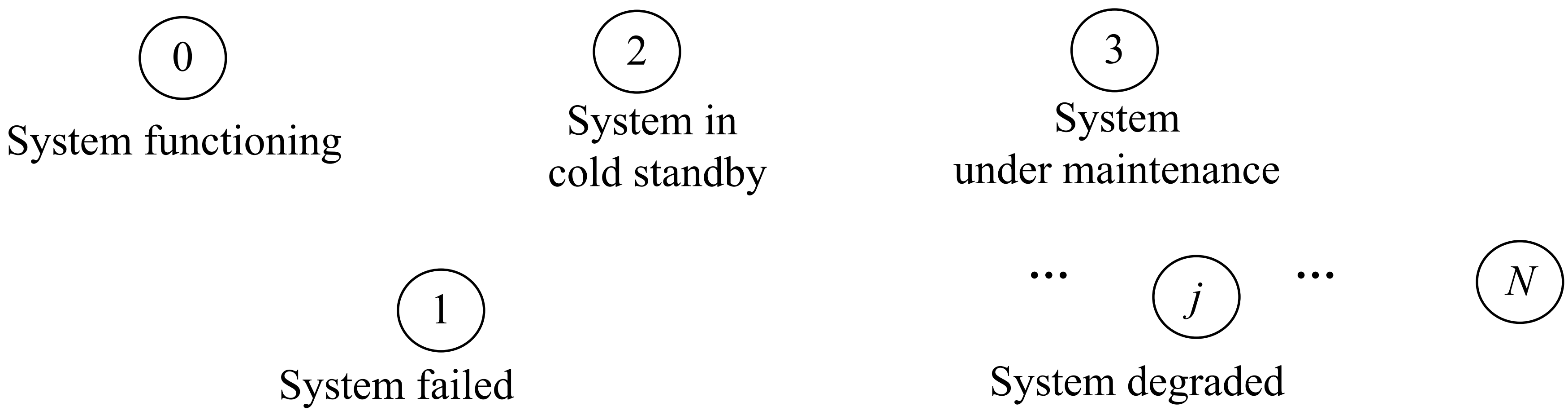
MARKOV PROCESS

Markov Processes: Basic Elements

Markov Processes: the System States (1)

- The **system** can occupy a **finite** or **countably infinite** number N+1 of states



Set of possible states $U = \{0, 1, 2, \ldots, N\}$

=

State-space of the random process

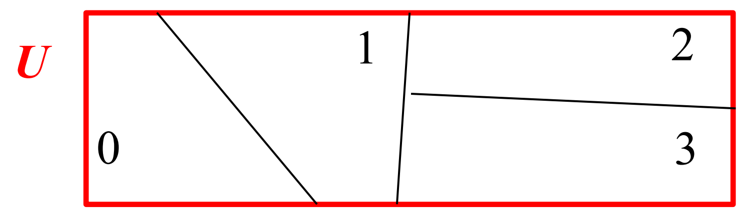
- The **States** are:
  - **Mutually Exclusive:** $P(\text{State} = i \cap \text{State} = j) = 0, \text{ if } \; i \neq j$

    (the system can be **only** in **one** state *at **each** time*)
  - **Exhaustive:** $P(U) = P\left(\bigcup_{i=1}^{N} \text{State} = i\right) = \sum_{i=1}^{N} P(\text{State} = i) = 1$

    (the system must be in **one** state *at **all** time*s

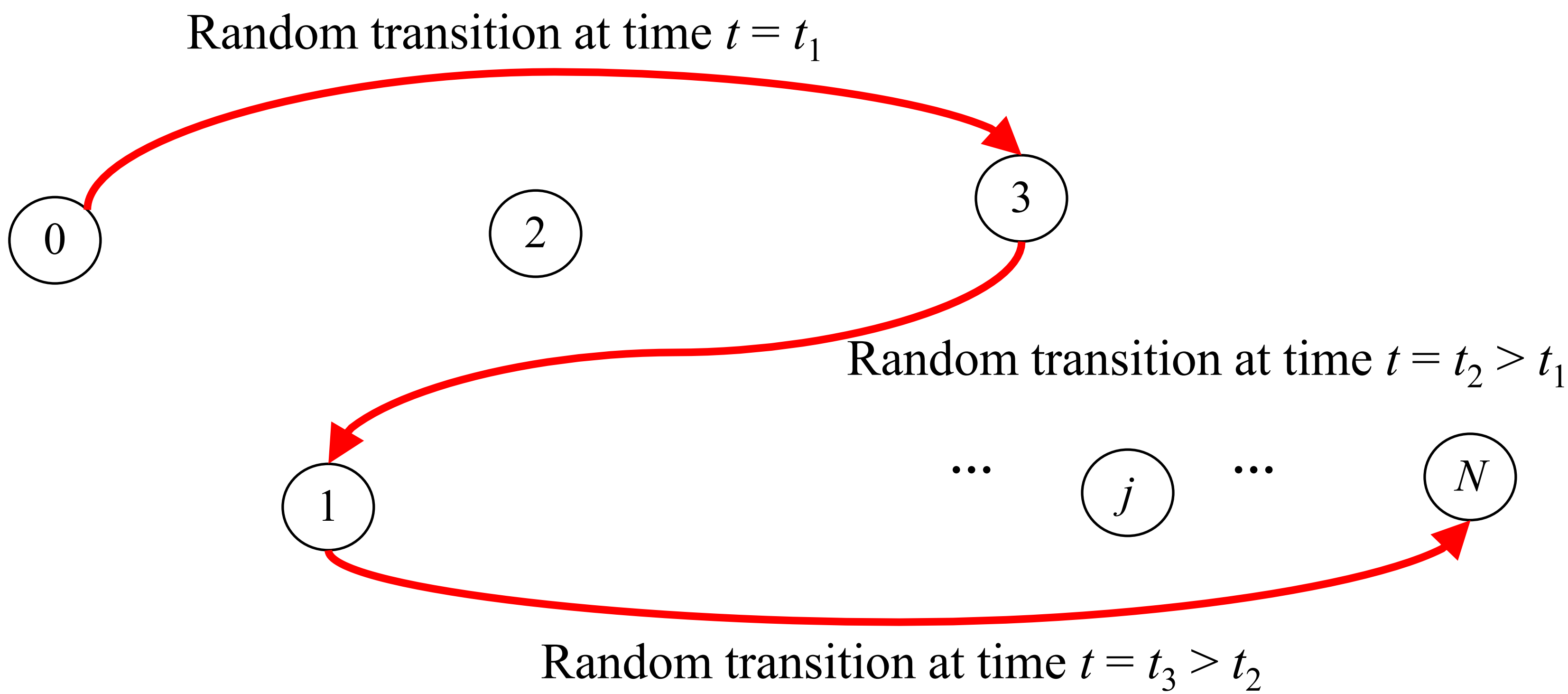
- **Example:**

Set of possible states $U = \{0, 1, 2, 3\}$



$$P(U) = P(\text{State} = 0 \cup \text{State} = 1 \cup \text{State} = 2 \cup \text{State} = 3)$$
$$= P(\text{State} = 0) + P(\text{State} = 1) + P(\text{State} = 2) + P(\text{State} = 3) = 1$$

- **Transitions** from one state to another occur **stochastically** (i.e., **randomly in time** and **in final transition state**)

