POLITECNICO DI MILANO

Basic notions of probability theory

Contents

- Boolean Logic
- Definitions of probability
- Probability laws

Why a Lecture on Probability?

Lecture 1, Slide 22:

Risk

RISK = POTENTIAL DAMAGE + UNCERTAINTY

Dictionary: RISK = possibility of damage or injury to people or things

1) What undesired conditions may occur? → Accident Scenario, S
2) With what probability do they occur? → Probability, p
3) What damage do they cause? → Consequence, x

$$\text{RISK} = \{S_i, p_i, x_i\}$$

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Basic Definitions

Definitions: experiment, sample space, event

- **Experiment ε:** process whose outcome is a priori unknown to the analyst (all possible outcomes are a priori known)

- **Sample space Ω:** the set of all possible outcomes of ε.

- **Event *E*:** a set of possible outcomes of the experiment ε (a subset of Ω):

  the event E occurs when the outcome of the experiment ε is one of the elements of E.

Boolean Logic

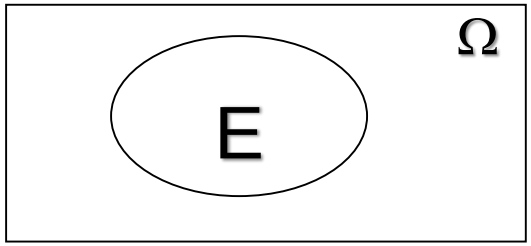
Definition: Certain events → Boolean Logic

***Logic of certainty*: an event** *E* can either **occur** or **not occur**

Indicator variable

$$X_E = \begin{cases} 0, \text{ when } E \text{ does not occur} \\ 1, \text{ when } E \text{ occurs} \end{cases}$$

Certain Events (Example)



(ε = die toss;Ω={1,2,3,4,5,6};E=Odd number)

I perform the experiment and the outcome is ‘3’

Event
E, X_E

True
$X_E=1$

Boolean Logic Operations

- **Negation:** $\overline{E}$
- **Union:** $X_{A \cup B}$
- **Intersection:** $X_{A \cap B}$

Boolean Logic Operations

- **Negation:** $\overline{E} \rightarrow \overline{X_E} = 1 - X_E$
- **Union:**

$$X_{A \cup B} = 1 - (1 - X_A)(1 - X_B) = 1 - \prod_{j=A,B} (1 - X_j) =$$

$$= \complement_{j=A,B} X_j = X_A + X_B - X_A X_B$$

- **Intersection:**

$$X_{A \cap B} = X_A X_B$$

- **Definition: *A* and *B* are mutually exclusive events if** $X_{A \cap B} = 0$

Exercise 1

Exercise 1

In an energy production plant there are two pumps: ‘pump P_1’ and ‘pump P_2’. Each pump can be in three different states:

‘0’= operating

‘1’= degraded

‘2’= failed

Consider the following events:

A = ‘pump P_1’ is degraded

B = both ‘pump P_1’ and ‘pump P_2‘ are failed

C = ‘pump P_2’ is not failed

Questions:

Which is the sample space Ω of the state of the pumps? Represent it graphycally?

