POLITECNICO DI MILANO

Basic notions of probability theory

Contents

- **Boolean Logic**
- **Definitions of probability**
- **Probability laws**

Why a Lecture on Probability?

Lecture 1, Slide 22:

Risk

RISK = **POTENTIAL DAMAGE** + UNCERTAINTY

Dictionary: RISK = possibility **of damage or injury to people or things**

1) What undesired conditions may occur? → Accident Scenario, **S**
2) With what probability do they occur? → Probability, **p**
3) What damage do they cause? → Consequence, **x**

$$\text{RISK} = \{S_i, p_i, x_i\}$$

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Basic Definitions

Definitions: experiment, sample space, event

- **Experiment ε:** process whose outcome is a priori unknown to the analyst (all possible outcomes are a priori known)

- **Sample space Ω:** the set of all possible outcomes of ε.

- **Event *E*:** a set of possible outcomes of the experiment ε (a subset of Ω):

the event E occurs when the outcome of the experiment ε is one of the elements of E.

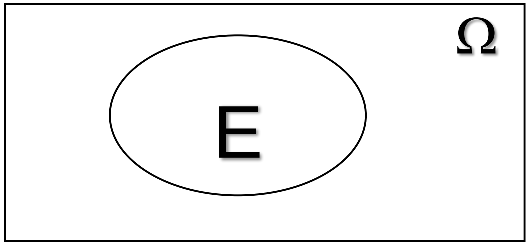
Boolean Logic

Definition: Certain events → Boolean Logic

***Logic of certainty*: an event** *E* can either **occur** or **not occur**

Indicator variable $X_E = \begin{cases} 0, \text{ when } E \text{ does not occur} \\ 1, \text{ when } E \text{ occurs} \end{cases}$

Certain Events (Example)



(ε = die toss;Ω={1,2,3,4,5,6};E=Odd number)

I perform the experiment and the outcome is ‘3’

Event
E, X_E

True
$X_E=1$

Boolean Logic Operations

- **Negation:** $\overline{E}$
- **Union:** $X_{A \cup B}$
- **Intersection:** $X_{A \cap B}$