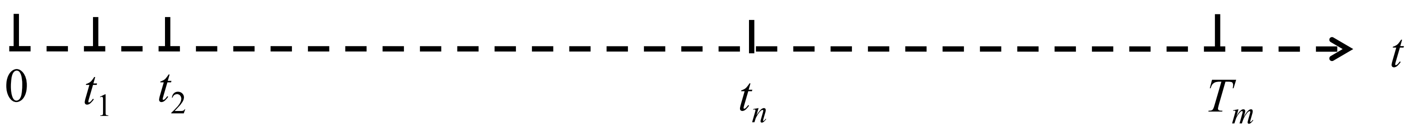


- The system state in **time** can be described by an **integer random variable** $X(t)$

$X(t)=5$ → the system occupies the **state** labelled by **number 5** at time t

- The **stochastic process** may be **observed** at:

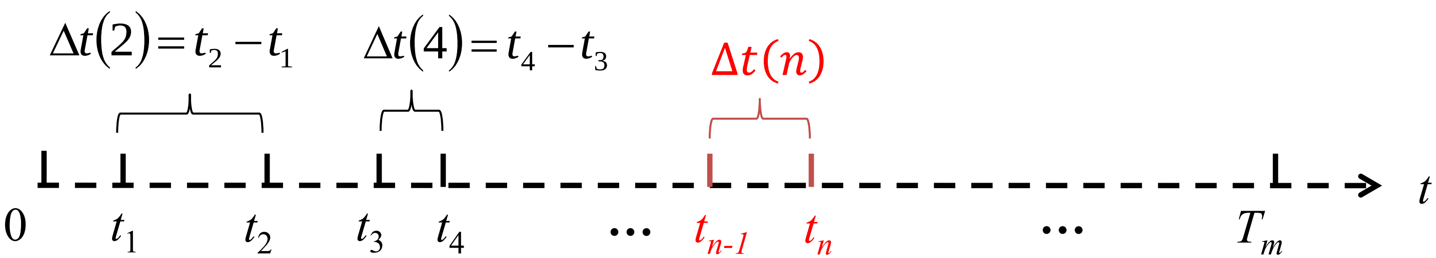
  - Discrete times → **DISCRETE-TIME DISCRETE-STATE MARKOV PROCESS**



  - Continuously → **CONTINUOUS-TIME DISCRETE-STATE MARKOV PROCESS**

Discrete-Time Markov Processes

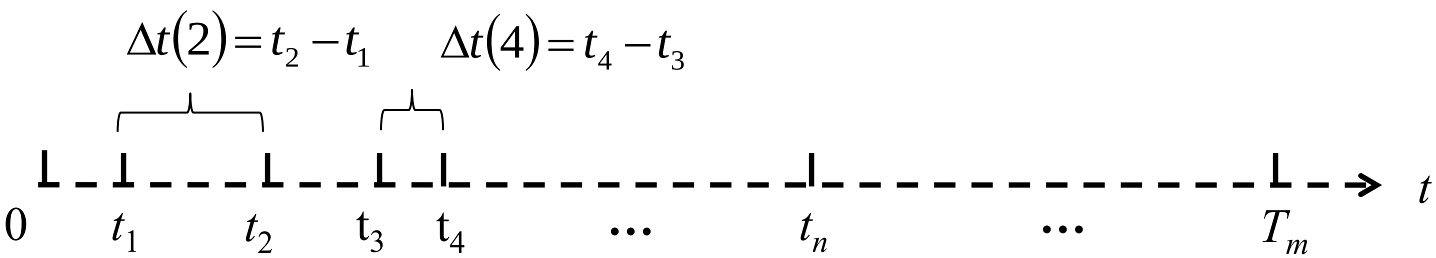
- The stochastic process is **observed** at **discrete** times



$$t_n = t_{n-1} + \Delta t(n)$$

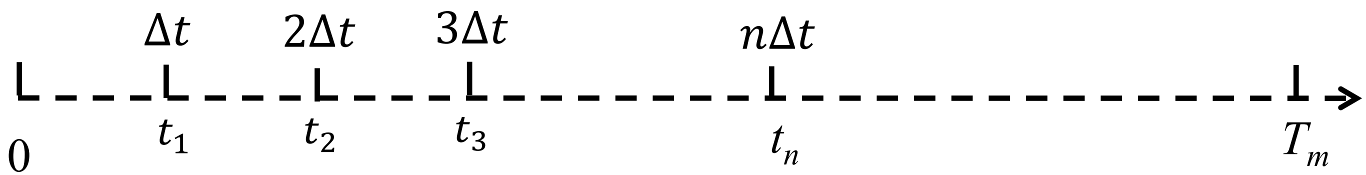
The Conceptual Model: Discrete Observation Times

- The stochastic process is **observed** at **discrete** times



$$t_n = t_{n-1} + \Delta t(n)$$

- **Hypotheses**:
  - The time interval $\Delta t(n)$ is **small** enough such that **only one** event (i.e., stochastic transition) can occur within it
  - For simplicity, $\Delta t(n) = \Delta t =$ constant



The Conceptual Model: Mathematical Representation

- The random process of system transition in time is described by an **integer random variable** $X(\cdot)$

- $X(n) \coloneqq$ **system state** at time $t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time t_3

The Conceptual Model: Objective

- The random process of system transition in time is described by an **integer random variable** $X(\cdot)$
- $X(n) :=$ **system state** at time $t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time t_3

OBJECTIVE:

Compute the probability that the system is in a given state at a given time, for all possible states and times

$$P[X(n) = j],\ n = 1, 2, \ldots, N_{time},\ j = 0, 1, \ldots, N$$

Objective:

$$P\left[X(n)=j\right], n=1,2,\ldots,N_{time}, j=0,1,\ldots,N$$

What do we need?

Objective:

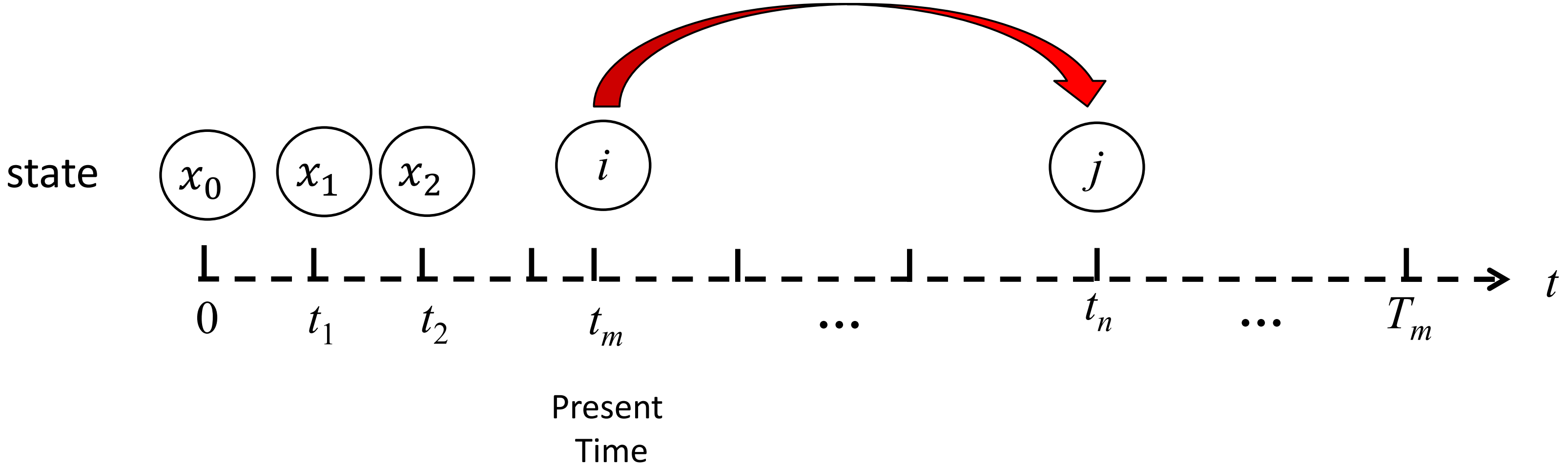
$$P\left[X\left(n\right)=j\right], n=1,2,\ldots,N_{time}, j=0,1,\ldots,N$$

What do we need?

