POLITECNICO DI MILANO

Interpretation of Probability in Reliability and Risk Analysis

Contents

- Boolean Logic
- Definitions of probability

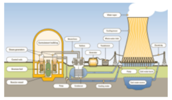
Why a Lecture on Probability?

Lecture 1, Slide 22:

*«Operative definition of **reliability**: **Probability** that an item performs its required function, under given environmental and operational conditions for a stated period of time”*

Lecture 1, Slide 42:

Probabilistic Risk Assessment




1. **What undesired conditions may occur?** → **Accident Scenario, S**
2. **With what probability do they occur?** → **Probability, p**
3. **What damage do they cause?** → **Consequence, x**

$$\textbf{RISK} = \{ S_i, p_i, x_i \}$$

Basic Definitions

Definitions: experiment, sample space, event

- **Experiment ε:** process whose outcome is a priori unknown to the analyst (all possible outcomes are a priori known)

- **Sample space Ω:** the set of all possible outcomes of ε.

- **Event *E*:** a set of possible outcomes of the experiment ε (a subset of Ω)

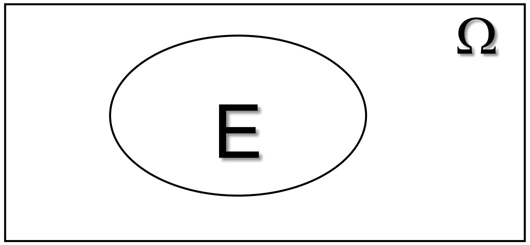
Boolean Logic

Definition: Certain events → Boolean Logic

Logic of certainty: **an event** *E* can either **occur** (the outcome of the experiment ε is one of the elements of *E*) or **not occur**

Indicator variable $\boldsymbol{X_E} = \begin{cases} 0, \text{ when } E \text{ does not occur} \\ 1, \text{ when } E \text{ occurs} \end{cases}$

Certain Events (Example)



(ε = die toss;Ω={1,2,3,4,5,6};E=Odd number)

I perform the experiment and the outcome is ‘3’

Event
E, X_E

True
$X_E=1$

Boolean Logic Operations

- **Negation:** $\bar{E}$

Boolean Logic Operations

- **Negation:** $\overline{E}$
- **Union:** $A \cup B$

Boolean Logic Operations

- **Negation:** $\overline{E}$
- **Union:** $A \cup B$
- **Intersection:** $A \cap B$

Boolean Logic Operations

- **Negation:** $\bar{E} \rightarrow X_{\bar{E}} = 1 - X_E$
- **Union:** $A \cup B \rightarrow X_{A \cup B} = X_A + X_B - X_A X_B =$
  $= 1 - (1 - X_A)(1 - X_B)$
- **Intersection:** $A \cap B \rightarrow X_{A \cap B} = X_A X_B$

Definition: *A* and *B* are mutually exclusive events if: $X_{A \cap B} = 0$

Boolean Logic Operations

- **Negation:** $\overline{E} \rightarrow \overline{X_E} = 1 - X_E$
- **Union:**

$$X_{A\cup B} = 1 - (1 - X_A)(1 - X_B) = 1 - \prod_{j=A,B} (1 - X_j) =$$

$$= \coprod_{j=A,B} X_j = X_A + X_B - X_A X_B$$

- **Intersection:**

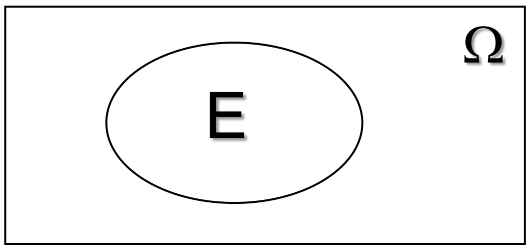
$$X_{A\cap B} = X_A X_B$$

- **Definition: *A* and *B* are mutually exclusive events if** $X_{A\cap B} = 0$

Uncertain Events

Uncertain Events

Let us consider: the experiment ε, its sample space Ω and the event *E*.



