Laboratory of Signal and Risk Analysis

POLITECNICO MILANO 1863

Exercise Session 23/02/2024
Probabilistic Models

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Exercise 1 (4.1 from green book)

Ten compressors, each one with a failure probability of 0.1, are tested independently.

1. What is the expected number of compressors that are found failed?
2. What is the variance of the number of compressors that are found failed?
3. What is the probability that none will fail?
4. What is the probability that two or more will fail?

Exercise 1 – Solution

$N = 10$ compressors, $p_i = 0.1\ \forall$ compressor $i = 1, \dots, N$, tested independently.

Compressor either failed or not. Let's call its state $Y = \begin{cases} 0, & failed\ with\quad p \\ 1, & not\ failed\ with\ \ 1-p \end{cases}$

Each compressor failure event is a Bernoulli trial. Given that they are independent, the binomial distribution can be used: $Bin(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$

1) $E[k] = \cdots = np = 10 * 0.1 = 1$

2) $Var[k] = \cdots = np(1-p) = 0.9$

3) $P(no\ failure) = P(Y_1 = 1, \dots Y_{10} = 1) = Bin(k = 0; n = 10, p = 0.1) = 0.349$

4) $P(two\ or\ more\ will\ fail) = P(k \geq 2|n, p) = \begin{cases} \sum_{k=2}^{10} Bin(k; n, p) \to too\ long \\ 1 - P(k < 2|n, p) \end{cases}$

$$= 1 - P(k < 2|n, p) = 1 - P(k = 0|n, p) - P(k = 1|n, p) = 0.264$$

(Note: $P(two\ or\ more\ will\ fail) \neq P(2\ compressors\ failed) \to P(k \geq 2) \neq P(k = 2)$)

Exercise 2 (4.9 from green book)

A machine has been observed to survive a period of 100 hours without failure with probability 0.5. Assume that the machine has a constant failure rate λ.

1. Determine the failure rate λ.
2. Find the probability that the machine will survive 500 hours without failure.
3. Determine the probability that the machine fails within 1000 hour, assuming that the machine has been observed to be functioning at 500 hours.

Exercise 2 – Solution

Constant failure rate → exponential distribution for the failure time: $T \sim e^{-\lambda t}$, such that P(T>100)=0.5. This type of data is usually called right censored: we only know that the component didn't fail during the test. This can happen for example if long tests are expensive or if limited amount of time is available for testing.

$$1) P(T > T_1) = 0.5 = R(T_1) = e^{-\lambda T_1} \rightarrow \lambda = \frac{ln2}{T_1} = 6.9 * 10^{-3}$$

$$2) P(T > T_2) = R(T_2) = e^{-\lambda T_2} = 0.032$$

$$3) P(T < 1000 | T > 500) = \frac{P(500<T<1000)}{P(T>500)} = \frac{R(500)-R(1000)}{R(500)} = 0.97$$

Exercise 3

Consider a system of two independent components with exponentially distributed failure times. The failure rates are λ_1 and λ_2 , respectively.

Determine the probability that component 1 fails before component 2.

Exercise 3 – Solution

Component 2 outlives component 1 if, for example, the failure of component 1 occurs at a time T_1 within a time interval (t, $t + dt$) and the failure of component 2 occurs at a time T_2 after t. The probability of this event is:

$$P\left(T_2 > t \middle| T_1 = t\right) \cdot f_{T_1}(t)\, dt$$

where

$$f_{T_1}(t)\, dt = \lambda_1 e^{-\lambda_1 t} dt$$

Furthermore, given the assumption of independent components, the conditional probability $P(T_2 \geq t \mid T_1 = t)$ is equal to $P(T_2 > t) = e^{-\lambda_2 t}$

Then, all the contributions for any time interval *(t, t + dt)* must be summed to give the required probability $P(T_2 > T_1)$:

$$P(T_2 > T_1) = \int_0^\infty P\left(T_2 > t \middle| T_1 = t\right) \cdot f_{T_1}(t)\, dt$$

$$= \int_0^\infty e^{-\lambda_2 t} \cdot \lambda_1 \cdot e^{-\lambda_1 t} dt = \lambda_1 \cdot \int_0^\infty e^{-(\lambda_1 + \lambda_2)t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Exercise 3 – Solution