Boolean Logic Operations

- **Negation:** $\overline{E} \rightarrow \overline{X_E} = 1 - X_E$
- **Union:**

$$X_{A\cup B} = 1-(1-X_A)(1-X_B) = 1-\prod_{j=A,B}(1-X_j) =$$

$$= \coprod_{j=A,B} X_j = X_A + X_B - X_A X_B$$

- **Intersection:**

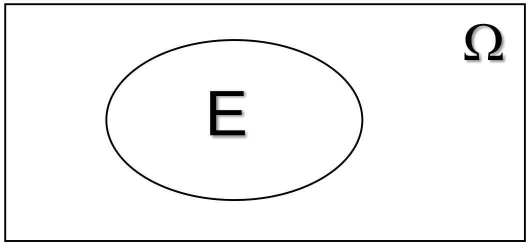
$$X_{A\cap B} = X_A X_B$$

- **Definition: *A* and *B* are mutually exclusive events if** $X_{A\cap B} = 0$

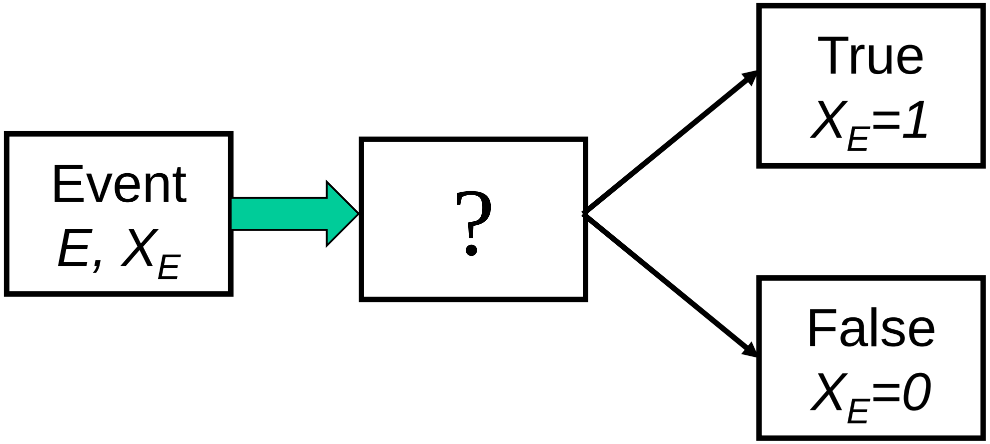
Uncertain Events

Uncertain Events

Let us consider: the experiment ε, its sample space Ω and the event *E*.

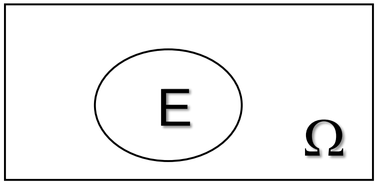


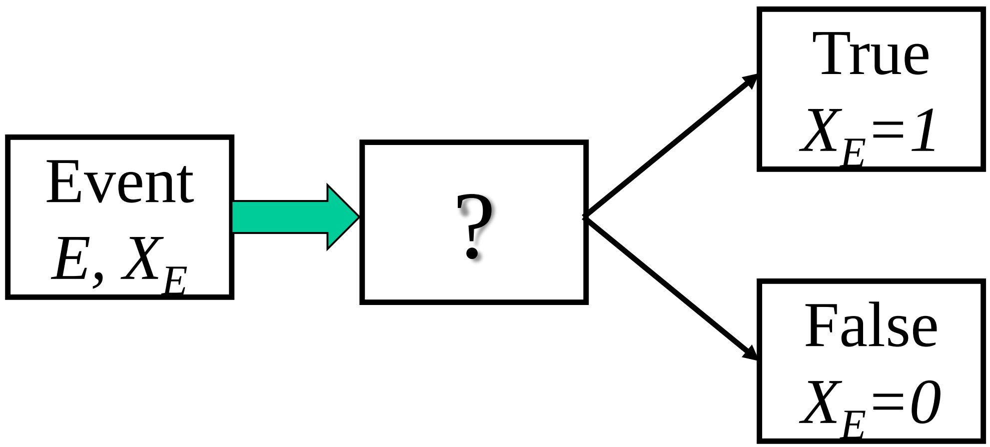
(ε = die toss; Ω={1,2,3,4,5,6}; E=Odd number)



Uncertain Events:

Let us consider: the experiment ε, its sample space Ω, event E.





Uncertain events can be compared → probability of E = p(E)

Probability for comparing the likelihood of events

Money for comparing the value of objects

Probability theory

Probability theory: Kolmogorov Axioms

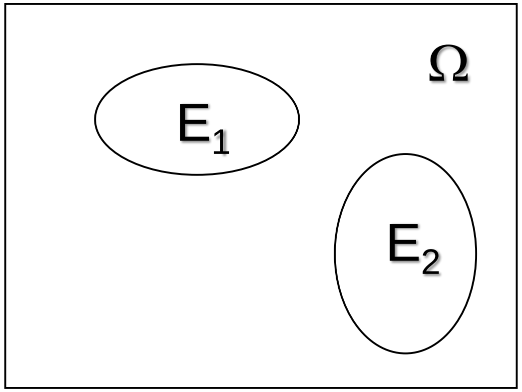
1. $0 \le p(E) \le 1$

2. $p(\Omega) = 1 \quad p(\varnothing) = 0$

3. Addition law:

   Let E_1, …, E_n be a finite set of mutually exclusive events:
   $(X_{E_i} \cap X_{E_j} = \emptyset)$.

$$p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} p(E_i)$$



Definitions of probability

Three definitions of probability

1. **Classical definition**
2. **Empirical Frequentist Definition**
3. **Subjective definition**

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$ and the event:

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is it applicable?