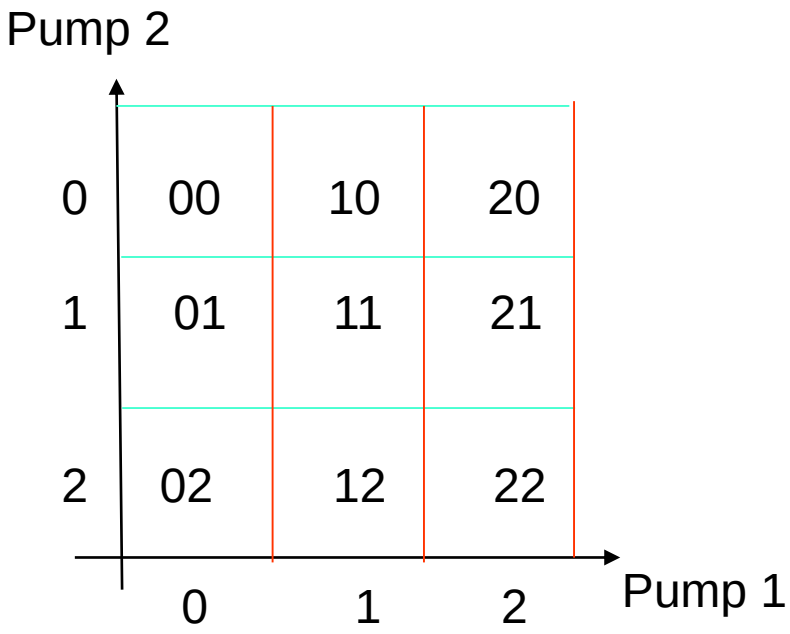
Represent events A, B and C

Find and represent $A \cap C$, $A \cup B$, $(A \cap C) \cup B$

Question 1: *Sample space Ω?*

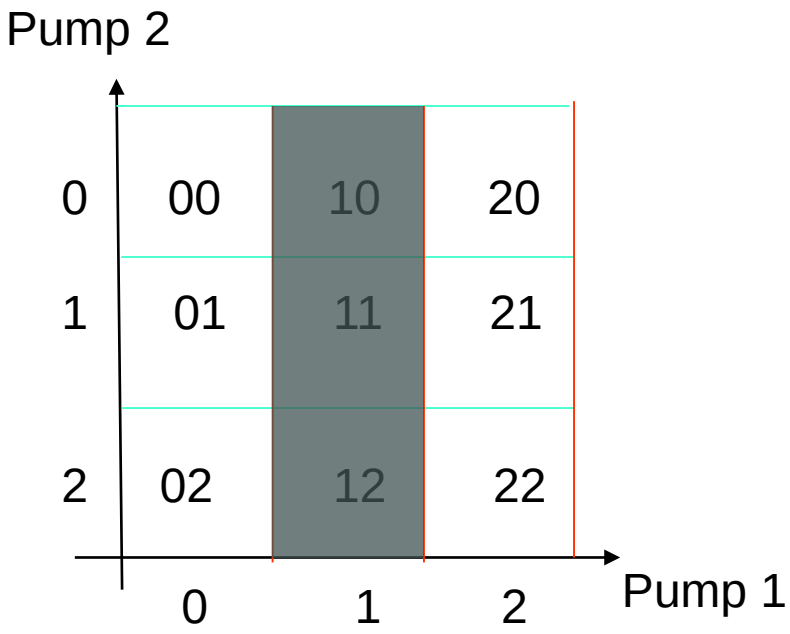
- Which is the sample space Ω?