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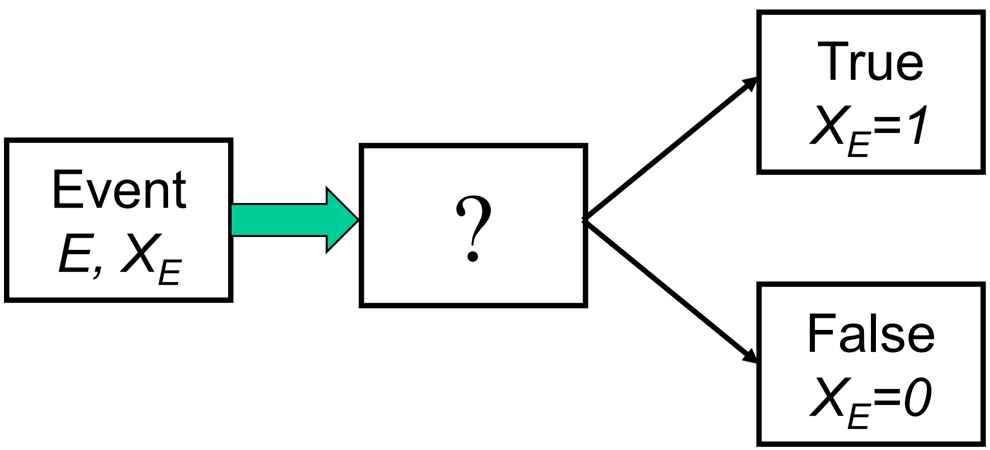
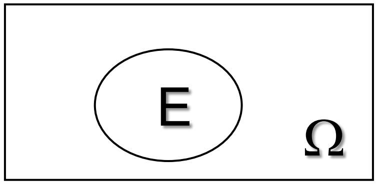
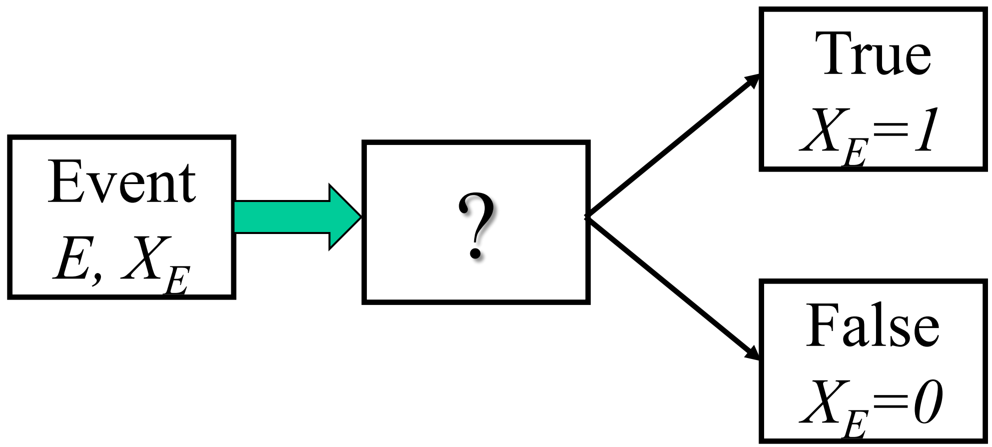


Uncertain Events:

Let us consider: the experiment ε, its sample space Ω, event *E*.