- Gambling (e.g. tossing of a die)
- If no evidence favouring one outcome over others

$$E = A_1 \, \mathrm{Y} \, A_2 \, \mathrm{Y} \ldots \mathrm{Y} \, A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability (criticisms)

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is this requirement met?

In most real life situations the outcomes are not equally probable!

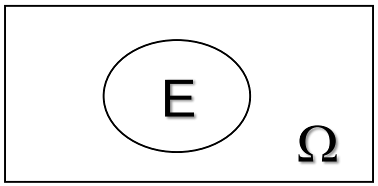
$$E = A_1 \,\mathrm{Y}\, A_2 \,\mathrm{Y} \ldots \mathrm{Y}\, A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

2. Frequentist Definition of Probability

Let us consider: the experiment ε, its sample space Ω and an event *E*.

ε = die toss;Ω={1,2,3,4,5,6};E={Odd number}



- *n* times ε, *E* occurs *k* times

  (*n* = 100 die tosses → *k*=48 odd numbers)

- *k/n* = the relative frequency of occurrence of *E*

  (k/*n* = 48/100=0.48)

$$\lim_{n \to \infty} \frac{k}{n} = p$$

p is defined as the probability of *E*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty}\frac{k}{n}=p$$

- This is not a limit from the matemathical point of view [limit of a numerical series]
- It is not possible to repeat an experiment an infinite number of times …
- We tacitly assume that the limit exists

Possible way out?

probability as a physical characteristic of the object:

- the physical characteristics of a coin (weight, center of mass, …) are such that when tossing a coin over and over again the fraction of 'head' will be *p*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

- Applicable only to those events for which we can conceive of a repeatable experiment (e.g. not to the event «your professor will be sick tomorrow»)
- The experiment conditions cannot be identical
  - let us consider the probability that a specific valve *V* of a specific Oil & Gas plant will fail during the next year
  - *what should be the population of similar valves?*
    - Large population: all the valves used in industrial plants. Considering data from past years, we will have a large number *n,* but data may include valves very different to *V*
    - Small population: valve used in Oil&Gas of the same type, made by the same manufacturer with the same technical characteristics $\rightarrow$ too small *n* for limit computation.

Similarity Vs population size dilemma

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

Some events (e.g. in the nuclear industry) have very low probabilities (e.g. $p \approx 10^{-6}$) (RARE EVENTS)