Transition Probabilities!

- **Transition probability:** conditional probability that the system moves to state j at time t_n given that it is in state i at current time t_m and given the previous system history

$$P[X(n) = j | X(0) = x_0, X(1) = x_1, X(2) = x_2, \ldots, X(m) = x_m = i]$$
$$\forall j = 0,1, \ldots, N$$



In general, for stochastic processes:

- the **probability** of a transition to a **future** state depends on its **entire life history**

$$P[X(n) = j | X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots, X(m) = x_m = i]$$

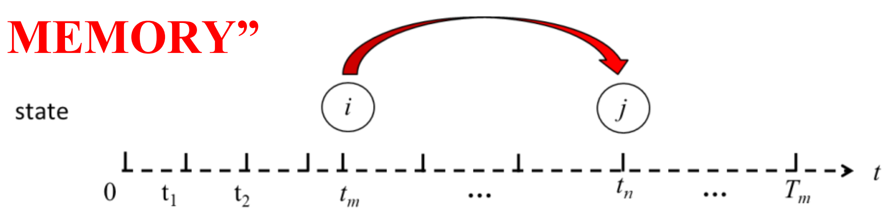
In Markov Processes:

- the **probability** of a transition to a **future** state **only** depends on its **present state**

$$P[X(n) = j | \cancel{X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots}, X_m = x_m = i]$$

$$=$$

THE PROCESS HAS "NO MEMORY"



$$p_{ij}(m,n) = P[X(n) = j | X(m) = i] \qquad n > m \geq 0$$

1. Transition probabilities $p_{ij}(m, n)$ are **larger than or equal to 0**

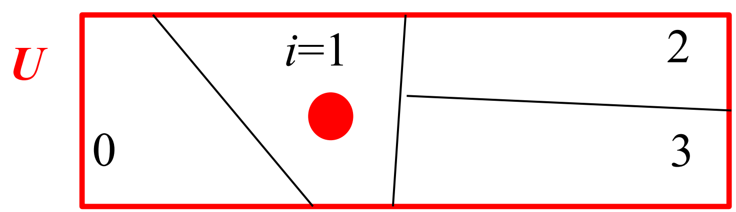
$$p_{ij}(m,n) \geq 0, \quad n > m \geq 0 \qquad i = 0, 1, 2, \ldots, N, \; j = 0, 1, 2, \ldots, N$$

(**definition** of **probability**)

2. Transition probabilities **must sum to 1**

$$\sum_{all\, j} p_{ij}(m,n) = \sum_{j=0}^{N} p_{ij}(m,n) = 1, n > m \geq 0 \qquad i = 0, 1, 2, \ldots, N$$

(the **set of states** is **exhaustive**)



$$\sum_{j=0}^{3} p_{1j}(m,n) = 1, n > m \geq 0$$

Starting from $i = 1$, the system either **remains in *i* = 1** or it goes **somewhere else, i.e.,** to ***j* = 0 or 2 or 3**

The Chapman-Kolmogorov Equation

The conceptual model: properties of the transition probabilities (2)

3. $p_{ij}(m,n)=\sum_k p_{ik}(m,r)p_{kj}(r,n) \quad i=0,1,2,\ldots,N, j=0,1,2,\ldots,N$

$$p[X(n)=j, X(m)=i]=\sum_k p[X(n)=j, X(r)=k, X(m)=i]$$ **(theorem of total probability)**

↓ **conditional probability**

$$=\sum_k p[X(n)=j \mid X(r)=k, X(m)=i]P[X(r)=k, X(m)=i]$$

↓ **Markov assumption**

$$=\sum_k p[X(n)=j \mid X(r)=k]P[X(r)=k, X(m)=i]$$

$$p_{ij}(m,n)=P[X(n)=j \mid X(m)=i]=\frac{P[X(n)=j, X(m)=i]}{P[X(m)=i]}$$ **(conditional probability)**

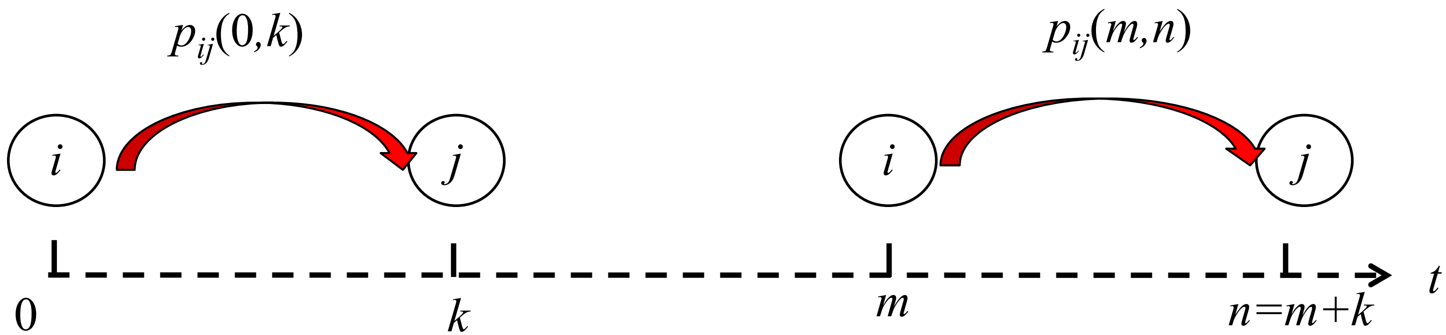
↓ **formula above**

$$=\sum_k p[X(n)=j \mid X(r)=k]\frac{P[X(r)=k, X(m)=i]}{P[X(m)=i]}$$

↓ **conditional probability**

$$=\sum_k P[X(n)=j \mid X(r)=k]P[X(r)=k \mid X(m)=i]=\sum_k p_{kj}(r,n)p_{ik}(m,r)$$

The Conceptual Model: Stationary Transition Probabilities



- If the **transition probability** $p_{ij}(m, n)$ depends on the **interval** $(t_n - t_m)$ and **not** on the **individual times** t_m and t_n, (**transition probabilities** are **stationary**)

