



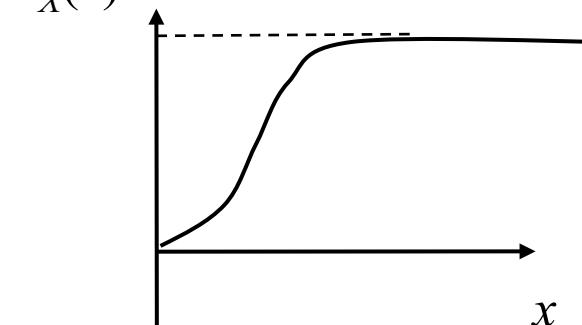
Basic notions of probability theory:  
• continuous probability distributions

# Probability distributions for reliability, safety and risk analysis:

- discrete probability distributions
- continuous probability distributions

# Probability functions (continuous random variables)

- Let  $X$  be a random variable which takes continuous values in  $\mathfrak{R}$
- Its cumulative distribution is  $F_X(x) = P(X \leq x)$

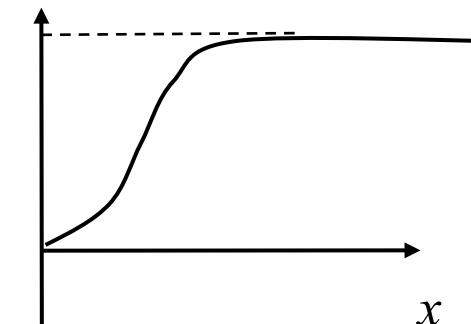


# Probability functions (continuous random variables)

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- Let us consider a small interval  $dx$ :

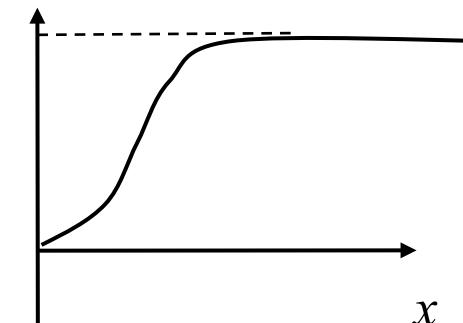
$$P(x \leq X < x + dx) = ???$$



# Probability functions (continuous random variables)

- Let  $X$  be a random variable which takes continuous values in  $\mathbb{R}$
- Its cumulative distribution is  $F_X(x) = P(X \leq x)$
- The probability density function  $f_X(x)$  is defined by:

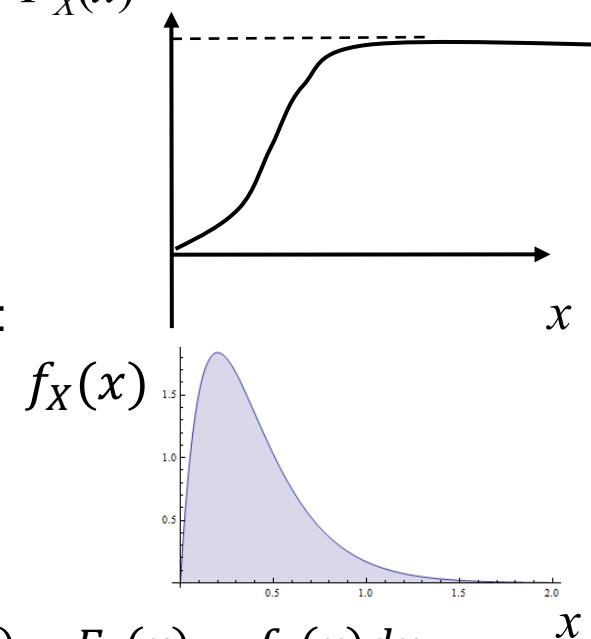
$$f_X(x) = \lim_{dx \rightarrow 0} \frac{F_X(x + dx) - F_X(x)}{dx}$$



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$$f_X(x) = \lim_{dx \rightarrow 0} \frac{F_X(x + dx) - F_X(x)}{dx} = \frac{dF_X}{dx}$$

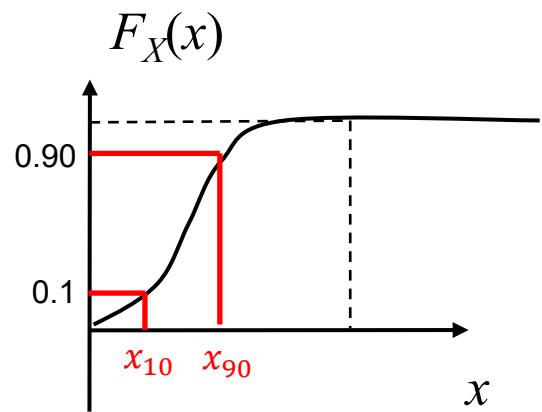


- Interpretation:  $P(x \leq X < x + dx) = F_X(x + dx) - F_X(x) = f_X(x)dx$
- $f_X(x)$  is **not** a probability but a probability per unit of  $x$  (probability density)
- $f_X(x) \geq 0$
- $\int_{-\infty}^{+\infty} f_X(x)dx = 1$
- $F_X(x) = \int_{-\infty}^x f_X(\xi)d\xi$

# Summary measures:*percentiles, median, mean, variance*

- Distribution Percentiles ( $x_\alpha$ ):

$$F_X(x_\alpha) = \frac{\alpha}{100}$$



# Summary measures:percentiles, median, mean, variance

- Distribution Percentiles ( $x_\alpha$ ):

$$F_X(x_\alpha) = \frac{\alpha}{100}$$

- Median of the distribution ( $x_{50}$ ):      **The probability to be below or above is equal**

$$F_X(x_{50}) = 0.5$$

- Mean Value (Expected Value):      **Where the probability mass is concentrated on average?**

$$\mu_X = E[X] = \langle X \rangle = \sum_{i=1}^n x_i p_i \quad (\text{discrete random variables})$$

$$= \int_{-\infty}^{\infty} x f_X(x) dx \quad (\text{continuous random variables})$$

- Variance ( $\text{var}[X]$ ):      **It is a measure of the dispersion of the values around the mean**

$$\sigma_X^2 = \sum_i (x_i - \mu_X)^2 p_i \quad (\text{discrete random variables})$$

$$= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx \quad (\text{continuous random variables})$$

## Exercise 1

Assume that a random variable  $X$  is described by a pdf of the form:

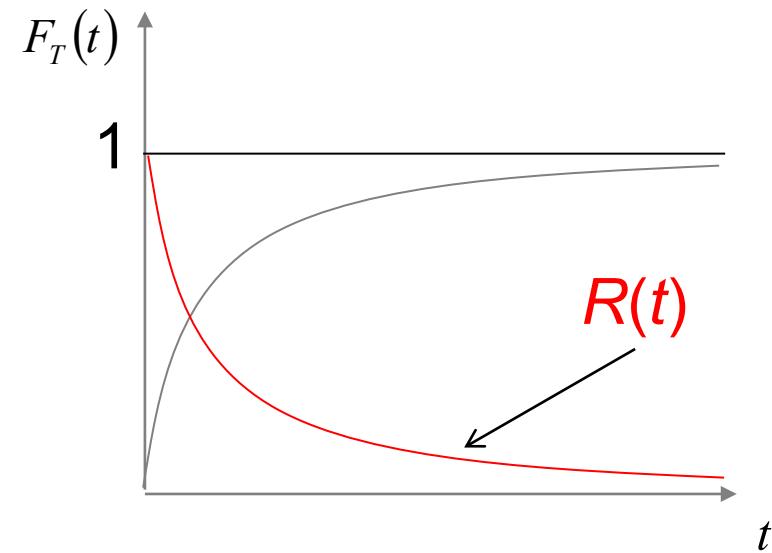
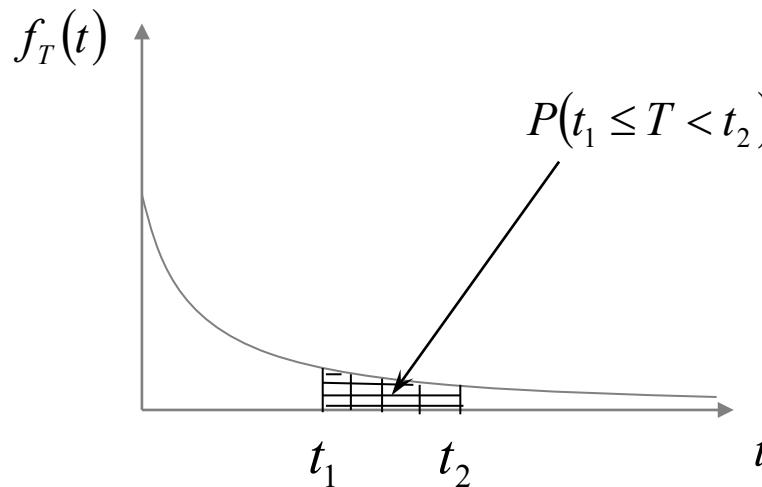
$$f_X(x) = \begin{cases} \alpha x^2 & 0 \leq x \leq 10 \\ 0 & \text{elsewhere} \end{cases}$$