Ω={‘00’,’01’,’02’,’10’,’11’,’12’,’20’,’21’,’22’}



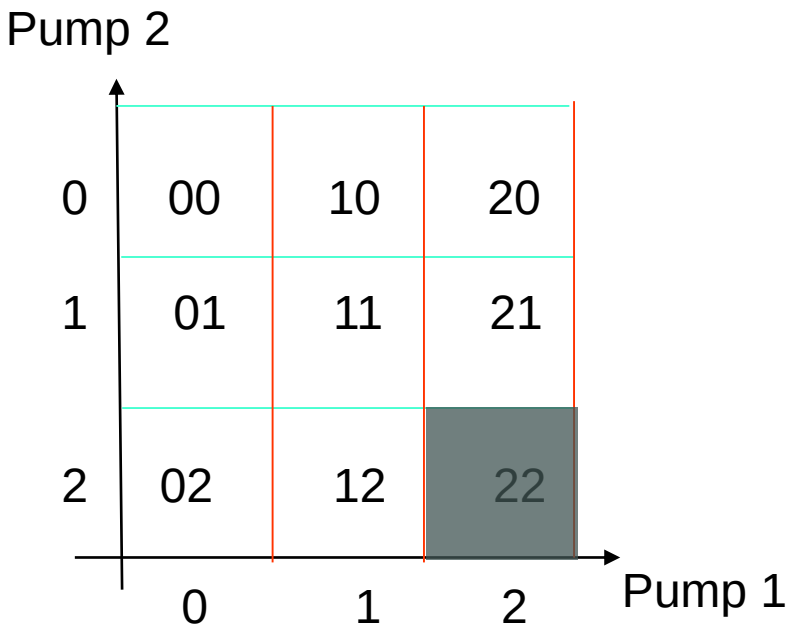
Question 2

- A= ‘pump 1’ is degraded



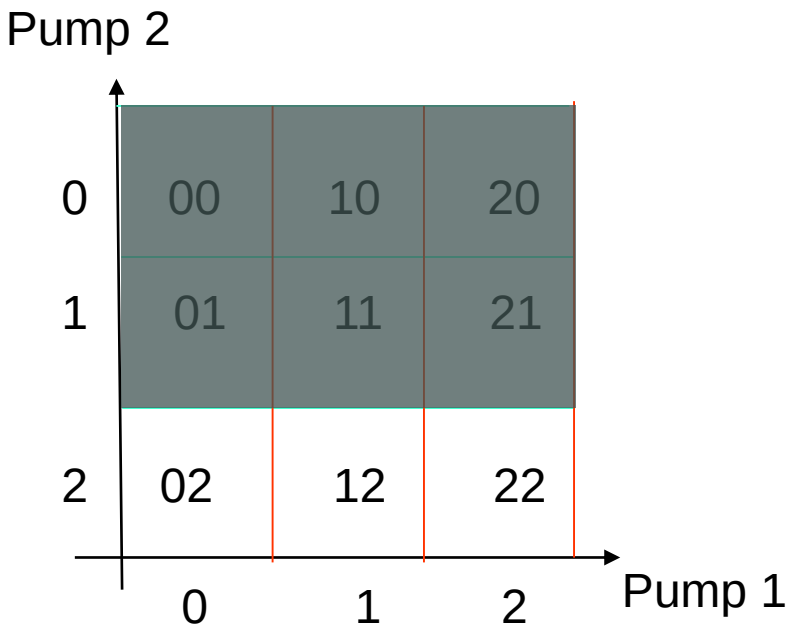
Question 2

- B = both ‘pump 1’ and pump 2 are failed



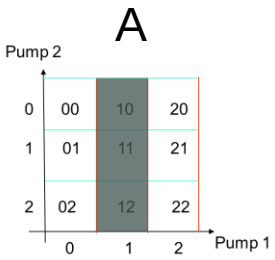
Question 2

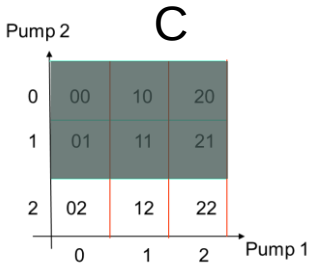
C ‘pump 2’ is not failed



Question 3

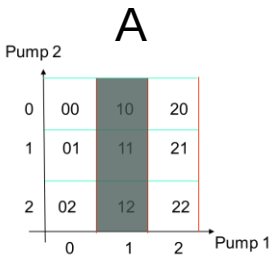
- A ∩ C

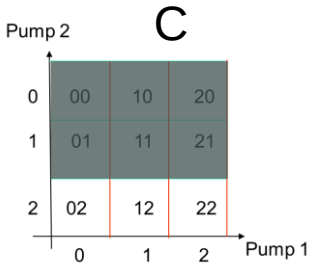


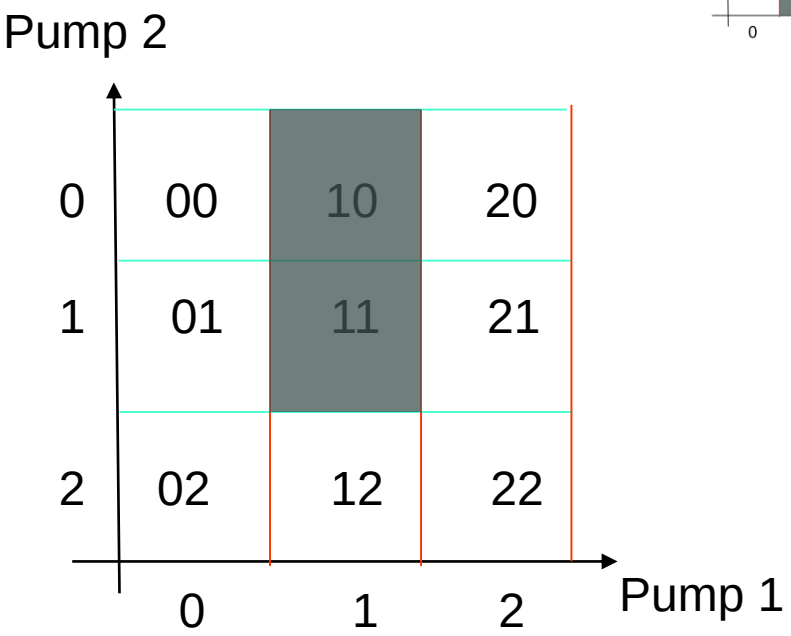


Question 3

- $A \cap C$

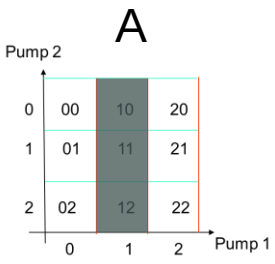


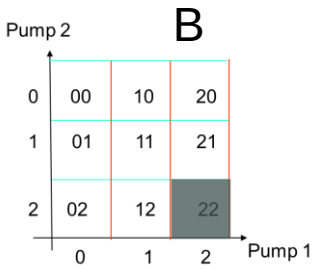




Question 3

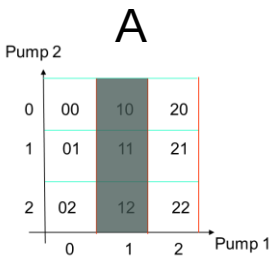
- A U B

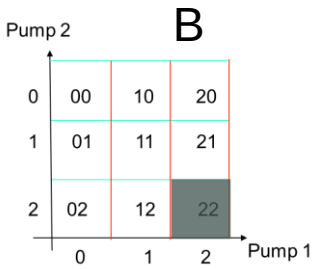


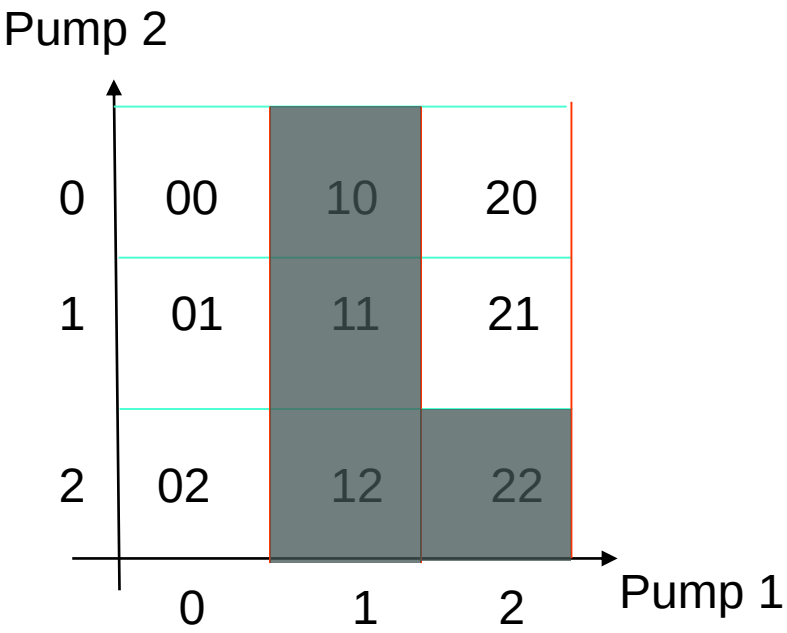


Question 3

- A U B

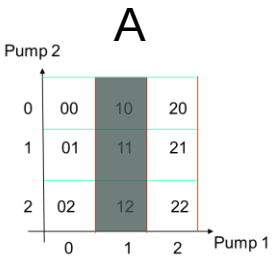


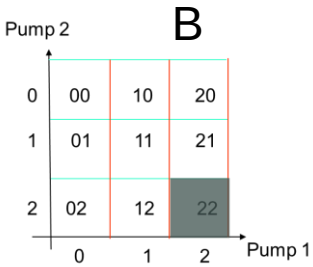


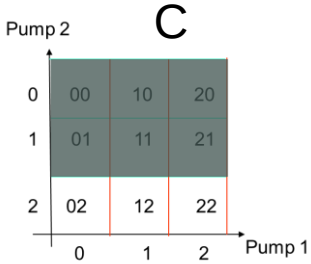


Question 3

- (A ∩ C) U B

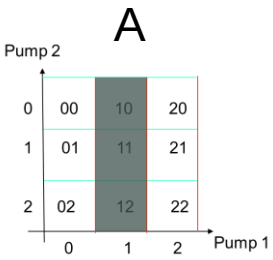


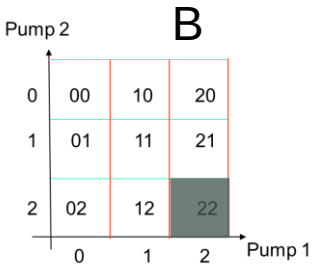