- **the Markov process is called "homogeneous in time"**

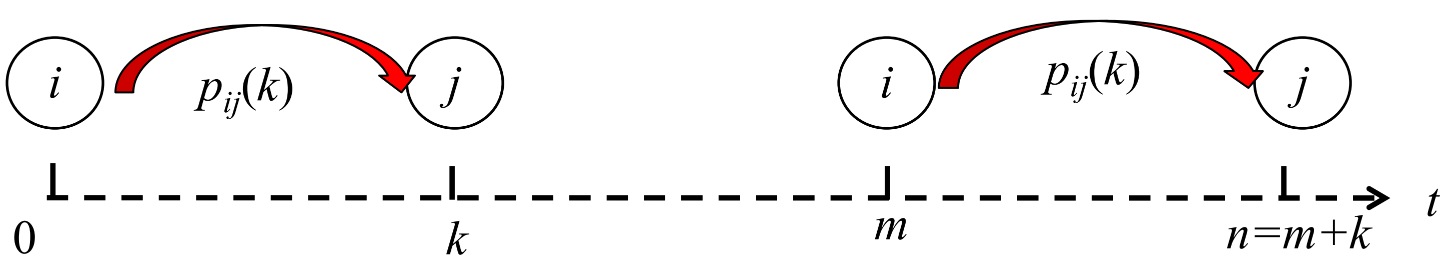
The Conceptual Model: Stationary Transition Probabilities- Notation

- If the **transition probability** $p_{ij}(m, n)$ depends on the **interval** $(t_n - t_m)$ and **not** on the **individual time** t_m then:

$$p_{ij}(m,n) = p_{ij}\big(m, m + \overbrace{(n-m)}^{k \text{ time steps}}\big) = p_{ij}(m, m+k) = P\left[X(m+k) = j \mid X(m) = i\right]$$

$$= P\left[X(k) = j \mid X(0) = i\right]$$

$$= p_{ij}(k), \quad k \geq 0 \quad i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N$$



The Chapman-Kolmogorov equation for homogeneous systems

- We know:
  - The one-step transition probabilities: $p_{ij}(1) = \boxed{p_{ij}}$

    $(i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N)$

  - The state probabilities at time $n = 0$ (initial condition):

    $\boxed{c_j} = P[X(0) = j]$

- Objective:
  - Compute the probability that the system is in a given state j at a given time t_n, for all possible states and times

$$P[X(n) = j] = \boxed{P_j(n)}, \; n = 1, 2, \ldots, N_{time}, \; j = 0, 1, \ldots, N$$

The Conceptual Model: Computation of the Unconditional State Probabilities

↓ Th. of Total Probability

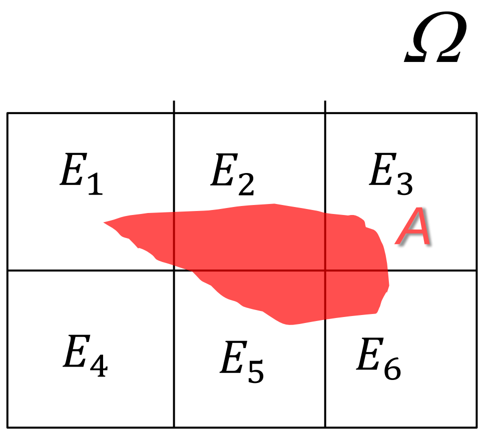
$$P_j(n) = P[X(n) = j] = \sum_{i=0}^{N} P[X(0) = i] \cdot P[X(n) = j | X(0) = i]$$

↓ homogeneous process

$$= \sum_{i=0}^{N} c_i \cdot p_{ij}(n)$$

Theorem of Total Probability (from Lecture 2)

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:



$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad\qquad \bigcup_{j=1}^{n} E_j = \Omega$$

- Given any event A in Ω,

$$A = \bigcup_{j=1}^{n} A \cap E_j$$

$$P(A) = \sum_{j=1}^{n} P\left(A \cap E_j\right) = \sum_{j=1}^{n} P\left(A|E_j\right) P(E_j)$$

The Conceptual Model: Computation of the Unconditional State Probabilities

↓ Th. of Total Probability

$$P_j(n) = P[X(n) = j] = \sum_{i=0}^{N} P[X(0) = i] \cdot P[X(n) = j | X(0) = i]$$

↓ homogeneous process

$$= \sum_{i=0}^{N} c_i \cdot \boxed{p_{ij}(n)}$$

… from Chapman-Kolmogorov equation using p_{ij}

Computation of the Unconditional State Probabilities at time 1

$$P_j(1) = P[X(1) = j] = \sum_{i=0}^{N} P[X(0) = i] \cdot P[X(1) = j | X(0) = i]$$

$$= \sum_{i=0}^{N} c_i \cdot p_{ij} = [c_0, \dots, c_i, \dots c_N] \cdot \begin{bmatrix} p_{0j} \\ \dots \\ p_{ij} \\ \dots \\ p_{Nj} \end{bmatrix}$$

Unconditional State Probabilities: Matrix Notation

- Introduce the row vectors:

$$\underline{P}(n) = [P_0(n), P_1\,(n), \dots, P_i(n), P_N(n)]$$

probabilities of the system being in state 0, 1, 2, …, N at the n-th time step

$$\underline{P}(0) = \underline{C} = [c_0, c_1, \dots, c_i, \dots, c_N]$$

initial condition

$$P_j(1) = \sum_{i=0}^{N} c_i \cdot p_{ij} = [c_0, \dots, c_i, \dots c_N] \cdot \begin{bmatrix} p_{0j} \\ \dots \\ p_{ij} \\ \dots \\ p_{Nj} \end{bmatrix}$$

$$\underline{P}(1) = [c_0, \dots, c_i, \dots c_N] \cdot \begin{bmatrix} p_{00} & \cdots & p_{0j} & \cdots & p_{0N} \\ \dots & \dots & \dots & \dots & \dots \\ p_{i0} & \cdots & p_{ij} & \cdots & p_{iN} \\ \dots & \dots & \dots & \dots & \dots \\ p_{N0} & \cdots & p_{Nj} & \cdots & p_{NN} \end{bmatrix} = \underline{C} \cdot \underline{\underline{A}}$$

The Conceptual Model: Notation - the Transition Probability Matrix

$$\underline{\underline{A}} = \begin{array}{c} i/j \\ 0 \\ 1 \\ \dots \\ N \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \dots & N \end{array} \\ \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0N} \\ p_{10} & p_{11} & \dots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \dots & p_{NN} \end{pmatrix} \end{array}$$

Properties:

- $\dim\left(\underline{\underline{A}}\right) = (N+1)\times(N+1)$
- $0 \le p_{ij} \le 1, \forall i, j \in \{0,1,2,\dots,N\}$

(all elements are **probabilities**)

The Conceptual Model: Notation - the Transition Probability Matrix

$$\begin{array}{c|ccccc}
i/j & 0 & 1 & \dots & N & \\
0 \; \Sigma & p_{00} & p_{01} & \dots & p_{0N} & =1 \\
1 & p_{10} & p_{11} & \dots & p_{1N} & \\
\dots & \dots & \dots & \dots & \dots & \\
N & p_{N0} & p_{N1} & \dots & p_{NN} &
\end{array}$$

$$\underline{\underline{A}} = \begin{pmatrix} p_{00} & p_{01} & \dots & p_{0N} \\ p_{10} & p_{11} & \dots & p_{1N} \\ \dots & \dots & \dots & \dots \\ p_{N0} & p_{N1} & \dots & p_{NN} \end{pmatrix}$$

Properties:

- $\dim\left(\underline{\underline{A}}\right) = (N+1) \times (N+1)$
- $0 \le p_{ij} \le 1, \forall i, j \in \{0, 1, 2, \dots, N\}$