1. Find the value of  $\alpha$  for which  $f_X(x)$  is a pdf?
2. What is  $P(X > 5)$ ?
3. Compute the following:
  - Mean of  $X$
  - Variance of  $X$
  - Standard Deviation of  $X$
  - Median of  $X$

# Reliability

- $T$  = Time to failure of a component (random variable)
  - Probability density function (pdf) at time  $t$ :  $f_T(t)$
  - Cumulative distribution function (cdf) at time  $t$  = probability of having a failure before  $t$  :  $F_T(t) = P(T < t)$
  - Reliability at time  $t$  = Probability that the component does not fail up to  $t$ :

$$R(t) = 1 - F_T(t)$$



# Mean Time To Failure

$$MTTF = E[T] = \int_0^{+\infty} f(t) \cdot t \, dt = -R(t) \cdot t \Big|_0^{+\infty} + \int_0^{+\infty} R(t)dt = \int_0^{+\infty} R(t)dt$$

by parts

$$\int_0^x h'(t) \cdot g(t)dt = h(t) \cdot g(t) \Big|_0^x - \int_0^x h(t) \cdot g'(t)dt$$

with

$$\begin{cases} h(t) = -R(t) \rightarrow h'(t) = -\frac{dR(t)}{dt} = -\frac{d[1 - F(t)]}{dt} = \frac{dF(t)}{dt} = f(t) \\ g(t) = t \rightarrow g'(t) = 1 \end{cases}$$

# Probability density function: interpretation

We start out a new item at time  $t = 0$  and at time  $t=0$ , we ask:  
«What is the probability that the item will fail in the interval  $[t, t+\Delta t]$ ?»

$$P(t \leq T < t + \Delta t) \approx f_T(t)\Delta t$$

# Another Question ...

- We start out a new item at time  $t = 0$  and at time  $t=0$ , we ask:  
«What is the probability that the item will fail in the interval  $[t, t+\Delta t]$ ?»  
$$P(t \leq T < t + \Delta t) \approx f_T(t)\Delta t$$
- We started out a new item at time  $t = 0$ ; **the item has survived until time  $t$** , we ask:  
«What is the probability that the item will fail in the **next** interval  $[t, t+\Delta t]$ ?»

# Hazard Function

- We start out a new item at time  $t = 0$  and at time  $t=0$ , we ask:  
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«What is the probability that the item will fail in the **next** interval  $[t, t+\Delta t]$ ?»

Hazard function

$$P(t \leq T < t + \Delta t | T > t) \approx h_T(t)\Delta t$$

# Hazard Function and Reliability

$$h_T(t)dt = P(t < T \leq t + dt | T > t) = \frac{P(t < T \leq t + dt)}{P(T > t)} = \frac{f_T(t)dt}{R(t)}$$



$$f(t) = \frac{dF(t)}{dt} = \frac{d(1 - R(t))}{dt} = -\frac{dR(t)}{dt}$$

$$h_T(t)dt = -\frac{dR(t)}{R(t)}$$



$$\int_0^t h_T(t^*)dt^* = \int_1^{R(t)} -\frac{dR(t^*)}{R(t^*)} = -\ln(R(t^*)) \Big|_1^{R(t)} = -\ln(R(t))$$

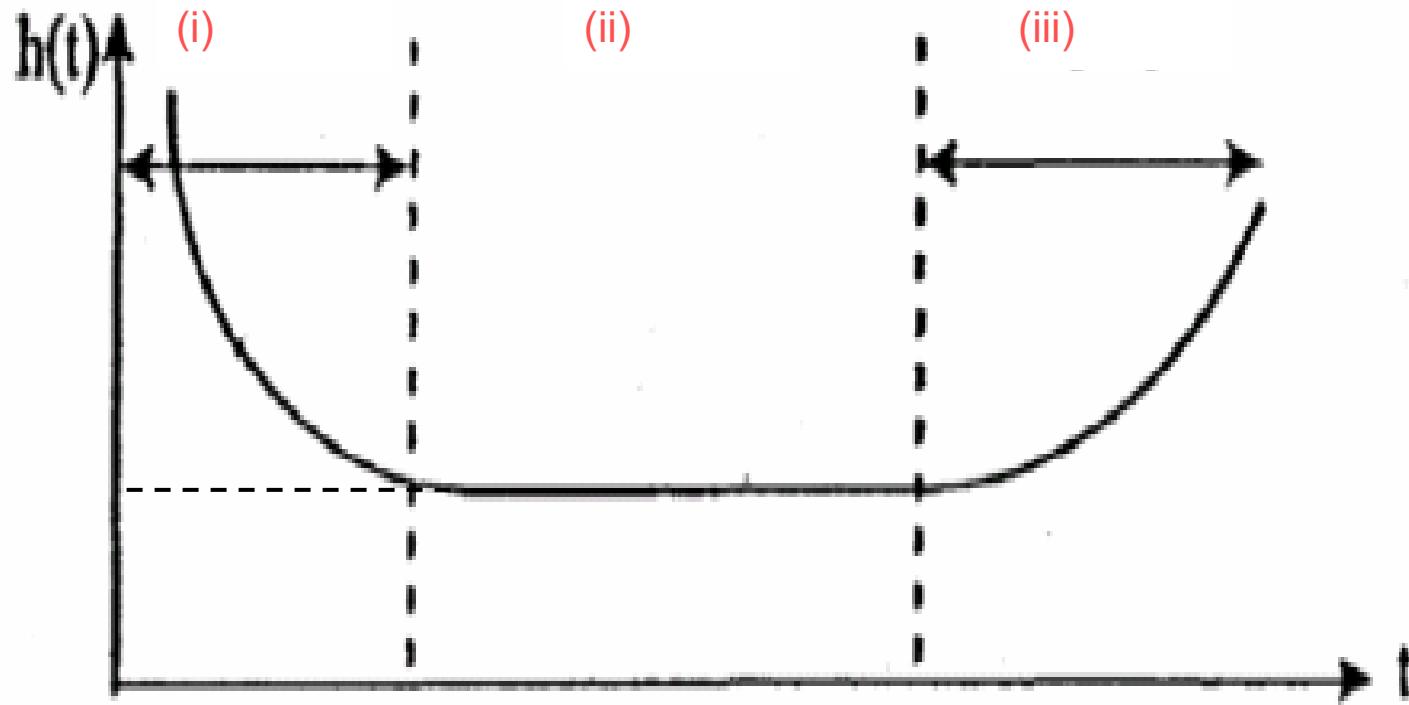


$$R(t) = e^{-\int_0^t h_T(t')dt'}$$

$$f(t) = h(t)R(t) = h(t)e^{-\int_0^t h_T(t')dt'}$$

# Hazard Function: the Bath-Tub Curve

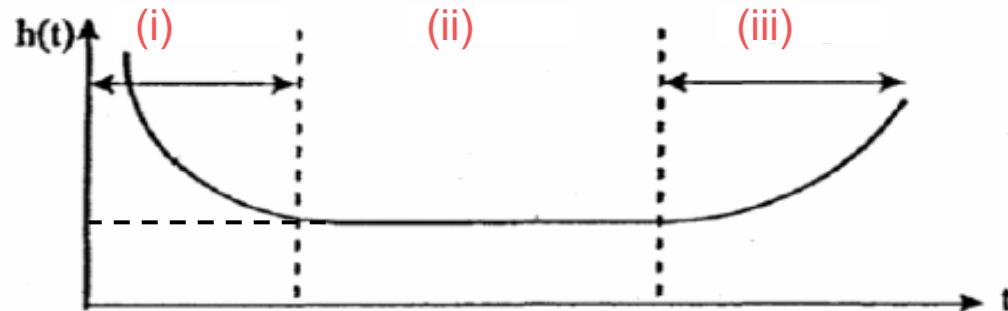
- Usually, the hazard function shows three distinct phases