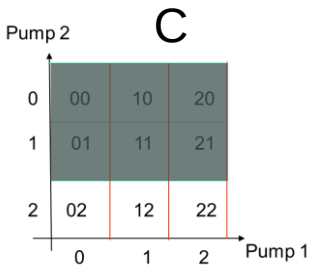


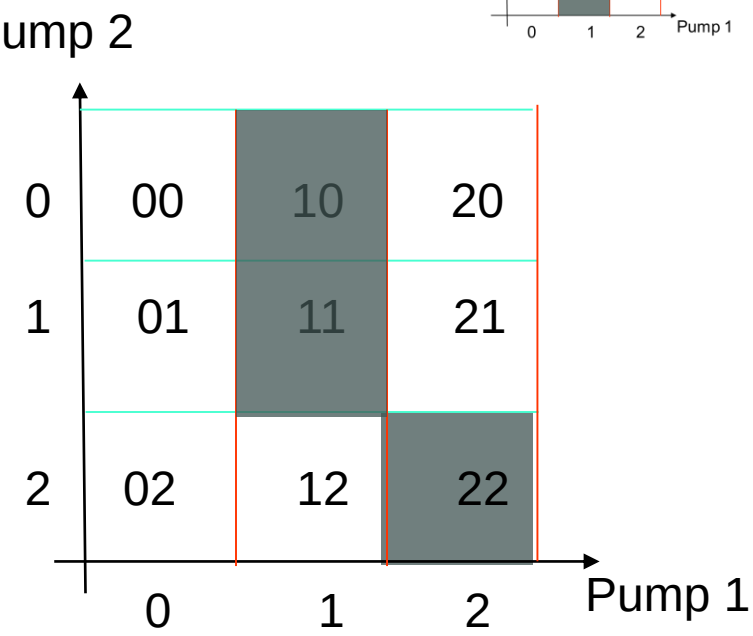
Question 3

- (A ∩ C) U B





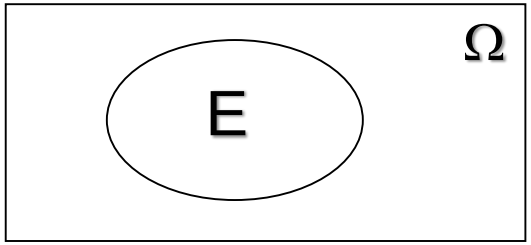




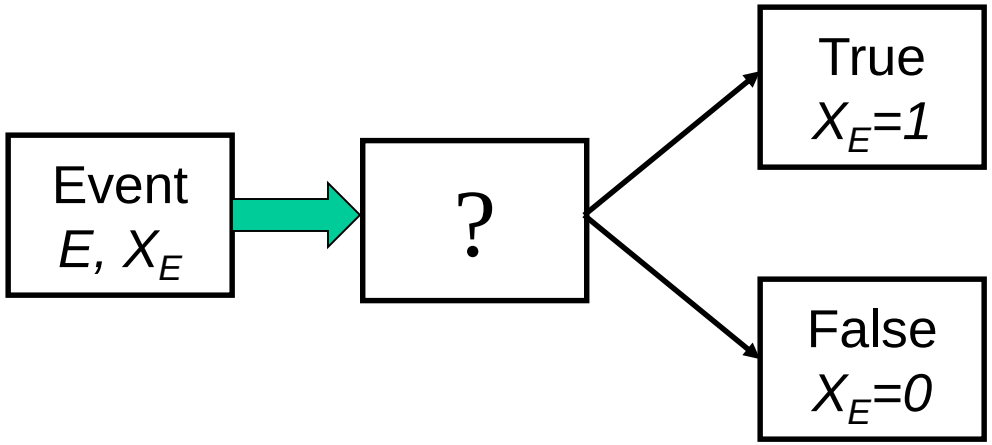
Uncertain Events

Uncertain Events

Let us consider: the experiment ε, its sample space Ω and the event *E*.

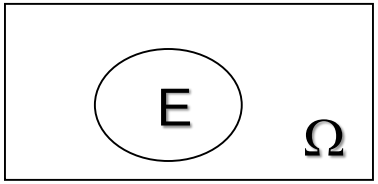


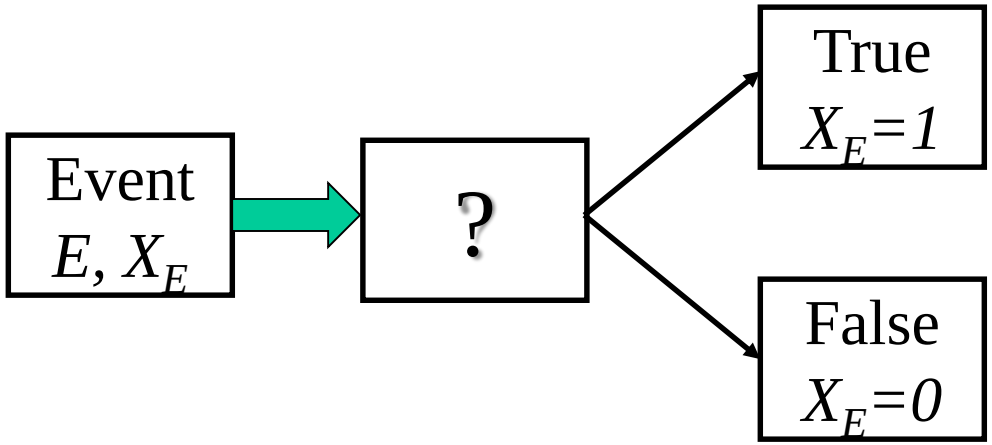
(ε = die toss; Ω={1,2,3,4,5,6}; E=Odd number)



Uncertain Events:

Let us consider: the experiment ε, its sample space Ω, event E.





Uncertain events can be compared → probability of E = p(E)

Probability for comparing the likelihood of events

Money for comparing the value of objects

Probability theory

Probability theory: Kolmogorov Axioms

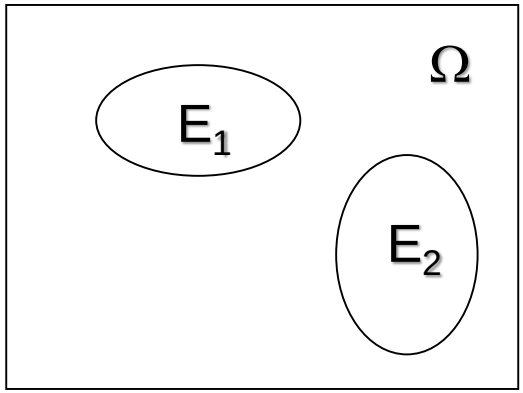
1. $0 \le p(E) \le 1$

2. $p(\Omega) = 1 \quad p(\varnothing) = 0$

3. Addition law:

   Let $E_1, \ldots, E_n$ be a finite set of mutually exclusive events: $(X_{E_i} \cap X_{E_j} = \emptyset)$.

$$p\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} p(E_i)$$



Definitions of probability

Three definitions of probability

1. **Classical definition**
2. **Empirical Frequentist Definition**
3. **Subjective definition**

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$ and the event:

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is it applicable?

- Gambling (e.g. tossing of a die)
- If no evidence favouring one outcome over others

$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

1. Classical Definition of Probability (criticisms)

- Let us consider an experiment with N possible elementary, mutually exclusive and equally probable outcomes: $A_1, A_2, \ldots, A_N$:

When is this requirement met?

In most real life situations the outcomes are not equally probable!

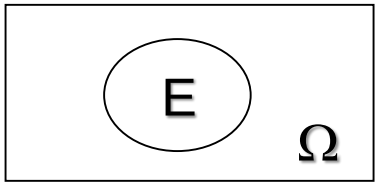
$$E = A_1 \cup A_2 \cup \ldots \cup A_M$$

$$p(E) = \frac{number\ of\ outcomes\ resulting\ in\ E}{total\ number\ of\ possible\ outcomes} = \frac{M}{N}$$

2. Frequentist Definition of Probability

Let us consider: the experiment ε, its sample space Ω and an event *E*.