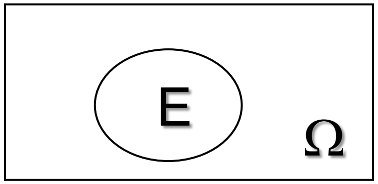


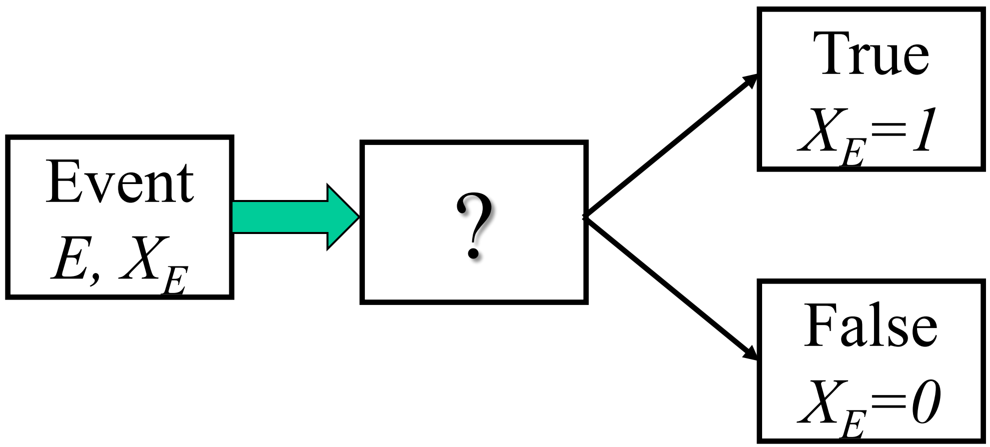
Uncertain events can be compared → probability of E = p(E)

Probability for comparing the likelihood of events

Uncertain Events:

Let us consider: the experiment ε, its sample space Ω, event E.





Uncertain events can be compared → probability of E = p(E)

Probability for comparing the likelihood of events

Money for comparing the value of objects

Definitions of probability

Three definitions of probability

1. **Classical definition**
2. **Empirical Frequentist Definition**
3. **Subjective definition**

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$ and the event:

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is it applicable?

- Gambling (e.g. tossing of a die)
- If no evidence favouring one outcome over others

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability (criticisms)

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is this requirement met?

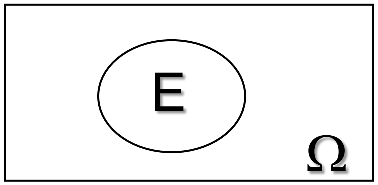
In most real life situations the outcomes are not equally probable!

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

2. Frequentist Definition of Probability

Let us consider: the experiment ε, its sample space Ω and an event *E*.



ε = die toss;Ω={1,2,3,4,5,6};E={Odd number}

- *n* times ε, *E* occurs *k* times

  (*n* = 100 die tosses → *k*=48 odd numbers)

- *k/n* = the relative frequency of occurrence of *E*

  (k/*n* = 48/100=0.48)

$$\lim_{n\to\infty}\frac{k}{n}=p$$

p is defined as the probability of *E*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n \to \infty} \frac{k}{n} = p$$

- This is not a limit from the mathematical point of view [limit of a numerical series]
- We tacitly assume that the limit exists
- It is not possible to repeat an experiment an infinite number of times …

Possible way out?

probability as a physical characteristic of the object:

- the physical characteristics of a coin (weight, center of mass, …) are such that when tossing a coin over and over again the fraction of 'head' will be *p*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n \to \infty} \frac{k}{n} = p$$

- Applicable only to those events for which we can conceive of a repeatable experiment (e.g. not to the event «Inter wins 2023/2024 champion league»)
- The experiment conditions cannot be identical
  - let us consider the probability that a specific valve *V* of a specific Oil & Gas plant will fail during the next year
  - *what should be the population of similar valves?*
    - Large population: all the valves used in industrial plants. Considering data from past years, we will have a large number *n,* but data may include valves very different to *V*
    - Small population: valve used in Oil&Gas of the same type, made by the same manufacturer with the same technical characteristics → too small *n* for limit computation.

Similarity Vs population size dilemma

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n \to \infty} \frac{k}{n} = p$$

Some events (e.g. in the nuclear industry) have very low probabilities (e.g. $p \approx 10^{-6}$)
(RARE EVENTS)

Very difficult to observe

The frequentist definition is not applicable

3. Subjective Definition of Probability

P(E) is the **degree of belief** that a person (assessor) has that E will occur, **given all the relevant information currently known to that person** (background knowledge)

- Probability is a numerical encoding of the state of knowledge of the assessor (De Finetti: "probability is the feeling of the analyst towards the occurrence of the event")
- P(E) is conditional on the background knowledge K of the assessor:
  $$\mathbf{P(E)= P(E|K)}$$
- Background knowledge typically includes data/models/expert knowledge
- If the background knowledge changes → the probability may change
- Two interpretations of subjective probability:
  - Betting interpretation
  - Reference to a standard for uncertainty