# Hazard Function: the Bath-Tub Curve

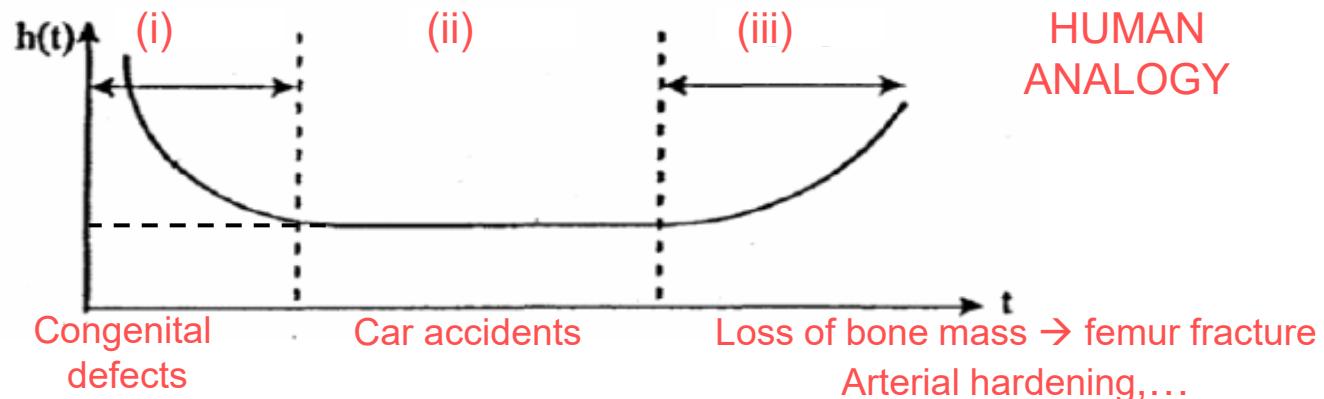
- Usually, the hazard function shows three distinct phases:
  - i. Decreasing - *infant mortality* or *burn in period*:
    - Failures due to defective pieces of equipment not manufactured or constructed properly (missing parts, substandard material batches, damage in shipping, ...)

The items are tested at the factory before they are distributed to the users → much of the infant mortality is removed before the items are delivered for use.



# Hazard Function: the Bath-Tub Curve

- Usually, the hazard function shows three distinct phases:
  - i. Decreasing - *infant mortality* or *burn in period*:
    - Failures due to defective pieces of equipment not manufactured or constructed properly (missing parts, substandard material batches, damage in shipping, ...)
  - ii. Constant - *useful life*
    - Random failures due to unavoidable loads coming from without (earthquakes, power surges, vibration, temperature fluctuations,...)
  - iii. Increasing – *ageing*
    - Aging failures due to cumulative effects such as corrosion, embrittlement, fatigue, cracking, ...



Univariate continuous probability distributions:

- 1) exponential distribution
- 2) Weibull distribution
- 3) Normal distribution

# Continuous Distributions: Exponential Distribution

- $T$ =failure time
- $h_T(t)=\lambda$  constant



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$$\begin{aligned} R(t) &= P\{T>t\} = P\{\text{no failure in } (0,t)\} = \\ &= \text{Poisson}(k=0;(0,t),\lambda) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t} \end{aligned}$$



$$\begin{aligned} F_T(t) &= 1 - P\{T > t\} = 1 - e^{-\lambda t} \\ f_T(t) &= \frac{dF(t)}{dt} = \lambda e^{-\lambda t} \end{aligned}$$

- It is the only distribution characterized by a constant failure rate

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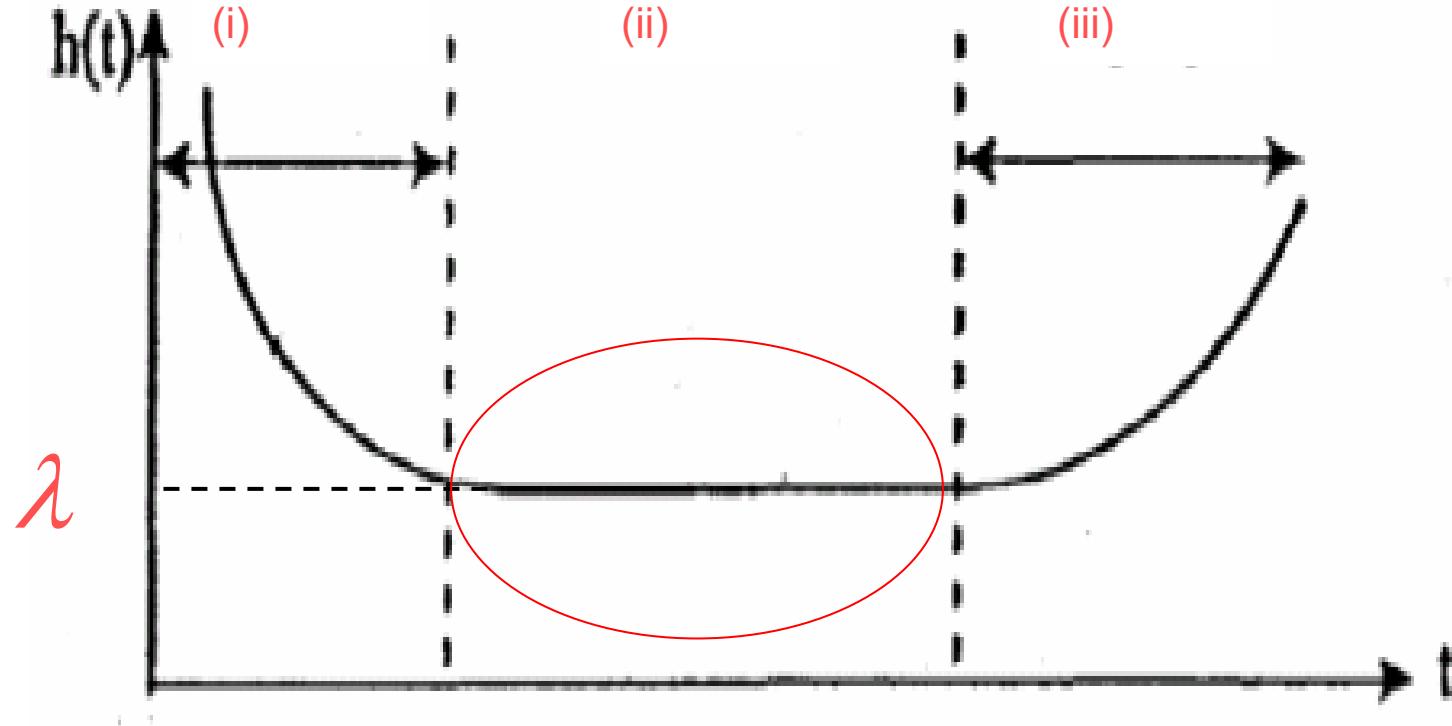
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# Exponential Distribution and bath tub curve



$$MTTF = E[T] = \int_0^{+\infty} R(t)dt = \int_0^{+\infty} e^{-\lambda t} dt = \frac{1}{\lambda}$$

$$\text{Var}[T] = \frac{1}{\lambda^2}$$

## Exercise 2

A rotary pump has a constant failure rate  $\lambda = 4.28 \cdot 10^{-4}$  hours $^{-1}$  (data from OREDA 2002). You are required to find:

- the probability that the pump survives 1 month (730 hours)
- the pump mean time to failure
- assume that the pump has been working without failures for two months (1460 hours), which is the probability that the pump will survive another month?