NOTE

This result can be easily generalized to a system of n independent components with failure rates $\lambda_1, \lambda_2, ..., \lambda_n$. The probability that component j is the first one to fail is:

$$P(\text{component } j \text{ fails first}) = \frac{\lambda_j}{\sum_{i=1}^{n} \lambda_i}.$$

Exercise 4 (HOMEWORK)

The reliability engineer of a nuclear power plant is unsure that the installed alarm system, composed by a single alarm, is reliable enough. If the reactor enters an unsafe condition, the probability that the alarm triggers is 0.99. Assume also that if the reactor is safe, the probability that the alarm will not trigger is still 0.99.

Suppose that the reactor is in unsafe conditions only one day out of 100.

1. What is the probability that the reactor is in unsafe conditions if the alarm goes off?
2. If we add a second alarm (identical to the first one), what is the probability that the reactor is in unsafe conditions if also the second alarm goes off? Comment the results.

Exercise 5 (4.3 from green book)

Consider the occurrence of misprints in a book and suppose that they occur at the rate of 2 per page.

1. What is the probability that the first misprint will not occur in the first page?
2. What is the expected number of pages until the first misprint appears?
3. Comment on the applicability of the Poisson assumption (independence, homogeneity, fixed period) in this case.

Exercise 5 – Solution

Occurrence of events (misprints) in a continuum period (pages, book). So this guides us towards the Poisson distribution: $Poi(k;(0,t),\lambda) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$, t: n° of pages, k: misprints, λ: rate of misprints.

1) $P(first\ misprint\ not\ in\ first\ page) = Poi(k = 0|(0, t = 1), \lambda = 2) = \frac{(2*1)^0}{0!} e^{-2*1} = 0.1353$
2) Be careful on the tricky question! The text asks for the expected number of pages, that we called t, until the first misprint (k) appears. Thus, we are checking for the first occurrence of the event “misprint” in a set number of pages. The misprint can either be present with a probability p or not with probability $1 - p$. The number of pages up until the event can be modeled through a geometric distribution: $(1 - p)^{t-1} p$.

   From the first point we know that we fail at finding the first occurrence of a misprint in the first page with probability 0.1353. If we assume that this probability holds for each page t, we can asses: 1-p=0.1353.

   Thus: E[t]=1/p=1.16.
3) We can assume that the misprint rate is constant and independent on the number of pages printed, that the misprint events are independent (and can be counted) and that two misprints cannot happen in the same page (is it realistic given the rate?).

Exercise 6 (4.7 from green book)

An aircraft flight panel is fitted with two types of artificial horizon indicators. The time to failure t of each indicator from the start of a flight follow and exponential distribution with a mean value of 15 hours for the first type and 30 hours for the second type. A flight lasts for a period of 3 hours.

1. What is the probability that the pilot will be without an artificial horizon indication by the end of a flight?
2. What is the mean time to this event, if the flight is of a long duration?

Exercise 6 – Solution

The flight panel is characterized by a flight indicator A with a redundancy B. The two indicators are both modeled with an exponential distribution (similar component in the system…) but with different parameter (for example we can think about one being newer that the other). We are interested in the reliability of the system. Therefore, we have to model their time to failure. From the text:

$f_{T_A}(t) = \lambda_A e^{-\lambda_A t}$, such that $E[T_A] = 15h = \frac{1}{\lambda_A}$

$f_{T_B}(t) = \lambda_B e^{-\lambda_B t}$, such that $E[T_B] = 30h = \frac{1}{\lambda_B}$

1) We can assume independence (no common cause failure from the text). So:

$$P(failure\ of\ panel\ before\ T = 3h) = P(failure\ of\ A\ before\ 3h\ AND\ failure\ of\ B\ before\ 3h)$$
$$= P(T_A \leq T = 3h)P(T_B \leq T = 3h) = \left(1 - e^{-\lambda_A T}\right)\left(1 - e^{-\lambda_B T}\right) = 0.01725$$