↓

Very difficult to observe

↓

The frequentist definition is not applicable

3. Subjective Definition of Probability

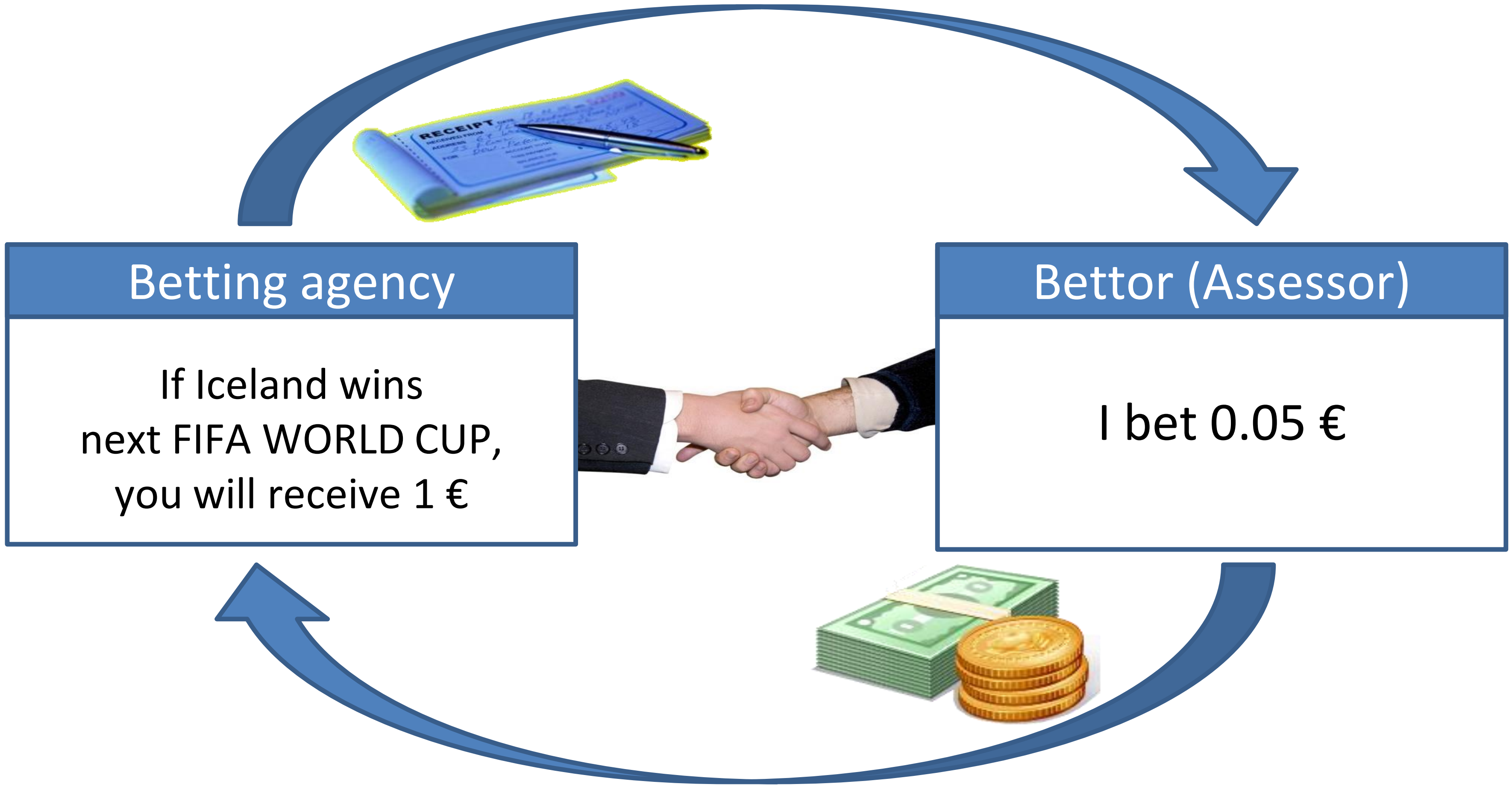
P(*E*) is the **degree of belief** that a person (assessor) has that E will occour, **given all the relevant information currently known to that person** (background knowledge)

- Probability is a numerical encoding of the state of knowledge of the assessor (De Finetti: “probability is the feeling of the analyst towards the occurrence of the event”)
- P(*E*) is conditional on the background knowledge *K* of the assessor:

  $$\mathbf{P(E) = P(E|K)}$$
- Background knowledge typically includes data/models/expert knowledge
- If the background knowledge changes → the probability may change
- Two interpretations of subjective probability:
  - Betting interpretation
  - Reference to a standard for uncertainty

