POLITECNICO DI MILANO

Basic notions of probability theory

- Probability Laws
- Discrete Random Variables

Flipped class

Piero Baraldi

Flipped Class (Part 1)

- **We will give you the material to be studied in the groups and the corresponding exercises.**
- **You will study the material and you will do the exercises with your teammates. You can send to Prof. Baraldi questions on doubts about the theory. The answers to your questions will be discussed during the lecture of Monday (piero.baraldi@polimi.it – write the group number in the title).**
- **Each group will upload the solutions on webeep within Monday Morning at 11:15. In case of problems using webeep, you can send the solution as an attachment to nicolasjavier.cardenas@polimi.it and piero.baraldi@polimi.it (in the e-mail subject write the number of the group)**
  - do not worry if you have not solved all exercises or if the solutions are wrong – It is normal.
- **We will publish the solution of the exercises during next week.**

Flipped Class (Part 2)

- The beginning of next Monday lecture will be dedicated to discuss your questions (sent by email). Please try to formulate the question in a general way (not ask the solution of the exercise).

- The beginning of the Exercise Session of next Friday (March 1st) will be dedicated to the correction of the exercises.

Flipped Class (Evaluation)

All members of the groups solving correctly all exercises will get +0.50 in the final exam (bonus points)

Practicalities

1) **We will take the attendance group by group:**
   - If the group is formed by at least 2 students, the group can start work together (in presence/hybrid/remotely connected) using Skype, Whatsapp, Microsoft Team, or whatever you prefer.
   - The single-person groups will be rearranged.
   - Inform us if your name is not in the list
2) **Send only one document per group, containing the solutions of all the exercises.**
3) **Solutions can be handwritten and should report all the intermediate steps. Please a single file, not up-down…**

- **Start of the slides of the flipped lecture**

- **Suggestion: do Exercise 0 in the exercise file**

Contents

Probability theory

Probability Theory: Kolmogorov Axioms

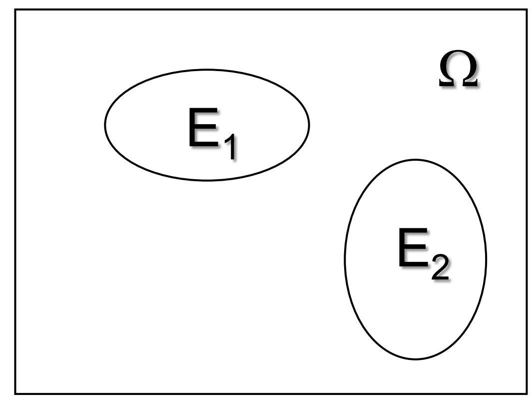
1. $0 \le p(E) \le 1$

2. $p(\Omega) = 1 \quad p(\varnothing) = 0$

3. Addition law:

   Let E_1, …, E_n be a finite set of mutually exclusive events:
   $(X_{E_i} \cap X_{E_j} = \emptyset)$.

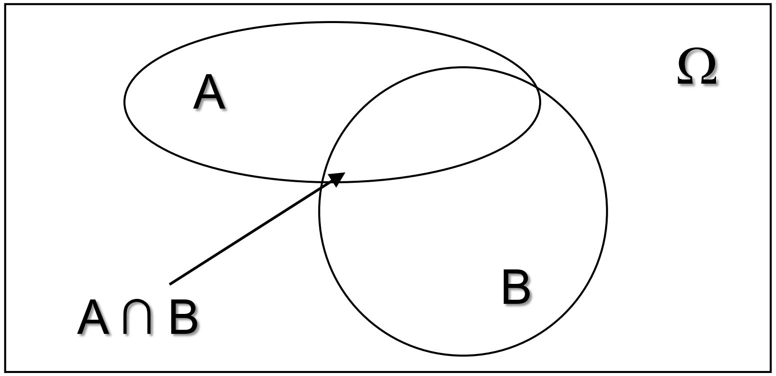
$$p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} p(E_i)$$



Probability Laws

Probability laws (1)

- Union of two non-mutually exclusive events



$$P_{A\cup B} = P_A + P_B - P_{A\cap B}$$

It can be demonstrated by using the three Kolmogorov axioms*

$$P_{A\cup B} \le P_A + P_B$$

- Rare event approximation: A and B events are considered as mutually exclusive ($A \cap B = \emptyset$) → $P(A \cap B) = 0$ →

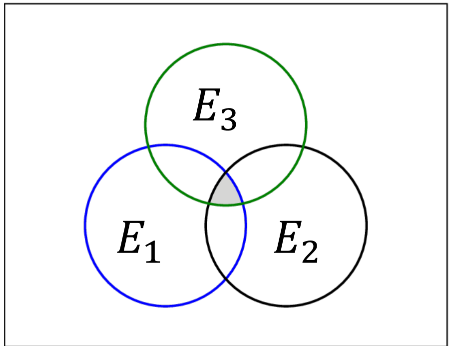
$$P_{A\cup B} = P_A + P_B$$

«conservative error = risk overstimation»

* http://www.ucs.louisiana.edu/~jcb0773/Berry_probbook/425chpt2.pdf

Probability laws (2)