(all elements are **probabilities**)

only (*N*+1)x*N* elements need to be known

- $\sum_{j=0}^{N} p_{ij} = 1, i = 0, 1, 2, \dots, N$

(the **set of states** is **exhaustive**)

$\underline{\underline{A}}$ **is a Stochastic Matrix**

Computation of the Unconditional State Probabilities (2)

- At the second time step $n = 2$:

$$P_j(2) = P\left[X(2) = j\right]$$

↓ theorem of total probability + Markov assumption

$$= \sum_{k=0}^{N} P\left[X(2) = j \middle| X(1) = k\right] \cdot P\left[X(1) = k\right]$$

↓ homogeneous process

$$= \sum_{k=0}^{N} p_{kj} \cdot P_k(1) =$$

$$= P_0(1) \cdot p_{0j} + P_1(1) \cdot p_{1j} + P_2(1) \cdot p_{2j} + \ldots + P_N(1) \cdot p_{Nj}, \quad = [P_1(0), \ldots, P_1(i), \ldots, P_1(N)] \cdot \begin{bmatrix} p_{0j} \\ \cdots \\ p_{ij} \\ \cdots \\ p_{Nj} \end{bmatrix}$$

$$with \;\; j = 0, 1, 2, \ldots, N$$

$$\underline{P}(2) = \underline{P}(1) \cdot \underline{\underline{A}} = \left(\underline{C}\,\underline{\underline{A}}\right)\underline{\underline{A}} = \underline{C}\,\underline{\underline{A}}^2$$

Proceeding in the same recursive way…

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$$

FUNDAMENTAL EQUATION OF THE HOMOGENEOUS DISCRETE-TIME DISCRETE-STATE MARKOV PROCESS

Problem Setting & Found Solution

- We know:
  - The one-step transition probabilities: p_{ij}
  - The initial condition $c_j = P[X(0) = j]$
- Objective:
  - Compute the probability that the system is in a given state j at a given time t_n, for all possible states and times: $\underline{P}(n)$
- Solution:

$$\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{A}^n = \underline{C} \cdot \underline{A}^n}$$

FUNDAMENTAL EQUATION

Multi-step Transition Probabilities: Interpretation

FUNDAMENTAL EQUATION $\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$

$$\underline{\underline{A}}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \dots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \dots & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & \dots & p_{NN}(n) \end{pmatrix}$$

***n*-th step transition probability matrix**

$p_{ij}(n)$ **???**

Multi-step Transition Probabilities: Interpretation

FUNDAMENTAL EQUATION $\boxed{\underline{P}(n) = \underline{P}(0) \cdot \underline{\underline{A}}^n = \underline{C} \cdot \underline{\underline{A}}^n}$

$$\underline{\underline{A}}^n = \begin{pmatrix} p_{00}(n) & p_{01}(n) & \dots & p_{0N}(n) \\ p_{10}(n) & p_{11}(n) & \dots & p_{1N}(n) \\ \dots & \dots & \dots & \dots \\ p_{N0}(n) & p_{N1}(n) & \dots & p_{NN}(n) \end{pmatrix}$$

***n*-th step transition probability matrix**

$$p_{ij}(n) = P[X(n) = j | X(0) = i]$$

probability of arriving in state *j* after *n* steps given that the initial state was *i*

Multi-step transition probabilities (2)

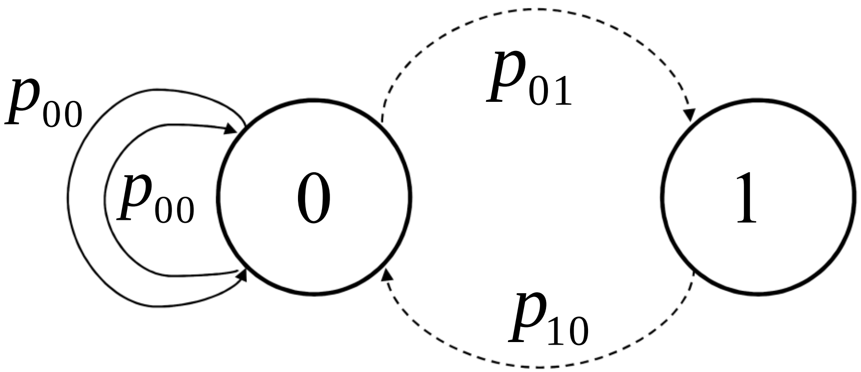
EXAMPLE WITH $N = 2$ STATES AND $n = 2$ time steps

$$\underline{\underline{A}} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad (i = 0,1, j = 0, 1)$$

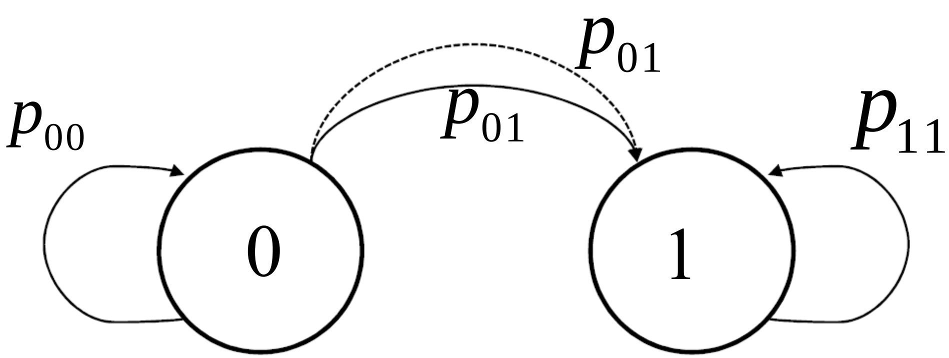
$$\underline{\underline{A}}^2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \cdot \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} \boxed{p_{00} \cdot p_{00} + p_{01} \cdot p_{10}} & \boxed{p_{00} \cdot p_{01} + p_{01} \cdot p_{11}} \\ p_{10} \cdot p_{00} + p_{11} \cdot p_{10} & p_{10} \cdot p_{01} + p_{11} \cdot p_{11} \end{pmatrix}$$

WHAT IS THE "PHYSICAL" MEANING?

Multi-step Transition Probabilities (3)



$$p_{00}(2) = p_{00} \cdot p_{00} + p_{01} \cdot p_{10}$$



$$p_{01}(2) = p_{00} \cdot p_{01} + p_{01} \cdot p_{11}$$

$p_{ij}(n) = P[X(n) = j \mid X(0) = i]$ **, $p_{ij}(n)$ is the sum of the probabilities of all trajectories with length n which originate in state i and end in state j**

Exercise 1: wet and dry days in a town

- Stochastic process of raining in a town (transitions between wet and dry days)

DISCRETE STATES

State 0: dry day

State 1: wet day

DISCRETE TIME

Time step = 1 day

TRANSITION MATRIX

$$\underline{\underline{A}} = \begin{array}{cc} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

You are required to:

1) Draw the Markov diagram
2) If today the weather is dry, what is the probability that it will be **dry two days from now**?