2) Losing the panel before time t = Failure of the system before time t = failure of both A and B before time t. If we call $f_{AB}(t)$ the system failure distribution the answer to the question is the expected value, which is called Mean Time To Failure:

$$MTTF = \int_0^\infty t f_{AB}(t)dt =^* \int_0^\infty R_{AB}(t)dt = \int_0^\infty 1 - F_{AB}(t)dt = \int_0^\infty 1 - \left(1 - e^{-\lambda_A T}\right)\left(1 - e^{-\lambda_B T}\right)dt = \int_0^\infty e^{-\lambda_A t} + e^{-\lambda_B t} - e^{-(\lambda_A+\lambda_B)t}dt = \left(-\frac{1}{\lambda_A}e^{-\lambda_A t} - \frac{1}{\lambda_B}e^{-\lambda_B t} + \frac{1}{\lambda_A+\lambda_B}e^{-(\lambda_A+\lambda_B)t}\right)\Big|_0^\infty = \frac{1}{\lambda_A} + \frac{1}{\lambda_B} + \frac{1}{\lambda_A+\lambda_B} = 55h$$

(Note: the equality $=^*$ can be demonstrated through the integration per part and the limit of the reliability function, and it holds for every distribution function).

Exercise 7 (4.11 from green book)

In considering the safety of a building, the total force acting on the columns of the building must be examined. This would include the effect of the dead load D (due to the weight of the structure), the live load L (due to the human occupancy, movable furniture…) and the wind load W. Assume that the load effects on the individual columns are statistically independent and follow a Gaussian distribution with:

$$\mu_D = 4.2\ kips \qquad \sigma_D = 0.3\ kips$$
$$\mu_L = 6.5\ kips \qquad \sigma_L = 0.8\ kips$$
$$\mu_W = 3.4\ kips \qquad \sigma_W = 0.7\ kips$$

1. Determine the mean and standard deviation of the total load acting on a column.
2. If the strength R of the column is also Gaussian with a mean equal to 1.5 times the total mean force, what is the probability of failure of the column? Assume that the coefficient of variation of the strength δ_R is 15% and the strength and load effects are statistically independent.

Exercise 7 – Solution

W,L,D are statistically independent such that:

$$W \sim \mathrm{N}(\mu_W, \sigma_W), \qquad L \sim \mathrm{N}(\mu_L, \sigma_L), \qquad D \sim \mathrm{N}(\mu_D, \sigma_D),$$

1) Total load : $T = W + L + D \sim N\left(\mu_W + \mu_L + \mu_D, \sqrt{\sigma_W^2 + \sigma_L^2 + \sigma_D^2}\right) = N(14.1, 1.1)$

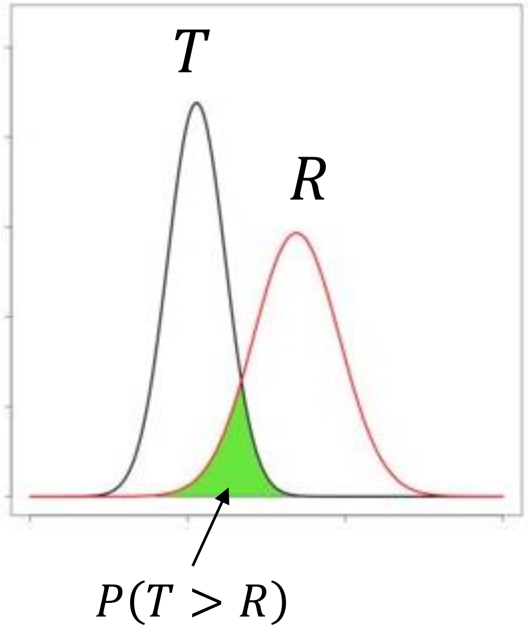
2) Strength of the column: $R \sim N(\mu_T * 1.5, \delta_R * \mu_T * 1.5)$

Column fails if T>R, so: P(failure)=P(T>R). T and R are gaussian random variables, so it is easier to work with their difference, which we know to be still gaussian, such that: $X = T - R \sim N\left(\mu_T - \mu_R, \sqrt{\sigma_T^2 + \sigma_R^2}\right) = N(-7.05, 3.35)$.