- Union of non-mutually exclusive events: $E_{\cup} = \bigcup_{i=1,\dots,n} E_i$



$$P(E_{\cup}) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P\left(E_i \cap E_j\right) + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

- Upper bound $P(E_U) \leq \sum_{j=1}^{n} P(E_j)$

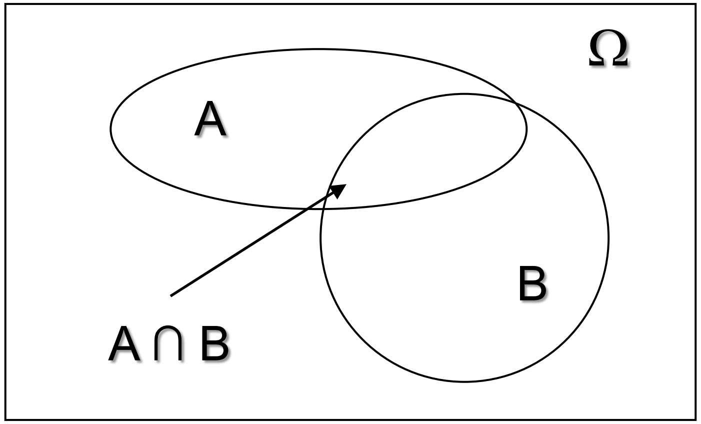
- Lower bound $P(E_U) \geq \sum_{j=1}^{n} P(E_j) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(E_i \cap E_j)$

- Rare event approximation: events are considered as mutually exclusive $(E_i \cap E_j = \emptyset, \forall i, j, i \neq j) \rightarrow P(E_{\cup}) = \sum_{i=1}^{n} P(E_i)$ «conservative error, risk overstimation»

Definition of Conditional Probability

- Conditional Probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



- Event A is said to be statistically independent from event B if:

$$P(A \mid B) = P(A)$$

- If A and B are statistically independent then:

$$P(A \cap B) = P(A)P(B)$$

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:

$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad \bigcup_{j=1}^{n} E_j = \Omega$$

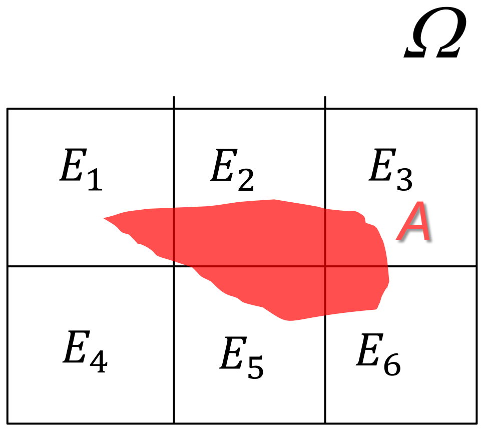
Ω

E_1	E_2	E_3
E_4	E_5	E_6

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:

$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad \bigcup_{j=1}^{n} E_j = \Omega$$

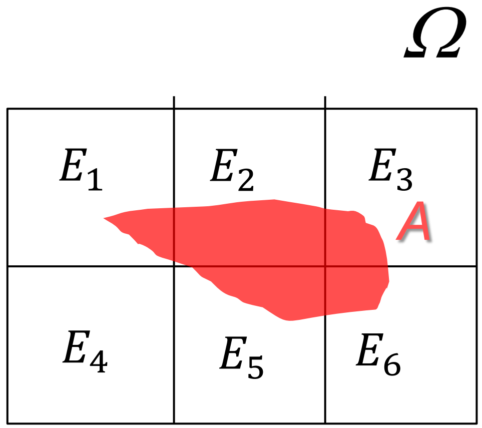


- Given any event A in Ω,

$$P(A) = ?$$

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:



$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad\qquad \bigcup_{j=1}^{n} E_j = \Omega$$

- Given any event A in Ω,

$$A = \bigcup_{j=1}^{n} A \cap E_j$$

$$P(A) = \sum_{j=1}^{n} P\left(A \cap E_j\right) = \sum_{j=1}^{n} P\left(A|E_j\right)P(E_j)$$