Open Problems

- We provided an analytical framework for computing the state probabilities
- Still open issues:
    1. Estimate the transition matrix A → Problem of parameter identification from data or expert knowledge
    2. Solve for a generic time n, i.e. find $P_j(n)$ as a function of n, without the need of multiplying n times the matrix A

Solution to the fundamental equation

Ergodic Markov Process

A Markov process is called ergodic if it is possible to eventually get from every state to every other state with positive probability

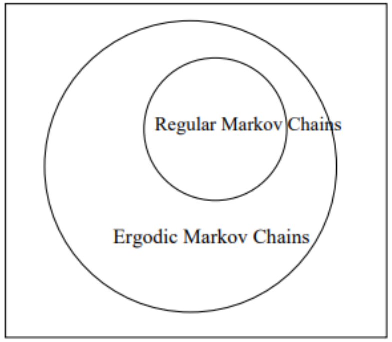
$$A = \begin{pmatrix} 0.8 & 0.2 \\ 0.50 & 0.5 \end{pmatrix}$$

Ergodic

$$A = \begin{pmatrix} 0.8 & 0.2 \\ 0 & 1 \end{pmatrix}$$

Non Ergodic

A Markov process is said to be regular if some power of the stochastic matrix A has all positive entries (i.e. strictly greater than zero).



$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = A^4 = \cdots = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^3 = A^5 = \cdots = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Ergodic – Non Regular

Solution to the Fundamental Equation (1)

$$\begin{cases} \underline{P}(n) = \underline{P}(0)\underline{\underline{A}}^n \\ \underline{P}(0) = \underline{C} \end{cases}$$

SOLVE THE <u>EIGENVALUE PROBLEM</u> ASSOCIATED TO MATRIX A

i) Set the **eigenvalue problem** $\underline{V} \cdot \underline{\underline{A}} = \omega \cdot \underline{V}$

ii) Write the **homogeneous form** $\underline{V} \cdot \left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$

iii) Find **non-trivial solutions** by setting $\det\left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$

iv) From $\det\left(\underline{\underline{A}} - \omega \cdot \underline{\underline{I}} \right) = 0$ compute the **eigenvalues** $\omega_j,\ j = 0,1,\ldots,N$

v) Set the ***N+1*** **eigenvalue problems** $\underline{V_j} \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V_j} \qquad j = 0,1,\ldots,N$

vi) From $\underline{V_j} \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V_j}$ compute the **eigenvectors** $\underline{V_j},\ j = 0,1,\ldots,N$

Eigenvalues of a Stocastic Matrix

- A is a stocastic matrix
- The Markov process is regular and Ergodic

$$\omega_0 = 1 \text{ and } \left|\omega_j\right| < 1, j = 1,2,\ldots,N$$

Solution to the fundamental equation (2)

The **eigenvectors** $\underline{V}_j$ span the $(N + 1)$-dimensional space and can be used as a **basis** to write **any** $(N + 1)$-dimensional vector as a **linear combination** of them

$$\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V_j} \qquad \text{AND} \qquad \underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V_j}$$

WE NEED TO FIND THE COEFFICIENTS α_j AND $c_j, j = 0,1,\ldots,N$

Solution to the fundamental equation (3)

FIND THE COEFFICIENTS $c_j,\ j=0,1,\ldots,N$ **FOR** $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$

SOLVE THE ASSOCIATED ADJOINT EIGENVALUE PROBLEM

i) Set the **adjoint eigenvalue problem**

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+$$

ii) Since for **real valued** matrices $\underline{\underline{A}}^+ = \underline{\underline{A}}^T$ then:

$$\underline{V}^+ \cdot \underline{\underline{A}}^+ = \omega^+ \cdot \underline{V}^+ \Rightarrow \underline{V}^+ \cdot \underline{\underline{A}}^T = \omega^+ \cdot \underline{V}^+$$

iii) Since the eigenvalues $\omega_j^+,\ j=0,1,\ldots,N$ depend **only** on $\det\left(\underline{\underline{A}}^T\right) = \det\left(\underline{\underline{A}}\right)$

$$\Rightarrow \omega_j^+ = \omega_j,\ j=0,1,\ldots,N$$

Solution to the fundamental equation (4)

iv) From $\underline{V}_j^+ \cdot \underline{\underline{A}}^+ = \omega_j \cdot \underline{V}_j^+,\ j = 0,1,\dots,N$ compute the adjoint eigenvectors

$$\underline{V}_j^+ \ ,\ j = 0,1,\dots,N$$

v) Adjoint problem

$$\Rightarrow \quad < \underline{V}_j^+, \underline{V}_i > \equiv \underline{V}_j^+ \cdot \underline{V}_i^T = \begin{cases} 0 \ \ if\ i \neq j \\ k\, otherwise \end{cases}$$

Solution of the fundamental equation (4)

iv) From $\underline{V}_j^+ \cdot \underline{\underline{A}}^+ = \omega_j \cdot \underline{V}_j^+, \; j = 0,1,\ldots,N$ compute the adjoint eigenvectors

$$\underline{V}_j^+ \;, \; j = 0,1,\ldots,N$$

v) By **definition** of the adjoint problem and since $\underline{V}_j^+$ and $\underline{V}_j$ are **orthogonal**

$$\Rightarrow \quad < \underline{V}_j^+, \underline{V}_i > \equiv \underline{V}_j^+ \cdot \underline{V}_i^T = \begin{cases} 0 \;\; if \; i \neq j \\ k \; otherwise \end{cases}$$

vi) Multiply the left-hand sides of $\underline{C} = \sum_{i=0}^{N} c_i \underline{V}_i$ by $\underline{V}_j^+$

$$\Downarrow$$

$$\left\langle \underline{V}_j^+, \underline{C} \right\rangle = \sum_{i=0}^{N} c_i \left\langle \underline{V}_j^+, \underline{V}_i \right\rangle = c_j \left\langle \underline{V}_j^+, \underline{V}_j \right\rangle \rightarrow c_j = \frac{\left\langle \underline{V}_j^+, \underline{C} \right\rangle}{\left\langle \underline{V}_j^+, \underline{V}_j \right\rangle}$$

(orthogonality)

Solution to the fundamental equation (5)

FIND THE COEFFICIENTS $\alpha_j, j = 0,1,\dots,N$ **FOR** $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$

USE $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$, $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ AND $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$

Solution to the fundamental equation (5)

FIND THE COEFFICIENTS $\alpha_j,\ j = 0,1,\dots,N$ **FOR** $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$

USE $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j$ **,** $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ AND $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$

i) Substitute $\underline{C} = \sum_{j=0}^{N} c_j \cdot \underline{V}_j$ into $\underline{P}(n) = \underline{C}\,\underline{\underline{A}}^n$ to obtain $P(n) = \left(\sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

ii) Set $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \underline{C} \cdot \underline{\underline{A}}^n = \left(\sum_{j=0}^{N} c_j \underline{V}_j \right) \cdot \underline{\underline{A}}^n$

Solution to the fundamental equation (6)

iii) Multiply $\underline{V}_j \cdot \underline{\underline{A}} = \omega_j \cdot \underline{V}_j$ by $\underline{\underline{A}}$ to obtain $\underline{V}_j \cdot \underline{\underline{A}} \cdot \underline{\underline{A}} = \omega_j \cdot \boxed{\underline{V}_j \cdot \underline{\underline{A}}}$

Since $\boxed{\underline{V}_j \cdot \underline{\underline{A}}} = \omega_j \cdot \underline{V}_j$ then $\underline{V}_j \cdot \underline{\underline{A}}^2 = \omega_j \cdot \boxed{\omega_j \cdot \underline{V}_j} = \omega_j^2 \cdot \underline{V}_j$