Therefore: $P(T > R) = P(X > 0)$, and we can solve the problem by standardizing the normal distribution, that is:

$$P(X > 0) = P\left(\frac{X - \mu_X}{\sigma_X} > \frac{0 - \mu_X}{\sigma_X}\right) = P\left(S > \frac{7.05}{3.35}\right) = 1 - P(S \leq 2.102)$$

$$= 1 - \phi(2.102) = 0.018$$



Exercise 8 (4.15 from green book)

The following relationship arises in the study of the earthquake-resistant design:

$$Y = ce^{X}$$

where Y is the ground motion intensity at the building site, X is the magnitude of an earthquake and c is related to the distance between the site and center of the earthquake.

If X is exponentially distributed,

$$f_X(x) = \lambda e^{-\lambda x}, \qquad x \geq 0$$

find the cumulative distribution function of Y, $F_Y(y)$.

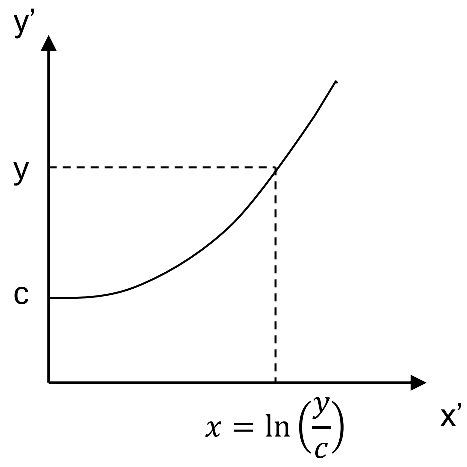
Exercise 8 – Solution

$$Y = ce^X \quad \rightarrow \quad \ln\left(\frac{y}{c}\right) = x \quad (1)$$

$$f_X(x) = \lambda e^{-\lambda x} \quad \rightarrow \quad F_X(x) = 1 - e^{-\lambda x} \quad (2)$$

$$F_Y(y) = P(Y \le y) = P(ce^X \le y) \Rightarrow P\left(X \le \ln\left(\frac{y}{c}\right)\right)$$

$$= F_X\left(\ln\left(\frac{y}{c}\right)\right) = 1 - e^{-\lambda \ln\left(\frac{y}{c}\right)} = 1 - e^{\ln\left(\frac{c}{y}\right)^\lambda} = 1 - \left(\frac{c}{y}\right)^\lambda$$



X=Magnitude

Y=Ground-motion intensity

Other solution (invariance under change of variable):

$$f_X(x)dx = f_Y(y)dy$$

$$f_Y(y) = f_X(x) \cdot \frac{1}{\frac{dy}{dx}} = \lambda e^{-\lambda x} \frac{1}{ce^x} = \lambda e^{-\lambda \ln\left(\frac{y}{c}\right)} \frac{1}{ce^{\ln\left(\frac{y}{c}\right)}} = \lambda e^{\ln\left(\frac{c}{y}\right)^\lambda} \frac{1}{y} = \lambda c^\lambda y^{-1-\lambda}$$

$$\overset{y \ge c}{F_Y(y)} = \int_c^y f_Y(y^*)dy^* = \lambda c^\lambda \int_c^y y^{*-1-\lambda} dy^* = \lambda c^\lambda \cdot \frac{y^{*-\lambda}}{-\lambda}\Big|_c^y = 1 - \left(\frac{c}{y}\right)^\lambda$$

Exercise 9 (HOMEWORK)

Suppose that, from a previous traffic count, an average of **60 cars per hour** was observed to make left turns at an intersection.

What is the probability that exactly 10 cars will be making left turns in a 10 minute interval?

Discretize the time interval of interest to approach the problem with the binomial distribution. Show that the solution of the problem tends to the exact solution obtained with the Poisson distribution as the time discretization gets finer.

Exercise 10 (HOMEWORK)

A capacitor is placed across a power source. Assume that surge voltages occur on the line at a rate of one per month and they are normally distributed with a mean value of 100 volts and a standard deviation of 15 volts. The breakdown voltage of the capacitor is 135 volts.

1. Find the Mean Time To Failure (MTTF) for this capacitor;
2. Find its reliability for a time period of one month.

Where to find more exercises…