DO EXERCISES 1 AND 2

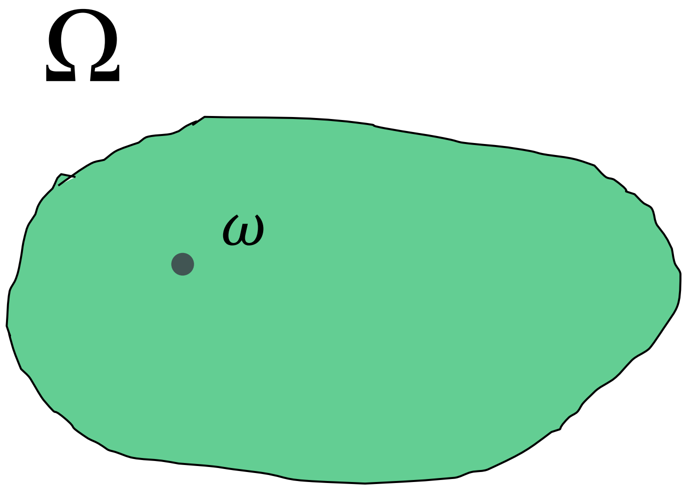
Random variables

Random variables

Experiment: ε

Sample space: Ω

Generic outcome: ω



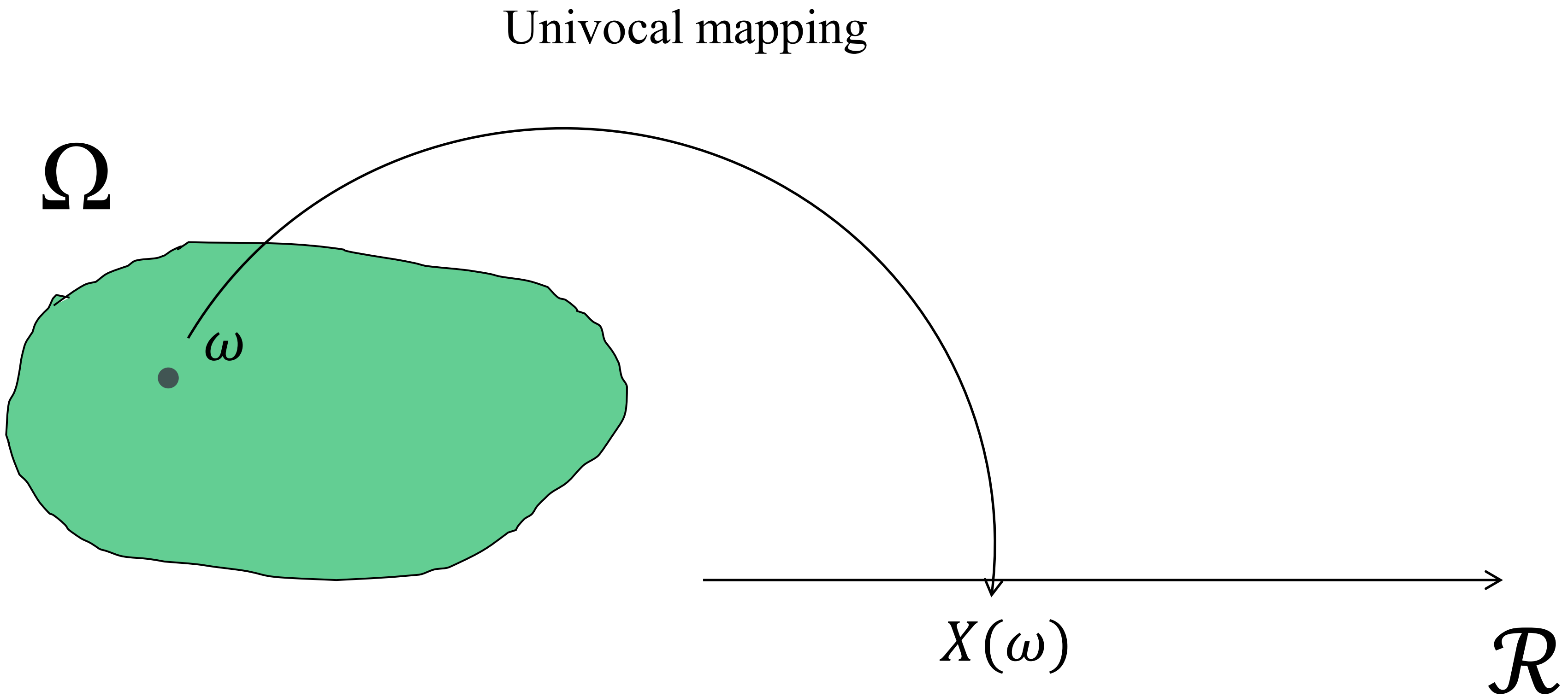
Random variables

Experiment: ε

Sample space: Ω

Generic outcome: ω

$X(\omega)$ random variable in $\mathcal{R}$



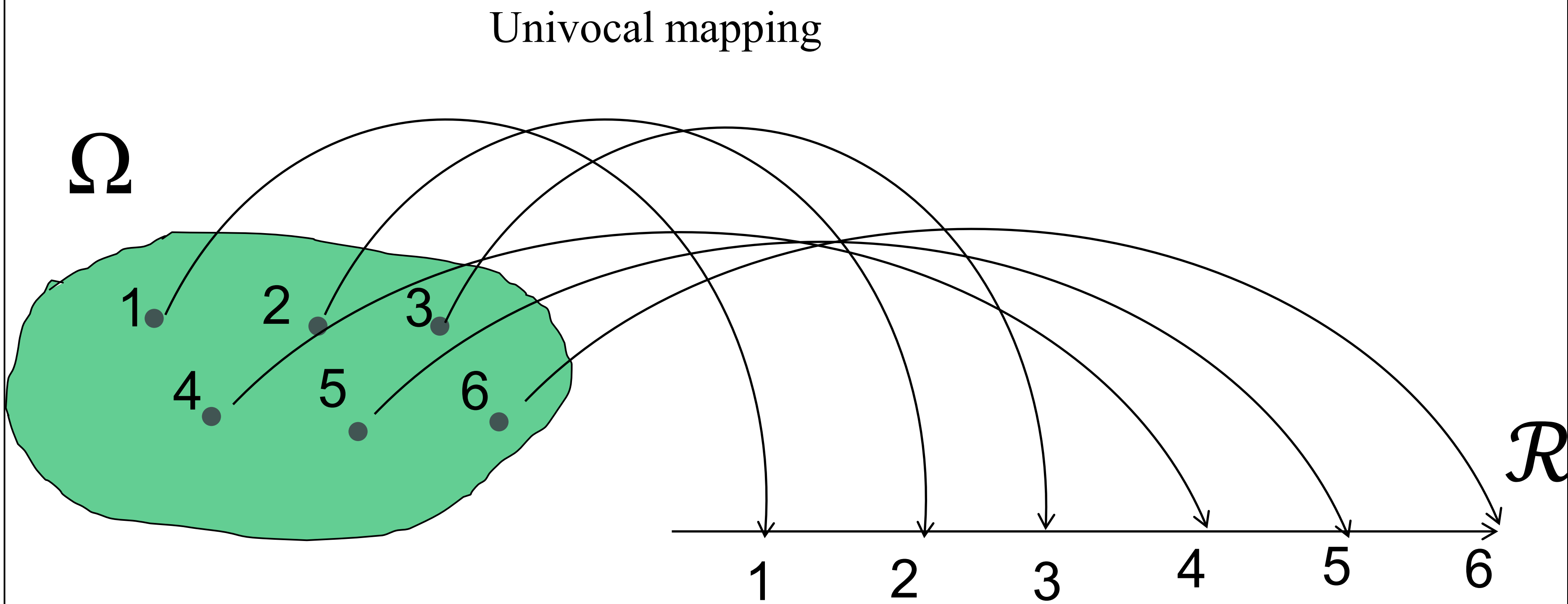
Random variable - Example

Experiment: ε = {throwing a die}

Sample space: Ω={1,2,3,4,5,6}

Generic outcome: ω

$X(\omega)$ in $\Re$



Random variable - Event