# Exponential distribution: memorylessness

- A component with constant failure rate,  $\lambda$ , is found still operational at a given time  $t_1$  (age of the component). What is the probability that it will fail in the next period of time of length  $\tau$ ?

$$\begin{aligned} P\{T \leq t_1 + \tau | T > t_1\} &= \frac{P(t_1 < T \leq t_1 + \tau)}{P(T > t_1)} = \\ &= \frac{F(t_1 + \tau) - F(t_1)}{R(t_1)} = \frac{(1 - e^{-\lambda(t_1 + \tau)}) - (1 - e^{-\lambda t_1})}{e^{-\lambda t_1}} \\ &= \frac{(e^{-\lambda t_1} - e^{-\lambda(t_1 + \tau)})}{e^{-\lambda t_1}} = 1 - e^{-\lambda \tau} = F(\tau) \end{aligned}$$

- Still exponential with failure rate  $\lambda$ !
- The probability that it will fail in some period of time of lengths  $\tau$  does not depend from the component age  $t_1$  (the component is always as good as new)

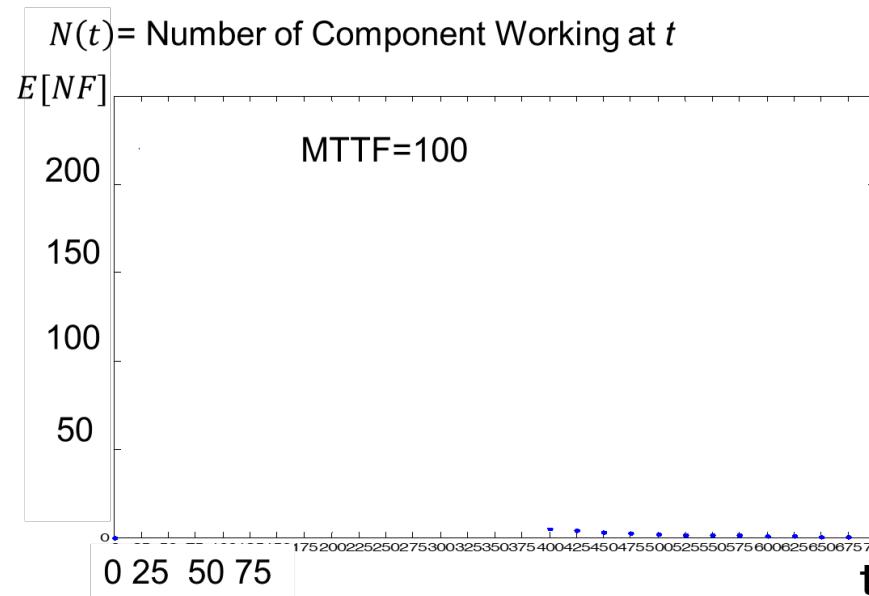
# Exponential Distribution: Interpretation of the pdf

## ■ Conceptual Experiment:

- $N_0$  identical components
- Constant failure rate:  $\lambda = 0.01$



? Expected Number of failures between  $(t - 25 \leq T < t)$  with  $t = 25, 50, 75, \dots, 700$



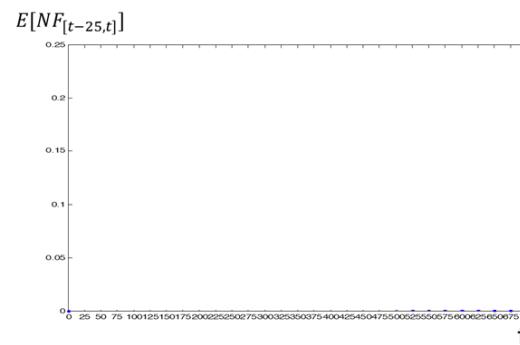
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$NF_{[t-25,t]}$  = number of failure in  $t - 25 \leq T < t$  = Random variable

### ■ Conceptual Experiment:

- $N_0$  identical components
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$NF_{[0,25]}$  = number of failure in  $0 \leq T < 25$  = Random variable  $\rightarrow$   
Binomial Distribution:  $b(NF_{[0,25]}, N_0, \lambda \cdot 25)$

$$E[NF_{[0,25]}] = N_0 P\{failure \text{ in } (t - \Delta t, t)\} = N_0 \lambda \Delta t = 1000 \cdot 0.01 \cdot 25 = 250$$

### ■ Conceptual Experiment:

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$NF_{[t-25,t]}$  = number of failure in  $t - 25 \leq T < t$  = Random variable  $\rightarrow$   
Binomial Distribution:  $b(NF_{[t-25,t]}, S(t - 25), \lambda \cdot 25)$

$$E[NF_{[t-25,t]}] = S(t - 25)P\{\text{failure in } (t, t + \Delta t)\} = S(t - 25)\lambda\Delta t$$

- $S(t)$  = number of component working at  $t$  [SURVIVED]

Number of failures between 25 and 50]  $\cong \lambda \cdot 25 \cdot S(25) = 0.01 \cdot 25 \cdot (1000 - 250) \cong 187$   
 $E[\text{Number of failures between 50 and 75}] \cong \lambda \cdot 25 \cdot S(50) = 0.01 \cdot 25 \cdot (1000 - 437) \cong 140$

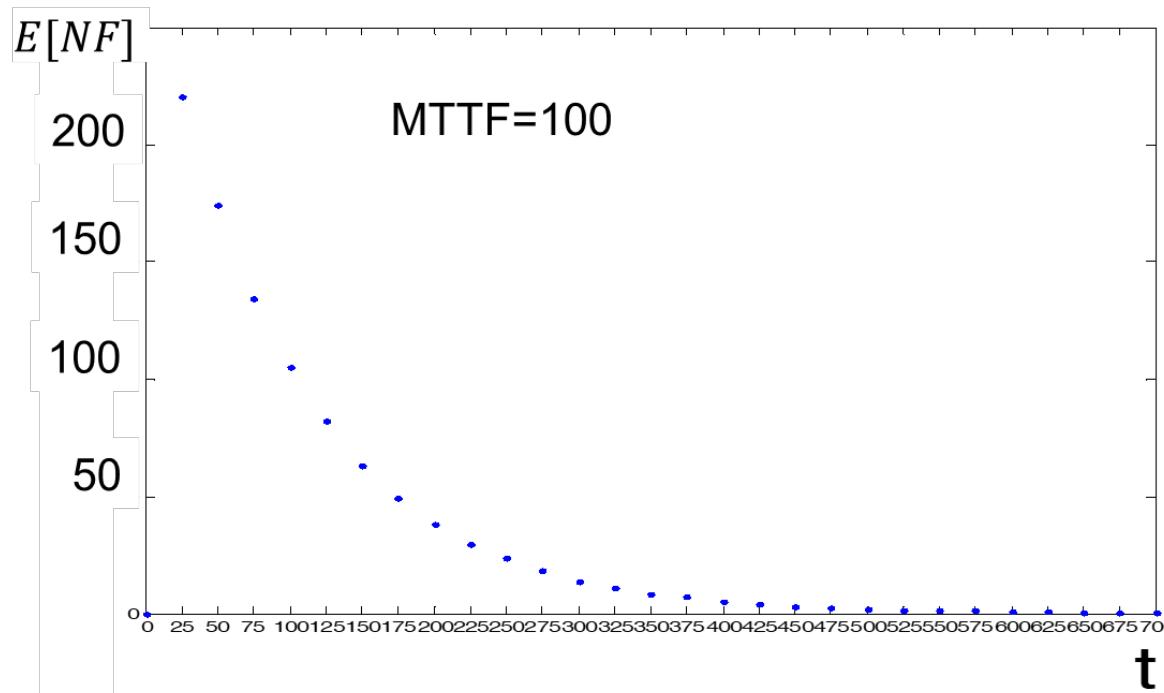
# Exponential Distribution: Interpretation of the pdf

## ■ Conceptual Experiment:

- $N_0$  identical components
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Number of failures for unit of time exponentially decreases

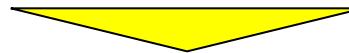


Univariate continuous probability distributions:

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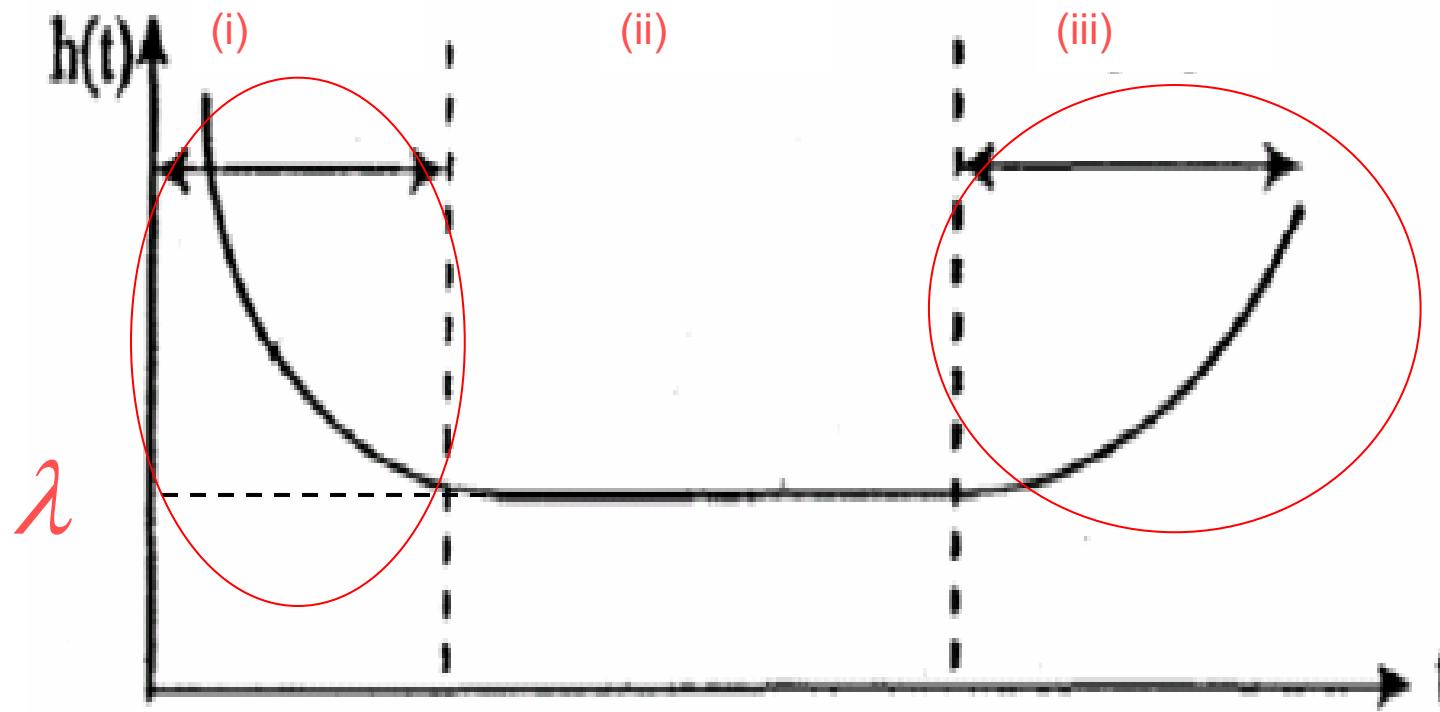
# Continuous Distributions : the Weibull Distribution

- The age of a component influences its failure process



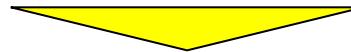
hazard rate does not remain constant throughout the lifetime

$$h(t) \neq \text{const}$$



# Continuous Distributions: the Weibull Distribution

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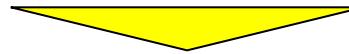
hazard rate does not remain constant throughout the lifetime

$$h(t) = \frac{\beta}{\tau} \left(\frac{t}{\tau}\right)^{\beta-1}, \quad t > 0$$

$\beta$  = shape parameter  
 $\tau$  = scale parameter

# Continuous Distributions: the Weibull Distribution

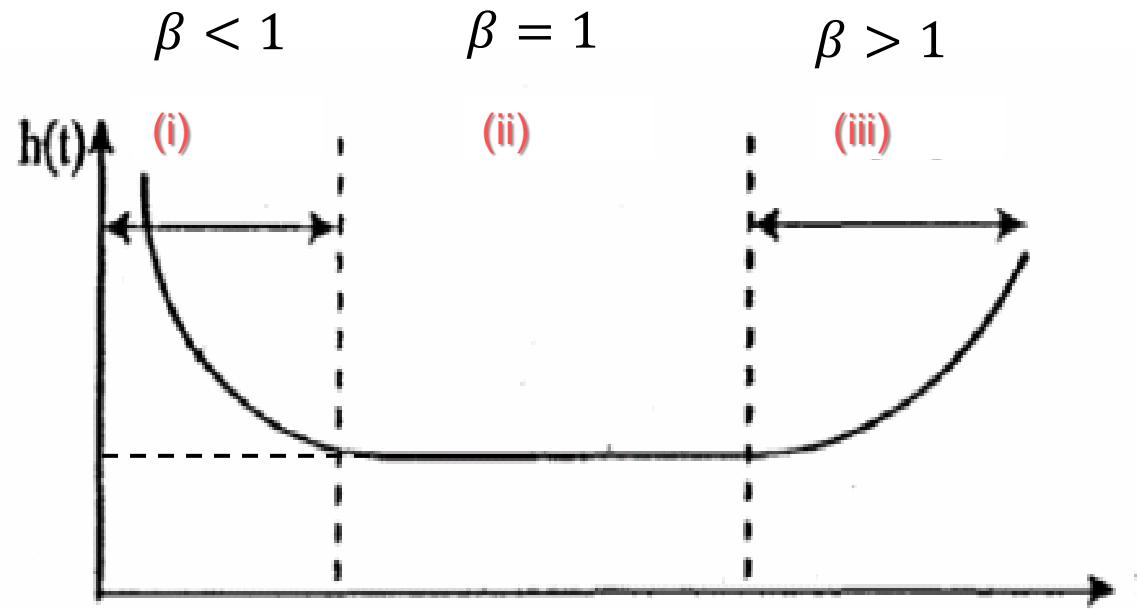
- The age of a component influences its failure process



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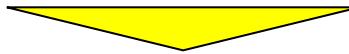
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$$F(t) = P(T < t) = 1 - e^{- \int_0^t h_T(t') dt'} = 1 - e^{- \int_0^t \frac{\beta}{\tau} \left( \frac{t'}{\tau} \right)^{\beta-1} dt'} = 1 - e^{- \left( \frac{t}{\tau} \right)^\beta}$$



$$f_T(t) = \frac{dF}{dt} = \frac{\beta}{\tau^\beta} t^{\beta-1} e^{- \left( \frac{t}{\tau} \right)^\beta}$$



$$E[T] = \tau \Gamma \left( \frac{1}{\beta} + 1 \right);$$

$$Var[T] = \tau^2 \left( \Gamma \left( \frac{2}{\beta} + 1 \right) - \Gamma \left( \frac{1}{\beta} + 1 \right) \right)^2$$

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx \quad k > 0$$

Univariate continuous probability distributions:

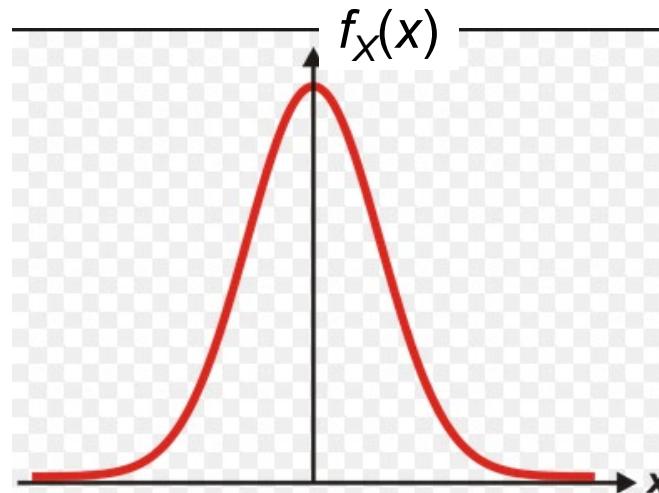
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# Continuous Distributions: Normal (or Gaussian) Distribution

Probability density function:

$$X \sim N(\mu_X, \sigma_X)$$

$$f_X(x; \mu_X, \sigma_X) = \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} \quad -\infty < x, \mu_X < \infty; \sigma_X > 0$$



It is the only distribution with a symmetric bell shape!

Expected value and variance:

$$E[X] = \mu_X$$

$$Var[X] = \sigma_X^2$$

# Transformations of random variables

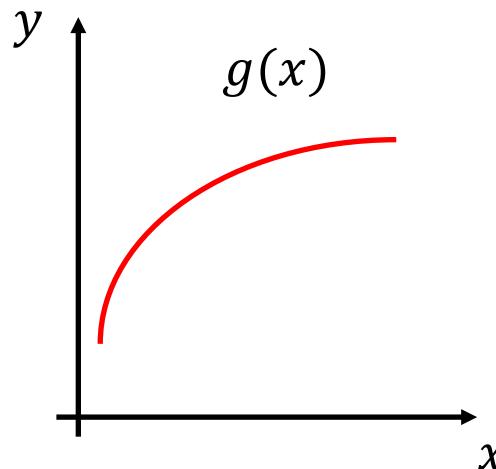
Random variables:

- $X \sim f_X(x)$
- $Y, y = g(x)$

Monotonically increasing



How to find the pdf of  $Y: f_Y(y) ?$



# Transformations of random variables

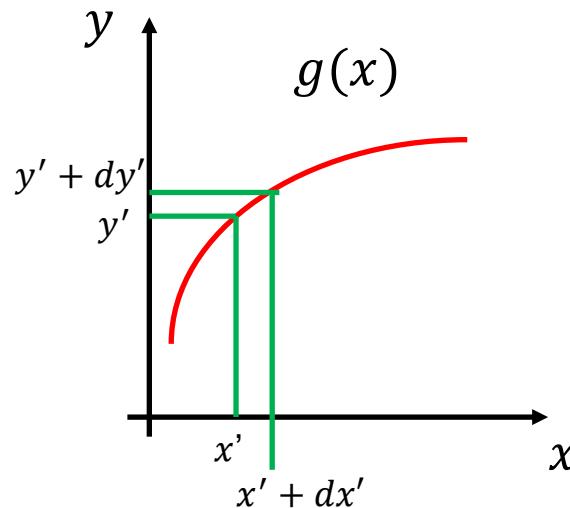
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How to find the pdf of  $Y$ :  $f_Y(y)$ ?