••• (proceeding in the same recursive way)

$$\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$$

iv) Substitute $\underline{V}_j \cdot \underline{\underline{A}}^n = \omega_j^n \cdot \underline{V}_j$ into $\underline{P}(n) = \sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j \quad = \underline{C} \cdot \underline{\underline{A}}^n = \sum_{j=0}^{N} c_j \cdot \underline{V}_j \underline{\underline{A}}^n$

$$\sum_{j=0}^{N} \alpha_j \cdot \underline{V}_j = \sum_{j=0}^{N} c_j \cdot \omega_j^{\,n} \cdot \underline{V}_j$$

$$\alpha_j = c_j \cdot \omega_j^{\,n}$$

Exercise 1: wet and dry days in a town

- Stochastic process of raining in a town (transitions between wet and dry days)

DISCRETE STATES

State 0: dry day

State 1: wet day

DISCRETE TIME

Time step = 1 day

TRANSITION MATRIX

$$\underline{\underline{A}} = \begin{array}{cc} & \begin{array}{cc} \mathit{dry} & \mathit{wet} \end{array} \\ \begin{array}{c} \mathit{dry} \\ \mathit{wet} \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

Today the weather is dry

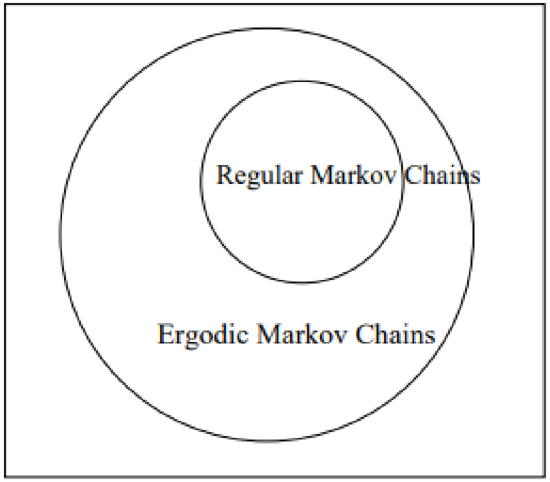
You are required to:

1) Estimate the probability that it will be dry ***n*** days from now?

Some Definitions

Quantity of Interest

Is it possible to make long-term predictions ($n \to +\infty$) of a Markov process?



It is possible to show that if the Markov process is regular then:

$$\exists \lim_{n \to +\infty} \underline{P}(n) = \Pi$$

Steady state probabilities

- **Steady state probabilities π_j:** probability of the system being in state *j* **asymptotically**

- **TWO ALTERNATIVE APPROACHES:**

1) Since $\omega_0 = 1$ and $|\omega_j| < 1,\ j = 1,2,\ldots,N$

AT STEADY STATE: $$\lim_{n\to\infty} \underline{P}(n) = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{\alpha_j} \cdot \underline{V_j} = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{c_j \cdot \omega_j^n} \cdot \underline{V_j} = c_0 \underline{V_0} = \underline{\Pi}$$

- **Steady state probabilities π_j**: probability of the system being in state j **asymptotically**
- **TWO ALTERNATIVE APPROACHES:**

1) Since $\omega_0 = 1$ and $|\omega_j| < 1,\ j = 1,2,\ldots,N$

AT STEADY STATE: $\lim_{n\to\infty} \underline{P}(n) = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{\alpha_j} \cdot \underline{V_j} = \lim_{n\to\infty} \sum_{j=0}^{N} \boxed{c_j \cdot \omega_j^n} \cdot \underline{V_j} = c_0 \underline{V_0} = \underline{\Pi}$

2) Use the recursive equation $\underline{P}(n) = \underline{P}(n-1) \cdot \underline{\underline{A}}$

AT STEADY STATE: $\underline{P}(n) = \underline{P}(n-1) = \underline{\Pi}$

SOLVE $\underline{\Pi} = \underline{\Pi} \cdot \underline{\underline{A}}$ subject to $\sum_{j=0}^{N} \Pi_j = 1$

Exercise 1: wet and dry days in a town (continue)

$$\underline{\underline{A}} = \begin{array}{c} \\ dry \\ wet \end{array} \begin{array}{c} \begin{array}{cc} dry & wet \end{array} \\ \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array} \qquad \underline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- *Question*: what is the probability that **one year from now** the day will be **dry**?
  - ❑ Use the approximation based on the recursive equation

First Passage Probabilities (1)

- **FIRST PASSAGE PROBABILITY AFTER *n* TIME STEPS:**

Probability that the system arrives **for the first time** in state *j* **after *n* steps**, given that it was in state *i* at the initial time 0

$$f_{ij}(n) = P\left[X(n) = j \ \ \textit{for the first time} \,\middle|\, X(0) = i\right]$$

=

$$f_{ij}(n) = P\left[X(n) = j, X(m) \neq j, 0 < m < n \mid X(0) = i\right]$$

NOTICE:

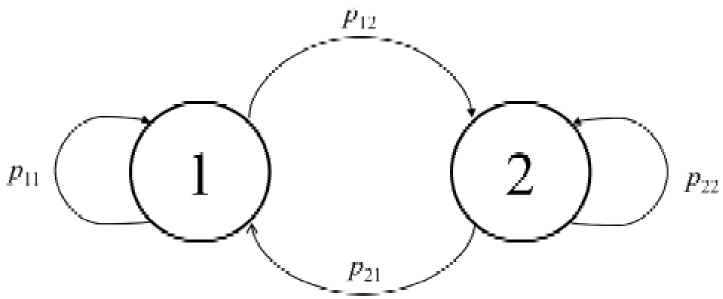
$$f_{ij}(n) \neq p_{ij}(n)$$

$p_{ij}(n)$ =probability that the system reaches state *j* **after *n* steps** starting from state *i*, but **not necessarily for the first time**

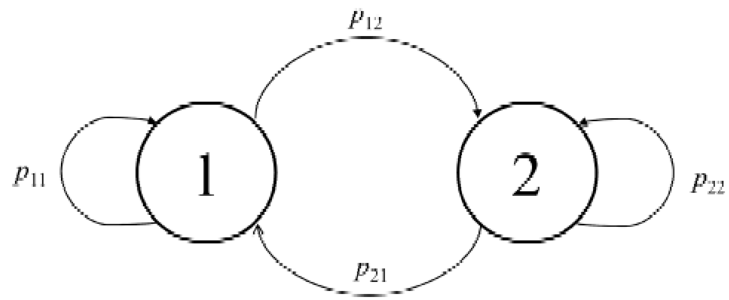
First Passage Probabilities: Exercise 3

Compute for the markov process in the Figure below:

- $f_{11}(1)$
- $f_{11}(n)$
- $f_{12}(n)$



EXAMPLE



$f_{11}(1) = p_{11}$

probability of going from state 1 to state 1 in 1 step for the first time

$f_{11}(n) = p_{12} \cdot p_{22}^{n-2} \cdot p_{21}$

probability that the system, starting from state 1, will return to the same state 1 for the first time after *n* steps: this is achieved by jumping in state 2 at the first step (p_{12}), remaining in state 2 during the successive *n*-2 steps (p_{22}^{n-2}) and moving back in the initial state 1 at the *n*-th step (p_{21}).