3. Betting Interpretation

P {Iran will win next FIFA world cup|K} = 0.05

Betting agency

If Iran wins
next FIFA WORLD CUP,
you will receive 1 €

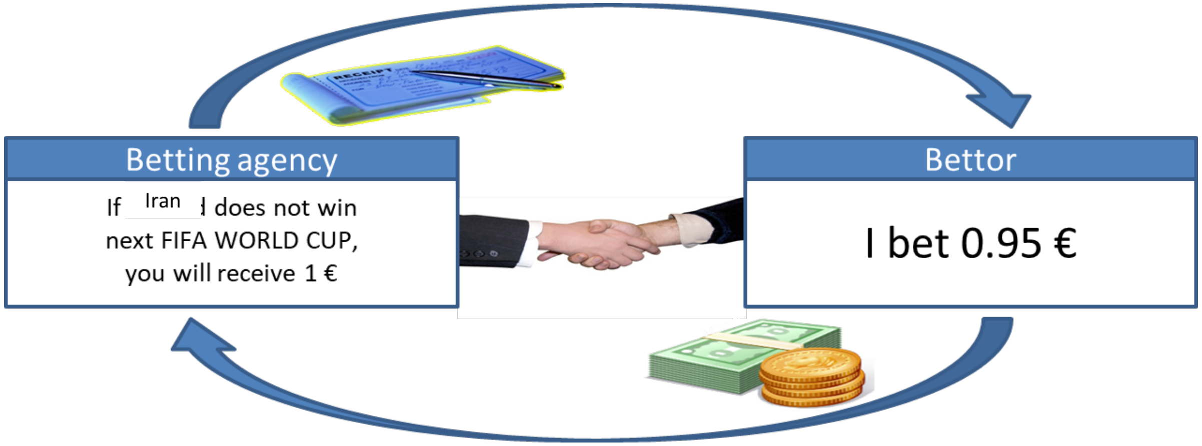
Bettor (Assessor)