ε = die toss;Ω={1,2,3,4,5,6};E={Odd number}



- *n* times ε, *E* occurs *k* times

  (*n* = 100 die tosses → *k*=48 odd numbers)

- *k/n* = the relative frequency of occurrence of *E*

  (k/*n* = 48/100=0.48)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

p is defined as the probability of *E*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

- This is not a limit from the matemathical point of view [limit of a numerical series]
- It is not possible to repeat an experiment an infinite number of times …
- We tacitly assume that the limit exists

Possible way out?

probability as a physical characteristic of the object:

- the physical characteristics of a coin (weight, center of mass, …) are such that when tossing a coin over and over again the fraction of 'head' will be *p*

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

- Applicable only to those events for which we can conceive of a repeatable experiment (e.g. not to the event «your professor will be sick tomorrow»)
- The experiment conditions cannot be identical
  - let us consider the probability that a specific valve *V* of a specific Oil & Gas plant will fail during the next year
  - *what should be the population of similar valves?*
    - Large population: all the valves used in industrial plants. Considering data from past years, we will have a large number *n,* but data may include valves very different to *V*
    - Small population: valve used in Oil&Gas of the same type, made by the same manufacturer with the same technical characteristics → too small *n* for limit computation.

Similarity Vs population size dilemma

2. Frequentist Definition of Probability (criticisms)

$$\lim_{n\to\infty} \frac{k}{n} = p$$

Some events (e.g. in the nuclear industry) have very low probabilities (e.g. $p \approx 10^{-6}$) (RARE EVENTS)

Very difficult to observe

The frequentist definition is not applicable

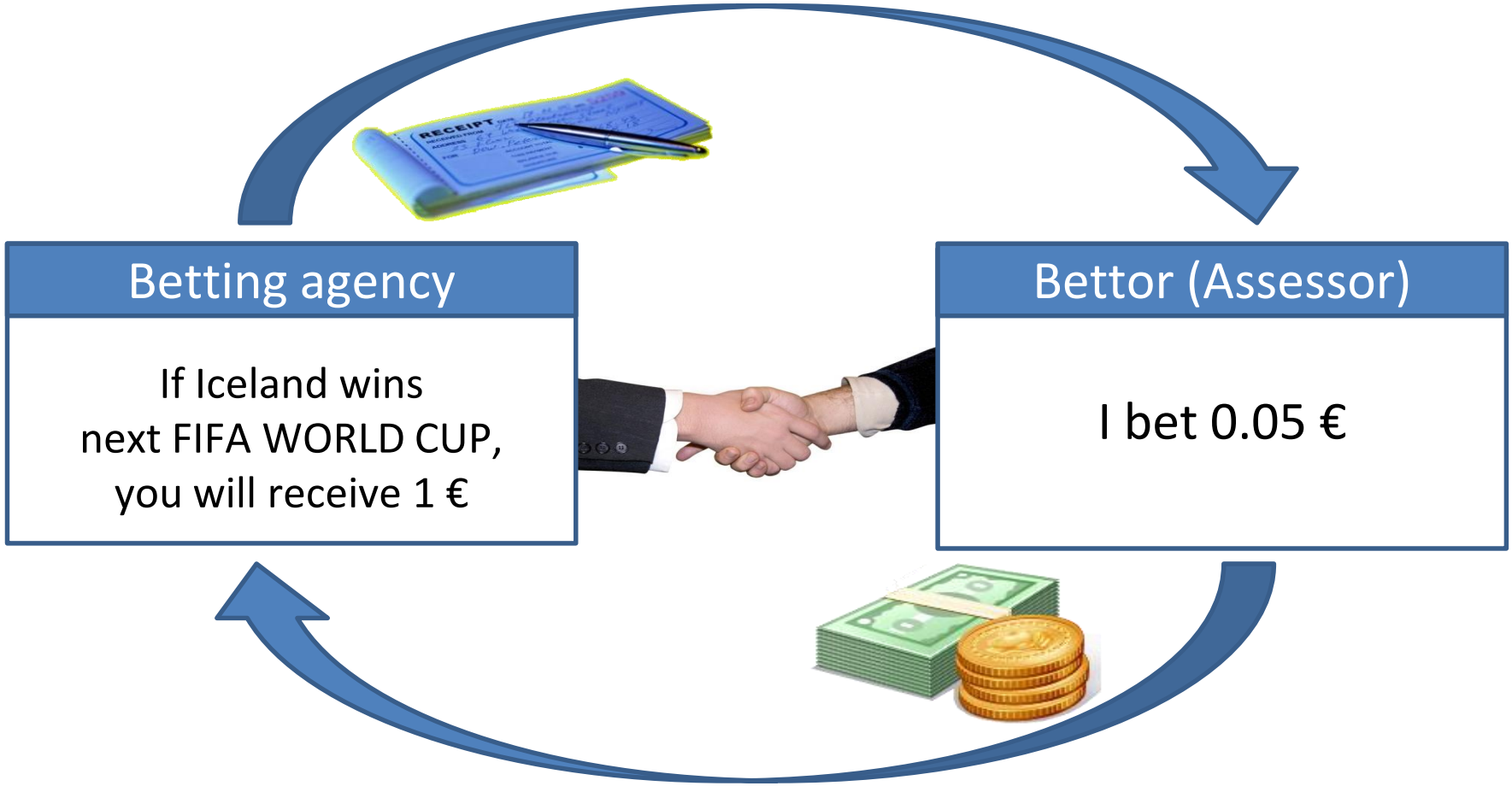
3. Subjective Definition of Probability

P(*E*) is the **degree of belief** that a person (assessor) has that E will occour, **given all the relevant information currently known to that person** (background knowledge)