$$P\{x' \leq X < x' + dx'\} = P\{y' \leq Y \leq y' + dy'\}$$

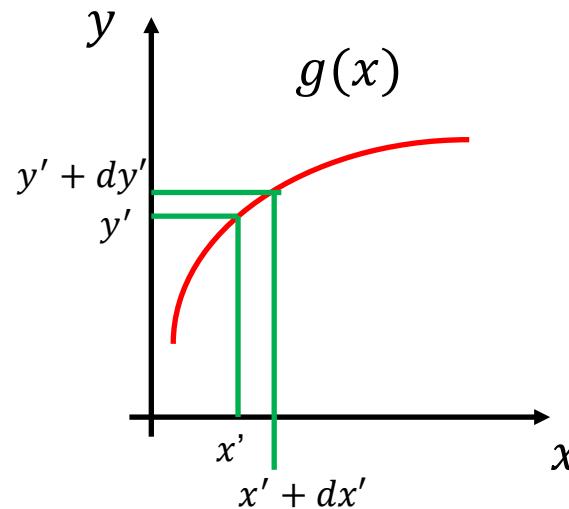
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- $X \sim f_X(x)$
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How to find the pdf of  $Y$ :  $f_Y(y)$ ?



$$P\{x' \leq X < x' + dx'\} = P\{y' \leq Y \leq y' + dy'\}$$

$$f_X(x')dx' = f_Y(y')dy'$$

$$f_Y(y) = f_X(x) \frac{dx}{dy} = f_X(x) \frac{1}{\frac{dg(x)}{dx}}$$

## Standard Normal Variable

$$X \sim N(\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$S = \frac{X - \mu}{\sigma} \quad \text{Standard Normal Variable}$$



What is the pdf of  $S$ ,  $f_S(s)$ ?

## Standard Normal Variable

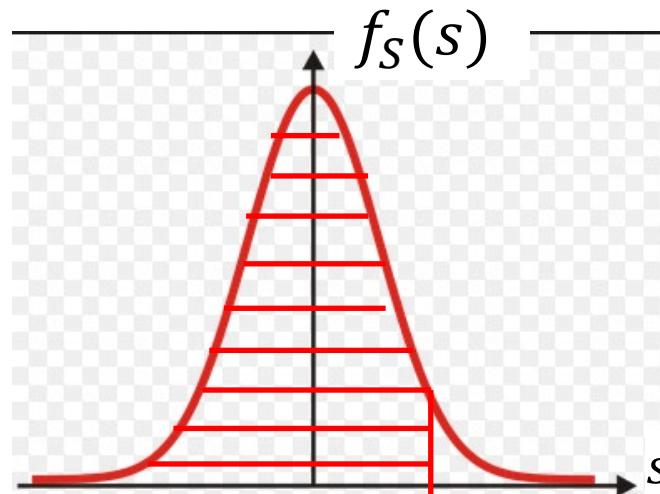
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$$S = \frac{X - \mu}{\sigma} \quad \text{Standard Normal Variable}$$



What is the pdf of  $S$ ,  $f_S(s)$ ?

$$f_S(s) = f_X(x) \frac{dx}{ds} = f_X(x) \frac{1}{\frac{d\left(\frac{x-\mu}{\sigma}\right)}{dx}} = \sigma f_X(x) = \sigma \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} = N(0,1)$$



$$F_S(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\frac{1}{2}\xi^2} d\xi$$

from tables

# Table of Standard Normal Probability

$s$	$\phi(s)$
0.00	0.500000
0.01	0.503989
0.02	0.507978
0.03	0.511966
0.04	0.515954
0.05	0.519939
0.06	0.523922
0.07	0.527904
0.08	0.531882
0.09	0.535857
0.10	0.539828
0.11	0.543796
0.12	0.547759
0.13	0.551717
0.14	0.555671
0.15	0.559618
0.16	0.563500
0.17	0.567494
0.18	0.571423
0.19	0.575345
0.20	0.579260
0.21	0.583166
0.22	0.587064
0.23	0.590954
0.24	0.549835
0.25	0.598706
0.26	0.602568
0.27	0.606420
0.28	0.610262
0.29	0.614092
0.30	0.617912
0.31	0.621720
0.32	0.623517
0.33	0.629301
0.34	0.633072
0.35	0.636831
0.36	0.640576
0.37	0.644309
0.38	0.648027
0.39	0.651732
0.40	0.655422
0.41	0.659097
0.42	0.662757
0.43	0.666402
0.44	0.670032
0.45	0.673645
0.46	0.677242
0.47	0.680823
0.48	0.684387
0.49	0.687933

$s$	$\phi(s)$
0.50	0.691463
0.51	0.694975
0.52	0.698468
0.53	0.701944
0.54	0.705401
0.55	0.708840
0.56	0.712260
0.57	0.715661
0.58	0.719043
0.59	0.722405
0.60	0.725747
0.61	0.729069
0.62	0.732371
0.63	0.735653
0.64	0.738914
0.65	0.742154
0.66	0.745374
0.67	0.748572
0.68	0.751748
0.69	0.754903
0.70	0.758036
0.71	0.761148
0.72	0.764238
0.73	0.767305
0.74	0.770350
0.75	0.773373
0.76	0.776373
0.77	0.779350
0.78	0.782305
0.79	0.785236
0.80	0.788145
0.81	0.791030
0.82	0.793892
0.83	0.796731
0.84	0.799546
0.85	0.802337
0.86	0.805105
0.87	0.807850
0.88	0.810570
0.89	0.813267
0.90	0.815940
0.91	0.818589
0.92	0.821214
0.93	0.823815
0.94	0.826391
0.95	0.828944
0.96	0.831473
0.97	0.833977
0.98	0.836457
0.99	0.838913

$$\Phi(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-\frac{1}{2}\xi^2} d\xi$$

1.03	0.848495
1.04	0.850830
1.05	0.853141
1.06	0.855428
1.07	0.857690
1.08	0.859929
1.09	0.862143
1.10	0.864334
1.11	0.866500
1.12	0.868643
1.13	0.870762
1.14	0.872857
1.15	0.874928
1.16	0.876976
1.17	0.878999
1.18	0.881000
1.19	0.882977
1.20	0.884930
1.21	0.886860
1.22	0.888767
1.23	0.890651
1.24	0.892512
1.25	0.894350
1.26	0.896165
1.27	0.897958
1.28	0.899727
1.29	0.901475
1.30	0.903199
1.31	0.904902
1.32	0.906583
1.33	0.908241
1.34	0.909877
1.35	0.911492
1.36	0.913085
1.37	0.914656
1.38	0.916207
1.39	0.917735
1.40	0.919243
1.41	0.920730
1.42	0.922196
1.43	0.923641
1.44	0.925066
1.45	0.926471
1.46	0.927855
1.47	0.929219
1.48	0.930563
1.49	0.931888

# Table of Standard Normal Probability

$s$	$\phi(s)$
1.50	0.933193
1.51	0.934478
1.52	0.935744
1.53	0.936992
1.54	0.938220
1.55	0.939429
1.56	0.940620
1.57	0.941792
1.58	0.942947
1.59	0.944083
1.60	0.945201
1.61	0.946301
1.62	0.947384
1.63	0.948449
1.64	0.949497
1.65	0.950529
1.66	0.951543
1.67	0.952540
1.68	0.953521
1.69	0.954486
1.70	0.955435
1.71	0.956367
1.72	0.957284
1.73	0.958185
1.74	0.959071
1.75	0.959941
1.76	0.960796
1.77	0.961636
1.78	0.962426
1.79	0.963273
1.80	0.964070
1.81	0.964852
1.82	0.965621
1.83	0.966375
1.84	0.967116
1.85	0.967843
1.86	0.968557
1.87	0.969258
1.88	0.969946
1.89	0.970621
1.90	0.971284
1.91	0.971933
1.92	0.972571
1.93	0.973197
1.94	0.973810
1.95	0.974412
1.96	0.975002
1.97	0.975581
1.98	0.976148
1.99	0.976705