3. Betting Interpretation

P {Iceland will win next UEFA EURO 2020|K} = 0.05

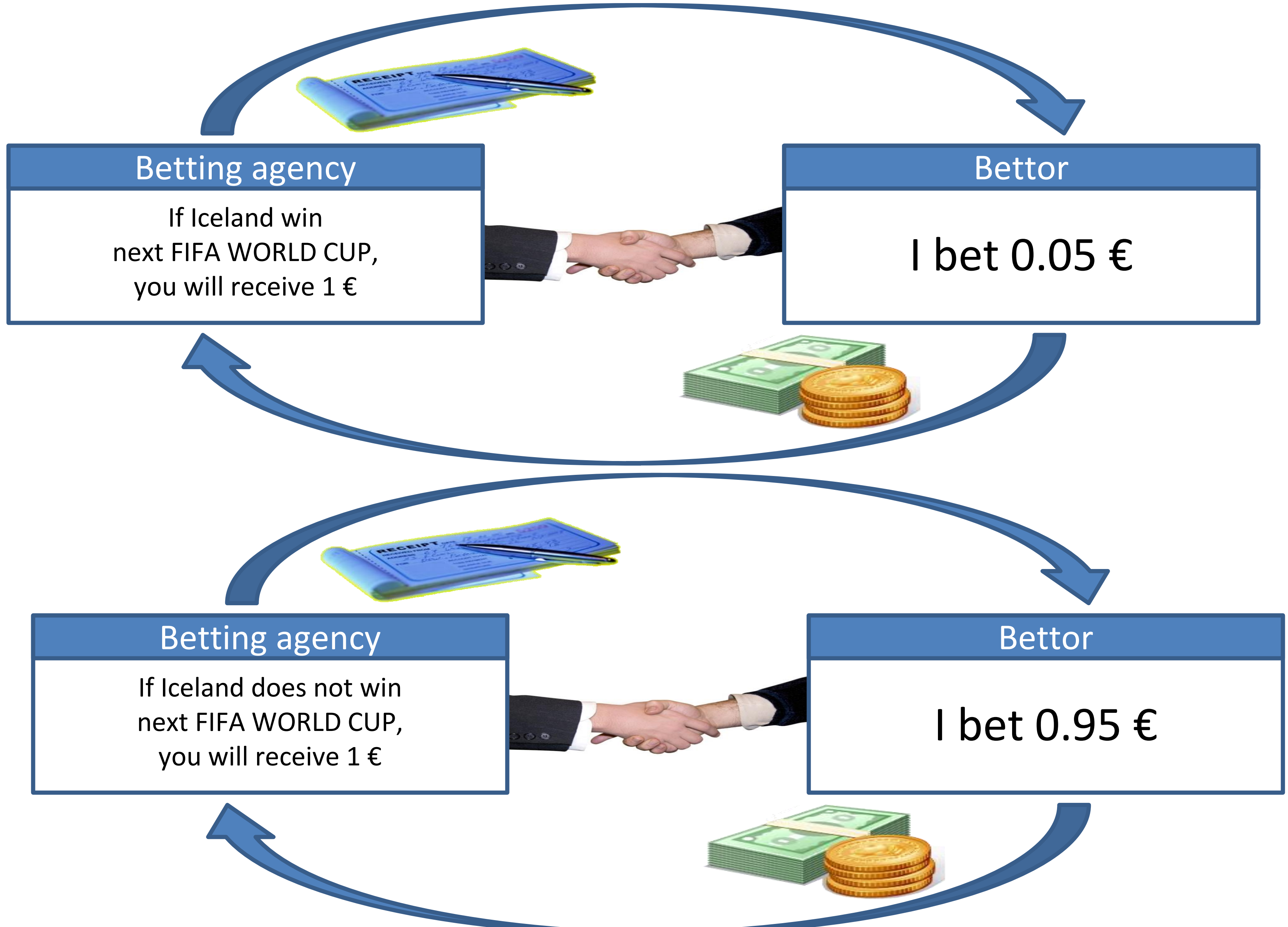


3. Betting Interpretation

P(E) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

The opposite must also hold: 1- P(*E*) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event E were not to occur and nothing otherwise.