Experiment: ε = {throwing a die}

Sample space: Ω={1,2,3,4,5,6}

Event: E_1= {1,2,3,4}

$E_2 = \emptyset$

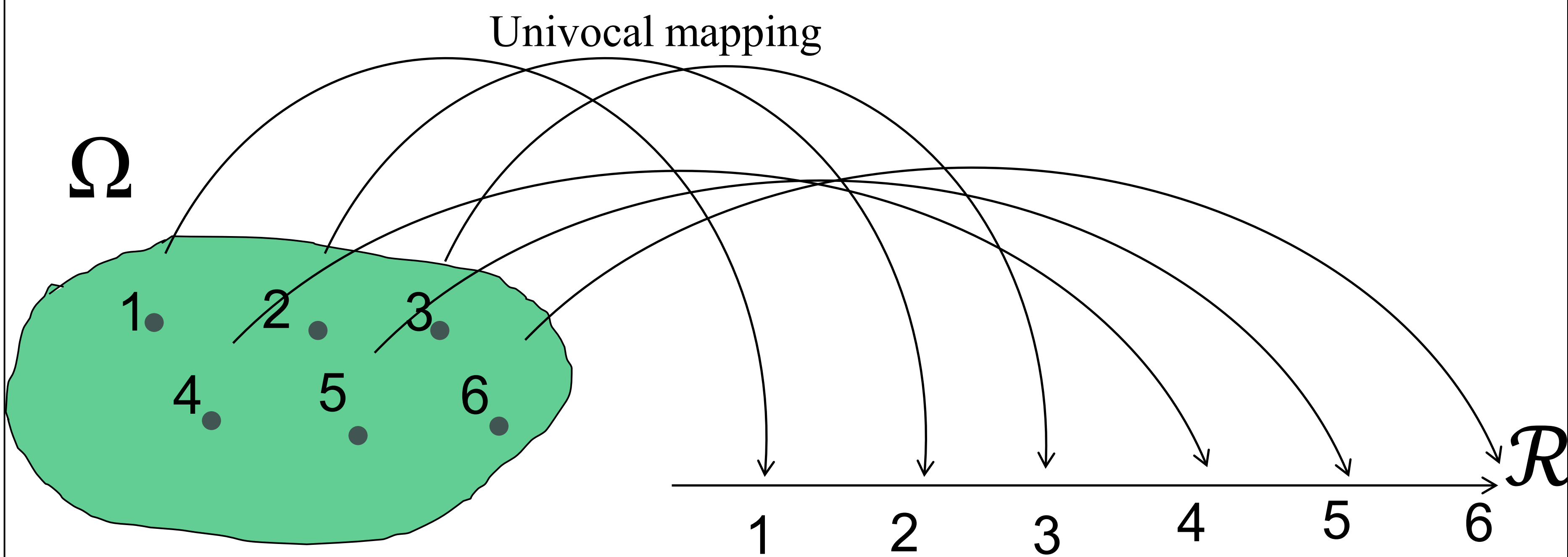
$E_3 = \Omega$

$X(\omega)$

$E_1=\{X<4.236\}$

$E_2=\{X<0\}$

$E_3=\{X<+\infty\}$



Random variables

Experiment: ε

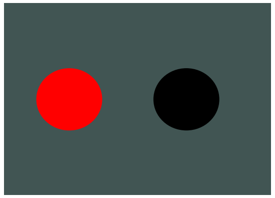
Sample space: Ω

Generic outcome: ω

$\rightarrow$ $X(\omega)$ random variable in $\mathcal{R}$

General mathematical models of random behaviours (It is not necessary to speak of the physical process)