$s$	$\phi(s)$
2.00	0.977250
2.01	0.977784
2.02	0.978308
2.03	0.978822
2.04	0.979325
2.05	0.979818
2.06	0.980301
2.07	0.980774
2.08	0.981237
2.09	0.981691
2.10	0.982136
2.11	0.982571
2.12	0.982997
2.13	0.983414
2.14	0.983823
2.15	0.984223
2.16	0.984614
2.17	0.984997
2.18	0.985371
2.19	0.985738
2.20	0.986097
2.21	0.986447
2.22	0.986791
2.23	0.987126
2.24	0.987455
2.25	0.987776
2.26	0.988089
2.27	0.988396
2.28	0.988696
2.29	0.988989
2.30	0.989276
2.31	0.989556
2.32	0.989830
2.33	0.990097
2.34	0.990358
2.35	0.990613
2.36	0.990863
2.37	0.991106
2.38	0.991344
2.39	0.991576
2.40	0.991802
2.41	0.992024
2.42	0.992240
2.43	0.992451
2.44	0.992656
2.45	0.992857
2.46	0.993053
2.47	0.993244
2.48	0.993431
2.49	0.993613

$s$	$\phi(s)$
2.50	0.993790
2.51	0.993963
2.52	0.994132
2.53	0.994267
2.54	0.994457
2.55	0.994614
2.56	0.994766
2.57	0.994915
2.58	0.995060
2.59	0.995201
2.60	0.995339
2.61	0.995473
2.62	0.995604
2.63	0.995731
2.64	0.995855
2.65	0.995975
2.66	0.996093
2.67	0.996207
2.68	0.996319
2.69	0.996427
2.70	0.996533
2.71	0.996636
2.72	0.996736
2.73	0.996833
2.74	0.996928
2.75	0.997020
2.76	0.997110
2.77	0.997197
2.78	0.997282
2.79	0.997365
2.80	0.997445
2.81	0.997523
2.82	0.997599
2.83	0.997673
2.84	0.997744
2.85	0.997814
2.86	0.997882
2.87	0.997948
2.88	0.998012
2.89	0.998074
2.90	0.998134
2.91	0.998193
2.92	0.998250
2.93	0.998305
2.94	0.998359
2.95	0.998411
2.96	0.998462
2.97	0.998511
2.98	0.998559
2.99	0.998605

# Table of Standard Normal Probability

$s$	$\phi(s)$
3.00	0.998630
3.01	0.998694
3.02	0.998736
3.03	0.998777
3.04	0.998817
3.05	0.998856
3.06	0.998893
3.07	0.998930
3.08	0.998965
3.09	0.998999
3.10	0.999032
3.11	0.999065
3.12	0.999096
3.13	0.999126
3.14	0.999155
3.15	0.992184
3.16	0.999119
3.17	0.999238
3.18	0.999264
3.19	0.999289
3.20	0.999313
3.21	0.999336
3.22	0.999359
3.23	0.999381
3.24	0.999402
3.25	0.999423
3.26	0.999443
3.27	0.999462
3.28	0.999481
3.29	0.999499
3.30	0.999516
3.31	0.999533
3.32	0.999550
3.33	0.999566
3.34	0.999581
3.35	0.999596
3.36	0.999610
3.37	0.999624
3.38	0.999637
3.39	0.999650
3.40	0.999663
3.41	0.999675
3.42	0.999687
3.43	0.999698
3.44	0.999709
3.45	0.999720
3.46	0.999730
3.47	0.999740
3.48	0.999749
3.49	0.999758

$s$	$\phi(s)$
3.50	0.999767
3.51	0.999776
3.52	0.999784
3.53	0.999792
3.54	0.999800
3.55	0.999807
3.56	0.999815
3.57	0.999821
3.58	0.999828
3.59	0.999835
3.60	0.999841
3.61	0.999847
3.62	0.999853
3.63	0.999858
3.64	0.999864
3.65	0.999869
3.66	0.999874
3.67	0.999879
3.68	0.999883
3.69	0.999888
3.70	0.999892
3.71	0.999896
3.72	0.999900
3.73	0.999904
3.74	0.999908
3.75	0.999912
3.76	0.999915
3.77	0.999918
3.78	0.999922
3.79	0.999925
3.80	0.999928
3.81	0.999931
3.82	0.999933
3.83	0.999936
3.84	0.999938
3.85	0.999941
3.86	0.999943
3.87	0.999946
3.88	0.999948
3.89	0.999950
3.90	0.999952
3.91	0.999954
3.92	0.999956
3.93	0.999958
3.94	0.999959
3.95	0.999961
3.96	0.999963
3.97	0.999964
3.98	0.999966
3.99	0.999967

$s$	$\phi(s)$
4.00	0.316712E-04
4.05	0.256088E-04
4.10	0.206575E-04
4.15	0.166238E-04
4.20	0.133458E-04
4.25	0.106883E-04
4.30	0.853906E-05
4.35	0.680688E-05
4.40	0.541234E-05
4.45	0.429351E-05
4.50	0.339767E-05
4.55	0.268230E-05
4.60	0.211245E-05
4.65	0.165968E-05
4.70	0.130081E-05
4.75	0.101708E-05
4.80	0.793328E-06
4.85	0.617307E-06
4.90	0.479183E-06
4.95	0.371067E-06
5.00	0.286652E-06
5.10	0.169827E-06
5.20	0.996443E-07
5.30	0.579013E-07
5.40	0.333204E-07
5.50	0.189896E-07
5.60	0.107176E-07
5.70	0.599037E-08
5.80	0.331575E-08
5.90	0.181751E-08
6.00	0.986588E-09
6.10	0.530343E-09
6.20	0.282316E-09
6.30	0.148823E-09
6.40	0.77688 E-10
6.50	0.40160 E-10
6.60	0.20558 E-10
6.70	0.10421 E-10
6.80	0.5231 E-11
6.90	0.260 E-11
7.00	0.128 E-11
7.10	0.624 E-12
7.20	0.301 E-12
7.30	0.144 E-12
7.40	0.68 E-13
7.50	0.32 E-13
7.60	0.15 E-13
7.70	0.70 E-14
7.80	0.30 E-14
7.90	0.15 E-14

## Exercise 8

- From historical data, the total annual rainfall in a catch basin is estimated to be normal (gaussian)  $N(60\text{cm}, 15^2 \text{ cm}^2)$ ,

What is the probability that in the next year the annual rainfall will be between 40 and 70 cm?

# Standard Normal Variable

$$P(a < X < b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



$$S = \frac{X - \mu}{\sigma}$$

$S \sim N(0,1)$



$$P(a < X < b) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{1}{2}s^2} \sigma ds =$$



$$\Rightarrow P(a < X < b) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{1}{2}s^2} ds = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

- For any sequence of  $n$  independent random variable  $X_i$ , their sum  $X = \sum_{i=1}^n X_i$  is a random variable which, for large  $n$ , tends to be distributed as a normal distribution

# Central limit theorem: Special Cases

- If  $X_i$  are independent, identically distributed random variables with mean  $\mu$  and finite variance given by  $\sigma^2$



$$S_n = \frac{\sum_{i=0}^n X_i}{n} \rightarrow N\left(\mu, \sqrt{\frac{\sigma^2}{n}}\right)$$

- If  $X_i$  are independent normal random variables with mean  $\mu_i$  and finite variance given by  $\sigma_i^2$ , and  $b_i \in \mathcal{R}$  are constants



$$Q_n = \sum_{i=1}^n b_i X_i \rightarrow N\left(\sum_{i=1}^n b_i \mu_i, \sqrt{\sum_{i=1}^n b_i^2 \sigma_i^2}\right)$$

# Exercise 9

- The daily concentration of a certain pollutant in a stream has the exponential distribution
  - 1. If the mean daily concentration of the pollutant is  $2 \text{ mg}/10^3 \text{ liter}$ , determine the constant  $c$  in the exponential distribution.
  - 2. Suppose that the problem of pollution will occur if the concentration of the pollutant exceeds  $6\text{mg}/10^3 \text{ liter}$ . What is the probability of a pollution problem resulting from this pollutant in a single day?
  - 3. What is the return period (in days) associated with this concentration level of  $6 \text{ mg}/10^3 \text{ liter}$ ? Assume that the concentration of the pollutant is statistically independent between days.
  - 4. What is the probability that this pollutant will cause a pollution problem at most once in the next 3 days?
  - 5. If instead of the exponential distribution, the daily pollutant concentration is Gaussian with the same mean and variance, what would be the probability of pollution in a day in this case?