3. Betting Interpretation: two sideness of the bet



3. Betting Interpretation (Criticism)

P(*E*) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

The opposite must also hold (1- P(*E*)) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event E were not to occur and nothing otherwise.

probability assignment depends from the value judgment about money and event consequences (the assessor may even think that in case of LOCA in a Nuclear Power Plant he/she will die and so the payment will be useless)

3. Reference to a standard for uncertainty

P(*E*) is the number such that the uncertainty about the occurrence of *E* is considered equivalent by the person assigning the probability (assessor) to the uncertainty about drawing a red ball from an urn containing P(*E*)*100% red balls

E={Germany will win next FIFA WORLD CUP}

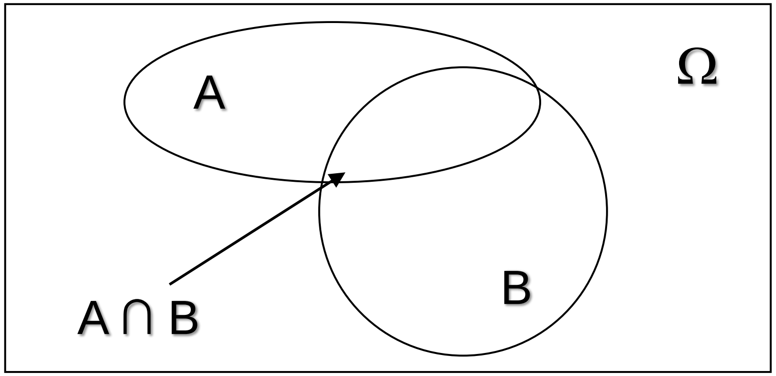
$$P(E)=0.33$$

urn

Probability laws

Probability laws (1)

- Union of two non-mutually exclusive events



$$P_{A\cup B} = P_A + P_B - P_{A\cap B}$$

It can be demonstrated by using the three Kolmogorov axioms*

$$P_{A\cup B} \leq P_A + P_B$$

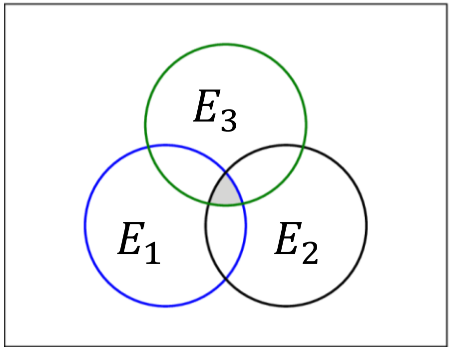
- Rare event approximation: A and B events are considered as mutually exclusive ($A \cap B = \emptyset$) → $P(A \cap B) = 0$ →

$$P_{A\cup B} = P_A + P_B$$

* http://www.ucs.louisiana.edu/~jcb0773/Berry_probbook/425chpt2.pdf

Probability laws (2)

- Union of non-mutually exclusive events: $E_{\cup} = \bigcup_{i=1,\dots,n} E_i$



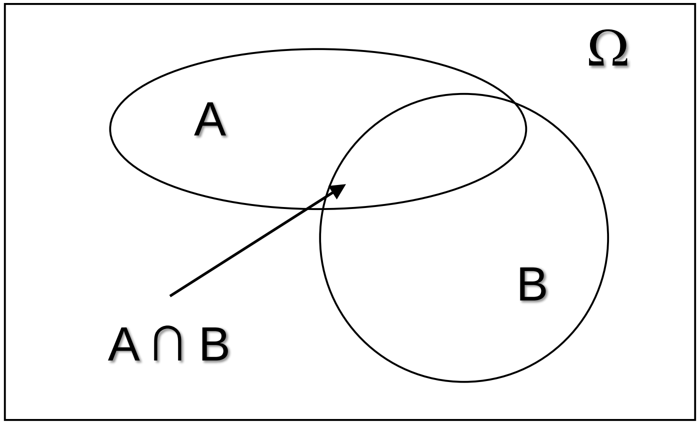
$$P(E_{\cup}) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P\left(E_i \cap E_j\right) + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