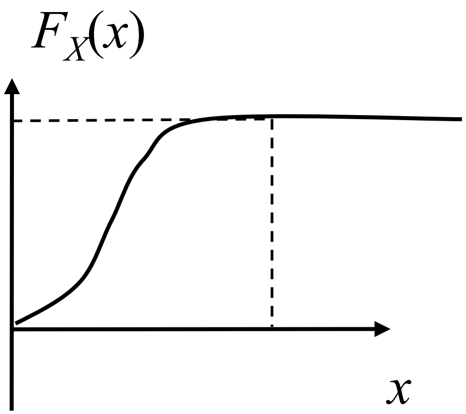
They apply to different physical phenomena which behave similarly



Probability distributions for reliability, safety and risk analysis

Probability functions (I)

- **Cumulative Distribution Function (cdf)**
  - $F_X(x) = P\{X \leq x\}$
  - Properties:



    - $\lim_{x \to -\infty} F_X(x) = 0$
    - $\lim_{x \to +\infty} F_X(x) = 1$
    - $F_X(x)$ is a non-decreasing function of x
    - The probability that X takes on a value in the interval $[a,b]$ is:

$$P\{a < X \leq b\} = F_X(b) - F_X(a)$$

Probability distributions for reliability, safety and risk analysis:

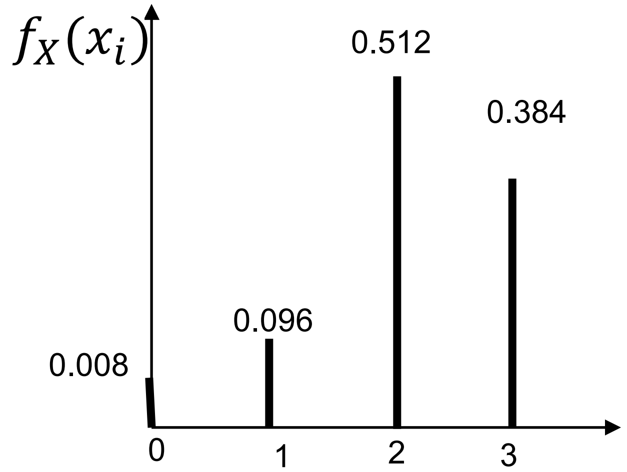
- discrete probability distributions
- continuous probability distributions

• Probability Mass Function (pmf)

- X – random variable takes discrete values $x_i, i = 0,1,\ldots,n$
  - ➢ Probability mass function:

$$f_X(x_i) = P\{X = x_i\} = p_i$$

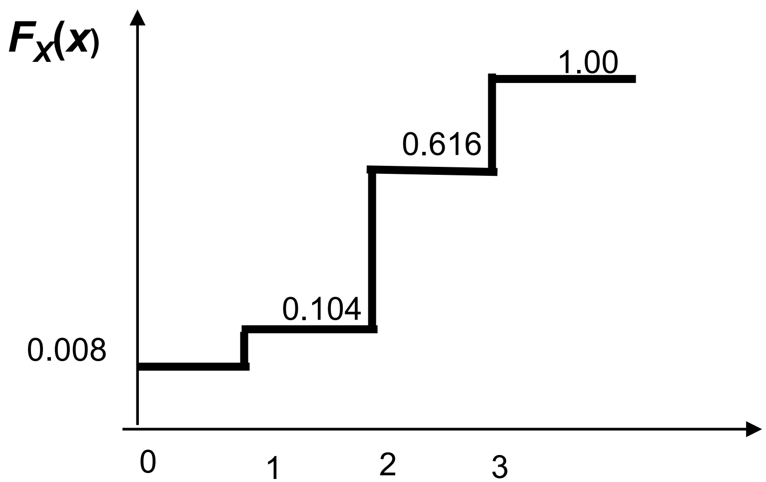
$$\sum_{i=0}^{n} f_X(x_i) = 1$$



  - ➢ Cumulative distribution function:

$$F_X(x) = P\{X \leq x\} =$$

$$= \sum_{i:\, x_i \leq x} f(x_i)$$



Summary measures: *median, variance, …*

- Mean Value (Expected Value):

$$\mu_X = E[x] = \sum_{i=1}^{n} x_i p_i$$

Where the probability mass is concentrated on average?