I bet 0.05 €

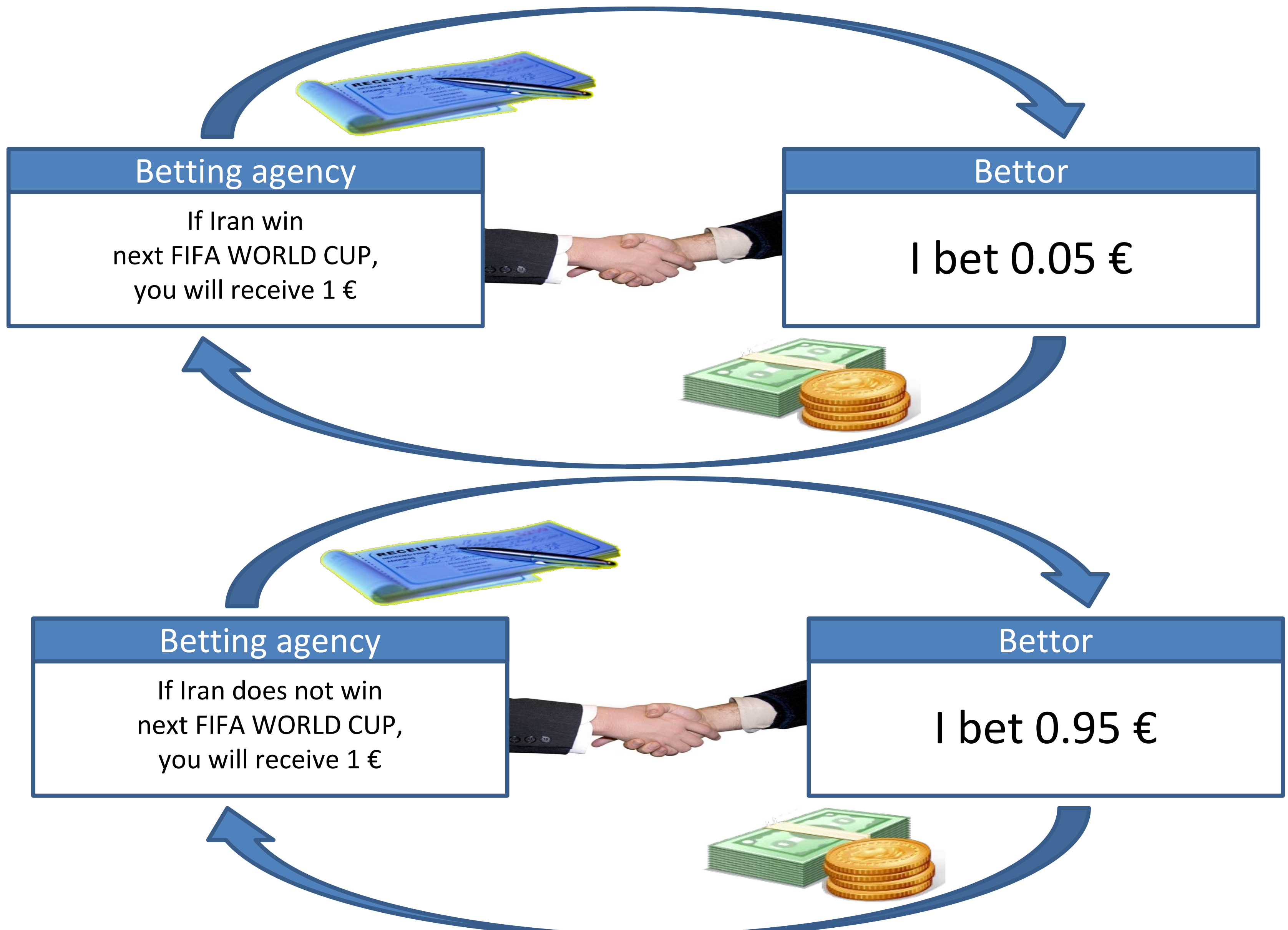
3. Betting Interpretation

P(E) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

Fair betting:
The opposite must also hold: 1- P(*E*) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were not to occur and nothing otherwise.



3. Betting Interpretation: two sideness of the bet



3. Betting Interpretation (Criticism)

P(*E*) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

The opposite must also hold (1- P(*E*)) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event E were not to occur and nothing otherwise.

- probability assignment depends from the value judgment about money and event consequences
  - Extreme case: the assessor may even think that in case of LOCA in a Nuclear Power Plant he/she will die and so the payment will be useless)

3. Reference to a standard for uncertainty

P(E|K) is the number such that the uncertainty about the occurrence of E is considered equivalent by the person assigning the probability (assessor) to the uncertainty about drawing a red ball from an urn of N balls containing P(E|K)*N red balls.

E={Germany will win next FIFA WORLD CUP}

$$P(E|K)=0.33$$

urn

NUREG 75/014

Nureg 75/014: Reactor Safety Study: An Assessment of Accident Risks in U.S. Commercial Nuclear Power Plants
(Rasmussen et al., 1975)

«The overall **probability** of a complete core meltdown is about $5 \cdot 10^{-5}$ per reactor per year»

NUREG 75/014: probability interpretation

Nureg 75/014: Reactor Safety Study: An Assessment of Accident Risks in U.S. Commercial Nuclear Power Plants (Rasmussen et al., 1975)

«The overall **probability** of a complete core meltdown is about $5 \cdot 10^{-5}$per reactor per year”

…

Interpretation?

…

“the likelihood of an average citizen’s being killed in a reactor accident is about the same as the chance of being killed by a falling meteorite”

«Widely quoted and much criticized statement»

Probability theory

Probability Theory: Kolmogorov Axioms

1. $0 \leq p(E) \leq 1$

2. $p(\Omega) = 1 \quad p(\emptyset) = 0$

3. Addition law:

   Let $E_1, \ldots, E_n$ be a finite set of mutually exclusive events: $(X_{E_i} \cap X_{E_j} = \emptyset)$.

$$p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} p(E_i)$$