$f_{12}(n) = p_{11}^{n-1} \cdot p_{12}$

probability that the system will arrive for the first time in state 2 after *n* steps; this is equal to the probability of remaining in state 1 for *n*-1 steps (p_{11}^{n-1}) and then jumping in state 2, at the final step (p_{12})

First Passage Probabilities (4)

- RELATIONSHIP WITH TRANSITION PROBABILITIES

$$f_{ij}(1) = p_{ij}(1) = p_{ij}$$

$$f_{ij}(2) = p_{ij}(2) - f_{ij}(1) \cdot p_{jj}$$

Probability that the system reaches state j at step 2, given that it was in state i at step 0

Probability that the system reaches state j for the first time at step 1 (starting from state i at 0) and that it remains in j at the successive step

$$f_{ij}(3) = p_{ij}(3) - f_{ij}(1) \cdot p_{jj}(2) - f_{ij}(2) \cdot p_{jj}$$

…

$$f_{ij}(k) = p_{ij}(k) - \sum_{l=1}^{k-1} f_{ij}(k-l) p_{jj}(l)$$ (Renewal Equation)

Exercise 1: wet and dry days in a town (Group Work –Part III)

$$\underline{\underline{A}} = \begin{array}{cc} & \begin{array}{cc} dry & wet \end{array} \\ \begin{array}{c} dry \\ wet \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} \end{array} \qquad \underline{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- *Question*: if today is dry, what is the probability that
  1) the first wet day will be Thursday?
  2) Wednesday will be wet?
  3) The first wet day will be within Thursday?

Recurrent, Transient and Absorbing States (1)

DEFINITIONS:

- First passage probability that the system goes to state *j* **within *m* steps** given that it was in *i* at time 0:

$$q_{ij}(m) = \sum_{n=1}^{m} f_{ij}(n)$$ = sum of the probabilities of the **mutually exclusive events** of reaching *j* for the first time after *n* = 1, 2, 3, …, *m* steps

- Probability that the system **eventually** reaches state *j* from state *i*:

$$q_{ij}(\infty) = \lim_{m \to \infty} q_{ij}(m)$$

- Probability that the system **eventually** returns to the initial state:

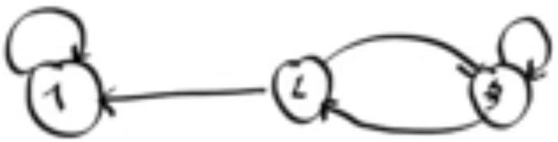
$$f_{ii} = q_{ii}(\infty)$$

Recurrent, transient and absorbing states (2)

- State i is **recurrent** if the system starting at such state will **surely** return to it (**sooner or later),** i.e., in finite time:
  $$f_{ii} = q_{ii}(\infty) = 1$$
  - For recurrent states $\Pi_i \neq 0$

Recurrent, transient and absorbing states (2)

- State i is **recurrent** if the system starting at such state will **surely** return to it **sooner or later** (i.e., in finite time):
  $f_{ii} = q_{ii}(\infty) = 1$
  - For recurrent states $\Pi_i \neq 0$
- State i is **transient** if the system starting at such state has a **finite probability** of **never** returning to it:
  $f_{ii} = q_{ii}(\infty) < 1$
  - For these states, at steady state $\Pi_i = 0$

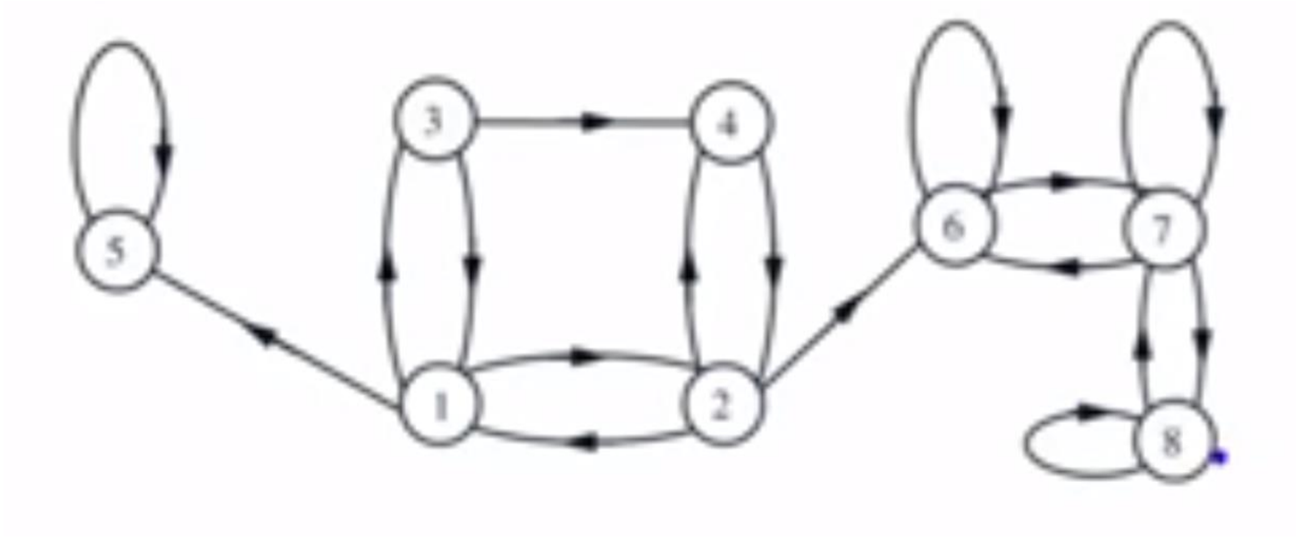


we **cannot** have a **finite Markov process** in which **all states** are **transients** because eventually it will leave them and **somewhere** it must go **at steady state**

- State i is **absorbing** if the system cannot leave it once it enters: $p_{ii} = 1$

Exercise 2

Classify the states of the following Markov Chain



Average Occupation Time of a State

S_i= number of consecutive time steps the system remains in state *i*

$\boldsymbol{E}[\boldsymbol{S_i}] = \boldsymbol{l_i}$= **Average occupation time** of state *i*

=

average number of time steps before the system exits state *i*

- Recalling that:

$p_{ii} =$ probability that the system "moves to" *i* in one time step, given that it was in *i*

$1 - p_{ii} =$ probability that the system exits *i* in one time step, given that it was in *i*

$$\mathrm{P}(S_i = n) = p_{ii}^n(1 - p_{ii})$$

$$S_i \sim \mathrm{Geom}(1 - p_{ii})$$

$$\boldsymbol{l_i} = \boldsymbol{E}[\boldsymbol{S_i}] = \frac{1}{1 - p_{ii}}$$

Univariate Discrete Distributions, Geometric Distribution

$p = \mathrm{P}\{\text{Failure}\}$ FAILURE=*Exit from the STATE*; $p = 1 - p_{ii}$

T= trail of the first experiment whose outcome is "failure" (or number of trials between two successive occurrences of failure);

S_i= number of consecutive time steps the system remains in state i → $S_i = T - 1$

The probability mass function:

$$\boxed{g(t; p) = (1-p)^{t-1} p}$$

t=1, 2,…

$$g(S_i, 1 - p_{ii}) = p_{ii}(1 - p_{ii})^{S_i}$$

S_i=0,1,…

Expected value of T (or return period):

$$E[T] = \sum_{t=1}^{\infty} t(1-p)^{t-1} p = p[1 + 2(1-p) + 3(1-p)^2 + \ldots] = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}$$

$$\boldsymbol{E}[\boldsymbol{S}_i] = \frac{1}{1 - p_{ii}}$$