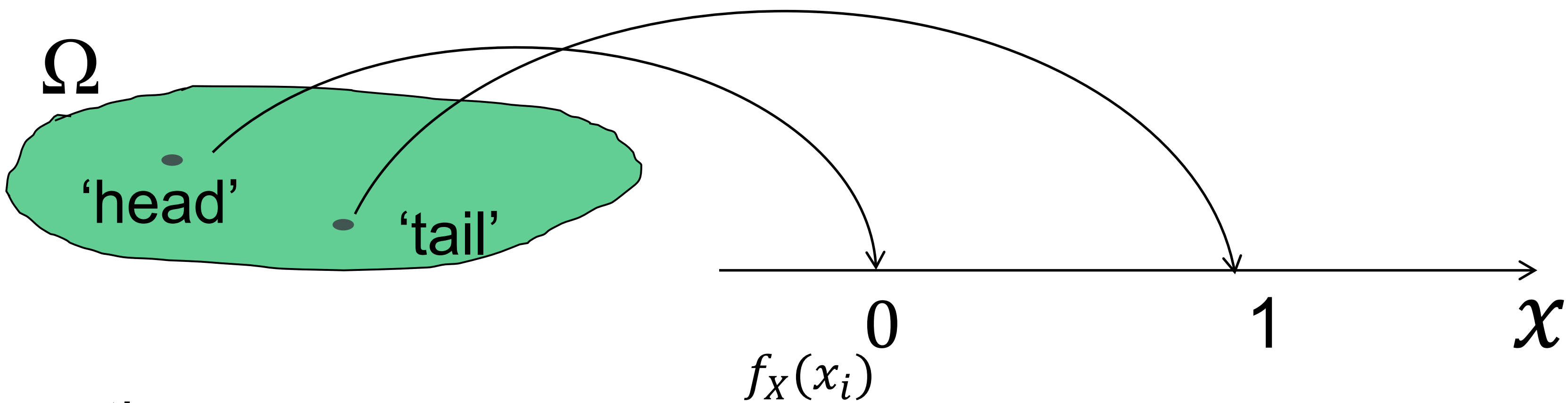
- Variance:

$$Var[X] = \sigma_X^2 = \sum_{i=1}^{n} (x_i - \mu_X)^2 p_i$$

It is a measure of the dispersion of the values around the mean

Example (Probabilty Mass Function)

Experiment: ε = {tossing a fair coin}

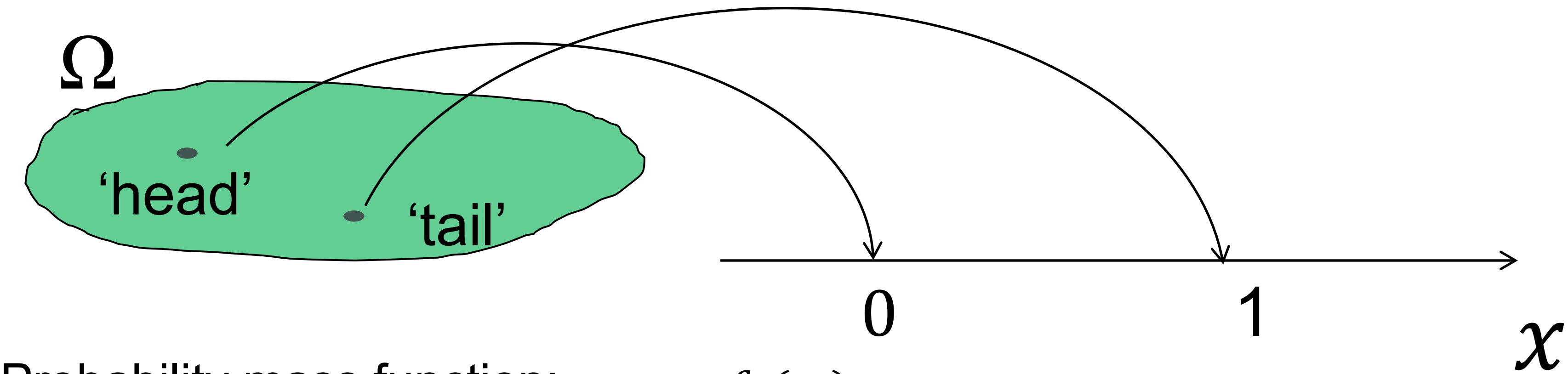


Questions:

1. Draw the probability mass function
2. Draw the cumulative distribution

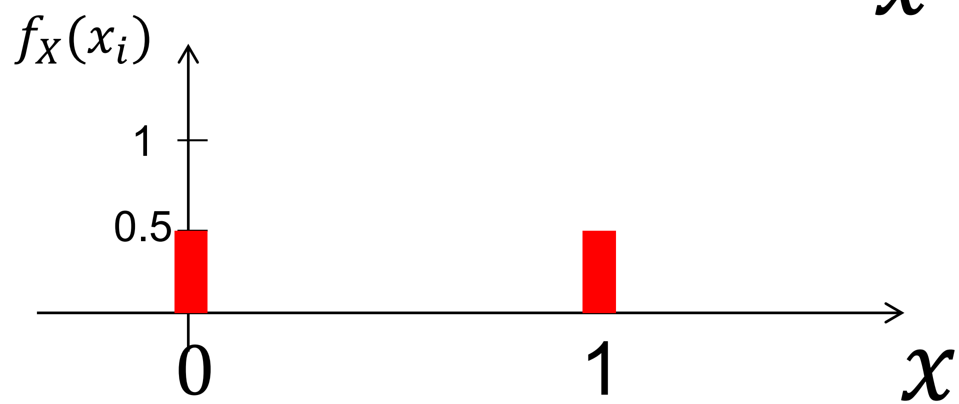
Example (Probabilty Mass Function)