- Probability is a numerical encoding of the state of knowledge of the assessor (De Finetti: "probability is the feeling of the analyst towards the occurrence of the event")
- P(*E*) is conditional on the background knowledge *K* of the assessor:
  $$\mathbf{P(E)= P(E|K)}$$
- Background knowledge typically includes data/models/expert knowledge
- If the background knowledge changes → the probability may change
- Two interpretations of subjective probability:
  - Betting interpretation
  - Reference to a standard for uncertainty

3. Betting Interpretation

P {Iceland will win next UEFA EURO 2020|K} = 0.05

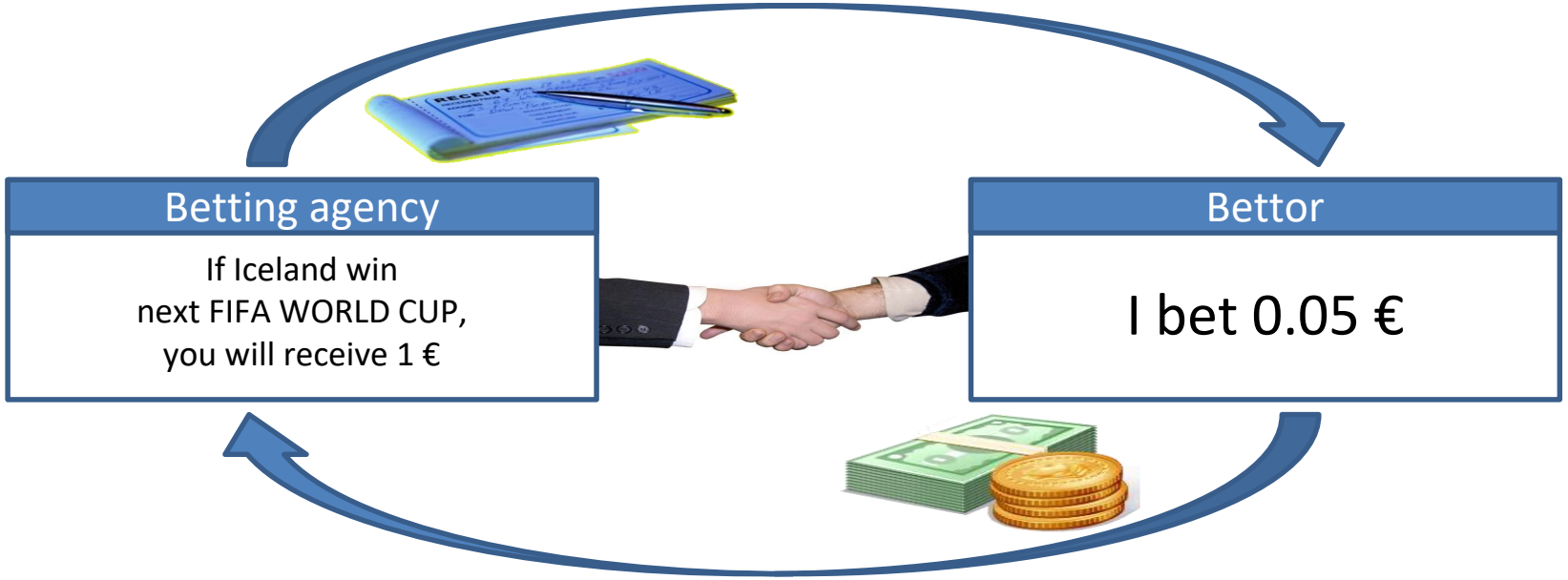


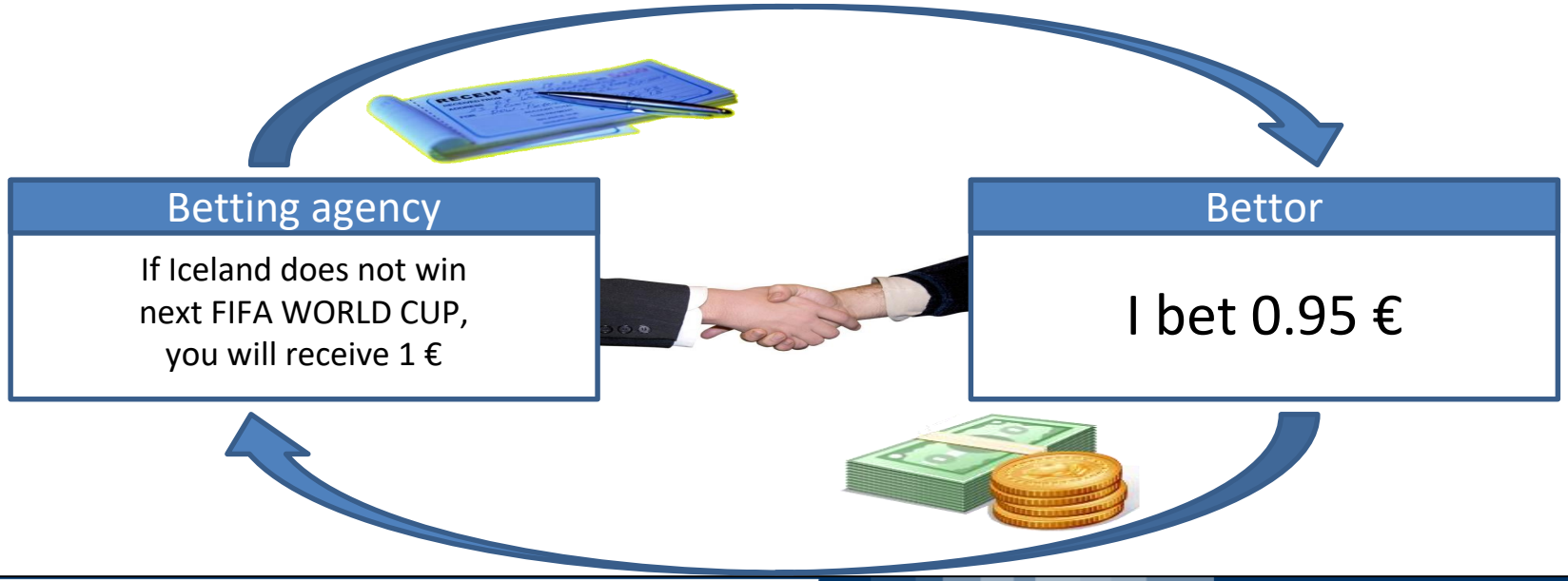
3. Betting Interpretation

P(E) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

The opposite must also hold: 1- P(*E*) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event E were not to occur and nothing otherwise.

3. Betting Interpretation: two sideness of the bet





3. Betting Interpretation (Criticism)

P(*E*) is the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event *E* were to occur and nothing otherwise.

The opposite must also hold (1- P(*E*)) is also the amount of money that the person assigning the probability would be willing to bet if a single unit of payment were given in return in case event E were not to occur and nothing otherwise.