## Exercise 9: Solution

- Verify the normalization of the probability density function

$$\int_0^{\infty} ce^{-cx} dx = 1 \Rightarrow -e^{-cx} \Big|_0^{\infty} = 1$$

Then, from the expected value of the exponential distribution:

$$E[X] = 1/c = 2 \rightarrow c = 0.5$$

$$E[X^2] = \int_0^{\infty} (x^2) 0.5 e^{-0.5x} dx = 8$$

$$\sigma_X^2 = E[X^2] - E^2[X] = 8 - 4 = 4$$

## Exercise 9: Solution

$$\begin{aligned} P(\text{pollution}) &= P(X > 6) = p_{X>6} = 1 - P(X \leq 6) \\ &= 1 - \int_0^6 0.5e^{-0.5x} dx = 1 + e^{-0.5x} \Big|_0^6 = 0.0498 \end{aligned}$$

$$E[T_{X>6}] = \frac{1}{p_{X>6}} = \frac{1}{0.0498} = 20 \text{ days}$$

$$\begin{aligned} P(\text{pollution at most once in 3 days}) &= \sum_{k=0}^1 \binom{3}{k} p_{X>6}^k (1 - p_{X>6})^{3-k} = \\ &= (1 - 0.0498)^3 + 3 \cdot 0.0498 \cdot (1 - 0.0498)^2 = \\ &= 0.993 \end{aligned}$$

$$P(X > 6) = 1 - P(X \leq 6) = 1 - P(v \leq 2) = 1 - \Phi(2) = 1 - 0.977 = 0.023$$

# Objectives of These Lectures

- **What is a random variable?**
- **What is a probability density function (pdf)?**
- **What is a cumulative distribution function (CDF)?**
- **What is the hazard function and its relationship with the pdf and CDF?**
- **The bath-tub curve**
- **Binomial, Geometric and Poisson Distribution**
- **Exponential Distribution and its memoryless property**
- **Weibull Distribution**
- **Gaussian Distribution and the central limit theorem**

- Slides
- Red book ('An introduction to the basics of reliability and risk analysis', E. Zio):
  - 4.1, 4.2, 4.3 (no 4.3.4), 4.4, 4.5, 4.6,
- Exercises on Green Book ('basics of reliability and risk analysis – Workout Problems and Solutions, E. Zio, P. Baraldi, F. Cadini)
  - All problems in Chapter 4
- If you are interested in probabilistic approaches for treating uncertainty, you can refer to:  

“Uncertainty in Risk Assessment – The Representation and Treatment of Uncertainties by Probabilistic and Non-Probabilistic Methods”

  - Chapter 2

**THE END (*probably* (-; )**

**Thank you for the attention!!!**

## Univariate continuous probability distributions:

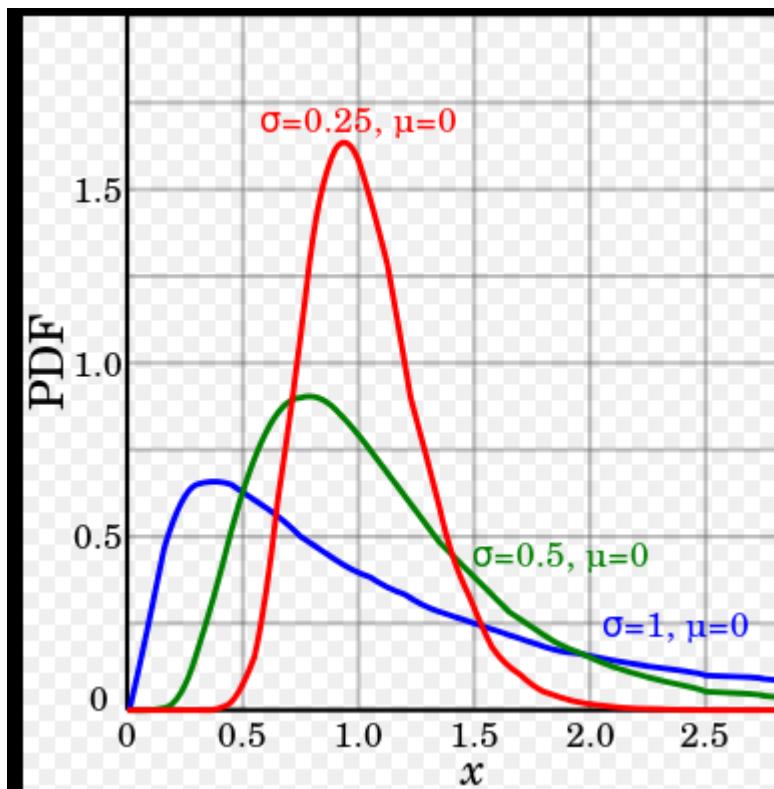
- 1) exponential distribution
- 2) Weibull distribution
- 3) Normal distribution
- 4) Lognormal distribution

# Univariate Continuous Distributions Log-normal Distribution

Probability density function:

$$g_X(x; \mu_Z, \sigma_Z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu_Z}{\sigma_Z}\right)^2} \quad x, \sigma_Z > 0$$

Notice that  
 $\sigma_Z$  and  $\mu_Z$  are not  
the expected value  
and standard deviation of  $X$



# Univariate Continuous Distributions Log-normal Distribution

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Notice that  
 $\sigma_Z$  and  $\mu_Z$  are not  
the expected value  
and standard deviation of  $X$

$$X \sim \text{Log-normal}(\mu_Z, \sigma_Z) \Rightarrow Z = \ln X \sim N(\mu_Z, \sigma_Z)$$

Expected value and variance:

$$E[X] = e^{\mu_Z + \frac{\sigma_Z^2}{2}}$$

$$\text{Var}[X] = e^{2\mu_Z + \sigma_Z^2} (e^{\sigma_Z^2} - 1)$$

# Univariate Continuous Distributions Log-normal Distribution

Probability density function:

$$g_X(x; \mu_Z, \sigma_Z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{x} e^{-\frac{1}{2}\left(\frac{\ln x - \mu_Z}{\sigma_Z}\right)^2} \quad x, \sigma_Z > 0$$

Note: if

$$X \sim \text{Log-normal}(\mu_Z, \sigma_Z) \Rightarrow Z = \ln X \sim N(\mu_Z, \sigma_Z)$$

Expected value and variance:

$$E[X] = e^{\mu_Z + \frac{\sigma_Z^2}{2}}$$

$$\text{Var}[X] = e^{2\mu_Z + \sigma_Z^2} (e^{\sigma_Z^2} - 1)$$

$$\sigma_Z^2 = \ln\left(1 + \frac{\sigma_X^2}{\mu_X^2}\right)$$

$$\mu_Z = \ln\left(\frac{\mu_X^2}{\sqrt{\sigma_X^2 + \mu_X^2}}\right)$$

# Example:

- With reference to previous Example, assume that the total annual rainfall is log-normally distributed (instead of normally) with the same mean and standard deviation of 60 cm and 15 cm, respectively.

What is the probability that in future years the annual rainfall will be between 40 and 70 cm, under this assumption?

## Solution:

Recall that if the distribution of a random variable  $X$  is log-normal ,

then the distribution of the variable  $Z = \ln X$  is normal  $[Z \sim N(\mu_z, \sigma_z)]$ .

The probability density function of the log-normal random variable  $X$  is:

$$f_X(x) = \frac{1}{\sigma_z \sqrt{2\pi}} \cdot \frac{1}{x} \cdot e^{-\frac{1}{2} \left( \frac{-\mu_z + \ln x}{\sigma_z} \right)^2}$$

# Example: Solution

$$\sigma_Z^2 = \ln \left( 1 + \frac{\sigma_X^2}{\mu_X^2} \right)$$
$$\mu_Z = \ln \left( \frac{\mu_X^2}{\sqrt{\sigma_X^2 + \mu_X^2}} \right)$$

First we compute the values of the two parameters  $\mu_z, \sigma_z$  of the distribution of the normal variable  $Z$ . With the data of the previous Example for the values of  $\mu_x$  and  $\sigma_x$  and equations, we have:

$$\mu_z = 4.06$$
$$\sigma_z = 0.25$$

Now, the probability that the annual rainfall will be between 40 cm and 70 cm, is

$$P(40 < X < 70) = \Phi \left( \frac{\ln(70) - 4.06}{0.25} \right) - \Phi \left( \frac{\ln(40) - 4.06}{0.25} \right) = \\ = \Phi(0.75) - \Phi(-1.48)$$

$$P(40 < X < 70) = 0.773373 - 0.069437 = 0.7039$$