- Upper bound $P(E_U) \le \sum_{j=1}^{n} P(E_j)$
- Lower bound $P(E_U) \ge \sum_{j=1}^{n} P(E_j) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(E_i \cap E_j)$
- Rare event approximation: events are considered as mutually exclusive $(E_i \cap E_j = \emptyset, \forall i, j, i \neq j) \rightarrow P(E_{\cup}) = \sum_{i=1}^{n} P(E_i)$

Probability laws (3)

- Conditional Probability of *A* given *B*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



- Event *A* is said to be statistically independent from event *B* if:

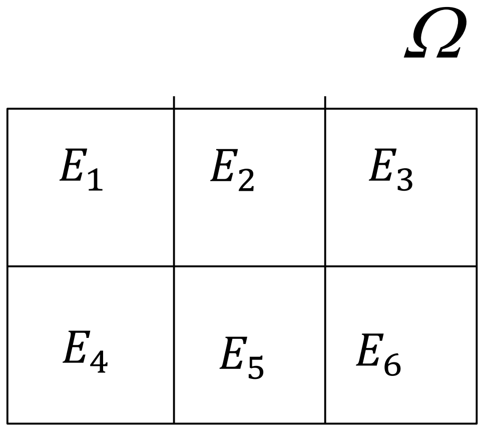
$$P(A|B) = P(A)$$

- If *A* and *B* are statistically independent then:

$$P(A \cap B) = P(A)P(B)$$

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:

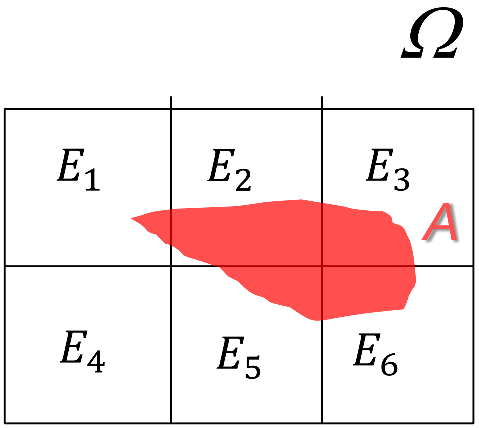


$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad\qquad \bigcup_{j=1}^{n} E_j = \Omega$$

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:

$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad\qquad \bigcup_{j=1}^{n} E_j = \Omega$$

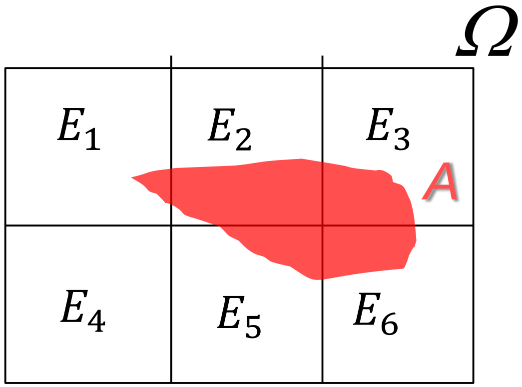


- Given any event A in Ω, its probability can be computed in terms of the partitioning events and the conditional probabilities of A on these events: $A = \cup_j \left(A \cap E_j\right) \rightarrow P(A) = \sum_j P(A \cap E_j)$

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n)$$

Bayes Theorem

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events E_j. We know



- Event A has occurred

Can I use this information to update the probability of $P(E_j)$?

$$P(E_i \mid A) = \frac{P(E_i A)}{P(A)} = \frac{P(A \mid E_i)P(E_i)}{\sum_{j=1}^{n} P(A \mid E_j)P(E_j)}$$

theorem of total probability

The Bayesian Subjective Probability Framework

P(E|K) is the **degree of belief** of the **assigner** with regard to the occurrence of E (numerical encoding of the **state of knowledge** – K - of the assessor)

Bayes Theorem to update the probability assignment in light of new information

Updated

Old

$$P(E_i|A,K) = \frac{P(A|E_i,K) \cdot P(E_i|K)}{\sum_{j=1}^{n} P(A|E_j,K) \cdot P(E_j|K)}$$

new information