Experiment: ε = {tossing a coin}



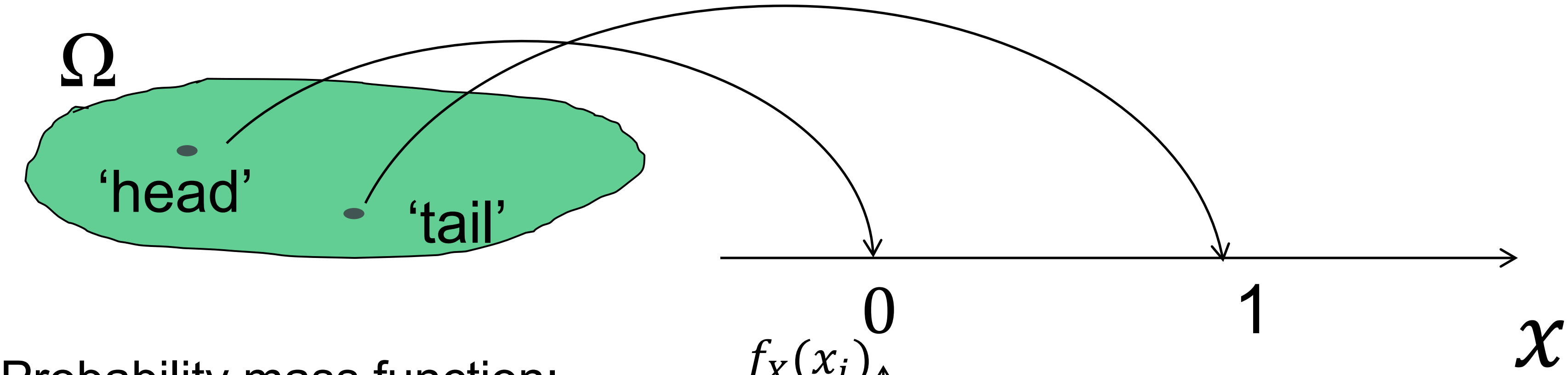
Probability mass function:

$$f_X(0) = P\{X = 0\} = 0.5$$
$$f_X(1) = P\{X = 1\} = 0.5$$



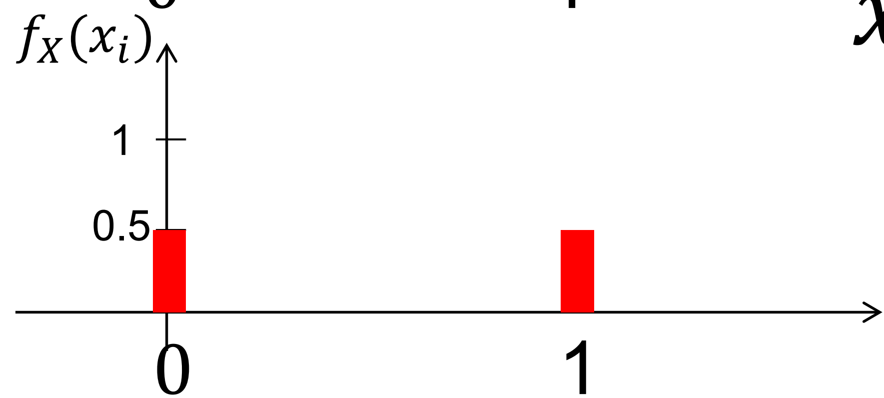
Example (Probabilty Mass Function)

Experiment: ε = {tossing a coin}

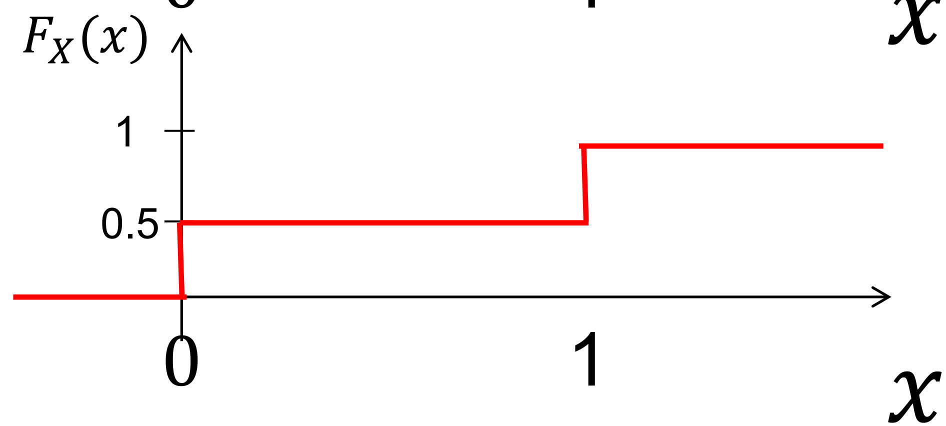


Probability mass function:

$$f_X(0) = P\{X = 0\} = 0.5$$
$$f_X(1) = P\{X = 1\} = 0.5$$



Cumulative distribution



SUGGESTION: DO EXERCISE 3

SUGGESTION FOR THE «BRAVE» STUDENTS: DO EXERCISE 4

Univariate discrete probability distributions

Univariate discrete probability distributions:

1) binomial distribution

2) geometric distribution

3) Poisson distribution

Univariate Discrete Distributions: Binomial Distribution (I)

Y = discrete random variable with only two possible outcomes:

- Y=1 (success) with $P\{Y=1\}=p$
- Y=0 (failure) with $P\{Y=0\}=1-p$

Bernoulli process

We perform n different trials of the experiment, $Y_1, \dots, Y_n$

X= discrete random variable counting the number of success out of the n trial (independently from the sequence with which successes appear):

$$X = \sum_{i=1}^{n} Y_i \qquad \Omega = \{0,1,2,\dots,n\}$$

The probability mass function: $\boxed{b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}}$ with $x=0,1,2,\dots,n$

$\binom{n}{x}$=binomial coefficient=$\frac{n!}{(n-x)!x!}$

Why?

- Probability of any specific sequence of *x* successes and *n*-*x* failures:

$$p^x(1-p)^{n-x}$$

- Number of sequences yielding to *x* successes out of *n* trials:

$$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

- All these sequences are mutually exclusive

$$b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

Univariate Discrete Distributions: Binomial Distribution (II)

$$\boxed{b(x;n,p) = \binom{n}{x} p^x (1-p)^{n-x}} \quad \text{with} \quad x=0,1,\ 2,\ldots n$$

$$\boxed{\begin{aligned} E[X] &= np \\ Var[X] &= np(1-p) \end{aligned}}$$

Univariate discrete probability distributions:

1) binomial distribution

2) geometric distribution

3) Poisson distribution

Univariate Discrete Distributions, Geometric Distribution

$p = \mathrm{P}\{\text{failure}\}$

T= trail of the first experiment whose outcome is "failure"

The probability mass function:

$$\boxed{g(t; p) = (1-p)^{t-1} p} \qquad t=1, 2, \ldots$$

Expected value:

$$\boxed{E[T] = \sum_{t=1}^{\infty} t(1-p)^{t-1} p = p[1 + 2(1-p) + 3(1-p)^2 + \ldots] = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}}$$

Univariate Discrete Distributions, Geometric Distribution

$p = \mathrm{P}\{\text{Failure}\}$

T= trail of the first experiment whose outcome is "failure" (or number of trials between two successive occurrences of failure);

The probability mass function:

$$\boxed{g(t;p) = (1-p)^{t-1}\,p} \qquad t=1, 2, \ldots$$

Expected value of T (or return period):

$$\boxed{E[T] = \sum_{t=1}^{\infty} t(1-p)^{t-1}\,p = p[1 + 2(1-p) + 3(1-p)^2 + \ldots] = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}}$$

SUGGESTION FOR THE «BRAVE» STUDENTS: DOUBLECHECK THE SOLUTION OF EXERCISE 4

Suggestion

DO EXERCISES 4 AND 5 (IN EXERCISE 5, WHEN NUMBERS ARE GETTING TOO BIG, JUST WRITE THE FORMULA WITHOUT COMPUTING THE NUMERICAL SOLUTION)

Univariate discrete probability distributions:

1) binomial distribution

2) geometric distribution

3) Poisson distribution

From the binomial to the poisson distribution

Approximation of the binomial distribution in the case of:

- $p \to 0$
- $n \to \infty$

$$b(x;n,p) = \frac{(np)^x}{x!} e^{-np}$$

It depends from only one parameter:

$$\mu = np=100 = \mathrm{E}[X]$$

which can be interpreted as the average number of successes in *n* experiments.

- $p \to 0$
- $n \to \infty$

$$b(x;n,p) = \frac{(np)^x}{x!} e^{-np} \to \pi(x;\mu) = \frac{(\mu)^x}{x!} e^{-\mu}$$

Suggestion

REVISE EXERCISE 5

Univariate Discrete Distributions, Poisson Distribution

Stochastic events that occur in a (continuous) period of time (e.g. failures, earthquakes,…):

- Rate of occurrence, λ, is constant

- Discrete Random Variable:

  K = number of events in the period of observation $(0, t)$

- Probability mass function: $\boxed{p(k;(0,t),\lambda) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}}$ $\quad k=0, 1, 2, \ldots$

$$\boxed{\begin{aligned} E[K] &= \lambda t \\ Var[K] &= \lambda t \end{aligned}}$$

Suggestion

DO EXERCISES 6 AND 7