probability assignment depends from the value judgment about money and event consequences (the assessor may even think that in case of LOCA in a Nuclear Power Plant he/she will die and so the payment will be useless)

3. Reference to a standard for uncertainty

P(*E*) is the number such that the uncertainty about the occurrence of *E* is considered equivalent by the person assigning the probability (assessor) to the uncertainty about drawing a red ball from an urn containing P(*E*)*100% red balls

E={Germany will win next FIFA WORLD CUP}

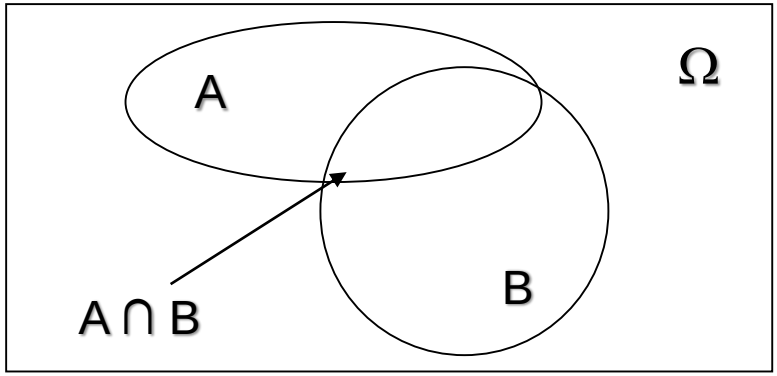
$$P(E)=0.33$$

urn

Probability laws

Probability laws (1)

- Union of two non-mutually exclusive events



$$P_{A \cup B} = P_A + P_B - P_{A \cap B}$$

It can be demonstrated by using the three Kolmogorov axioms*

$$P_{A \cup B} \leq P_A + P_B$$

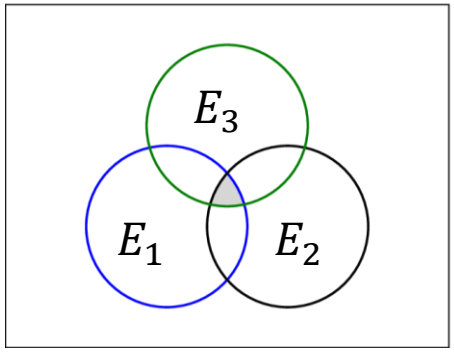
- Rare event approximation: A and B events are considered as mutually exclusive ($A \cap B = \emptyset$) → $P(A \cap B) = 0$ →

$$P_{A \cup B} = P_A + P_B$$

* http://www.ucs.louisiana.edu/~jcb0773/Berry_probbook/425chpt2.pdf

Probability laws (2)

- Union of non-mutually exclusive events: $E_{\cup} = \bigcup_{i=1,\dots,n} E_i$



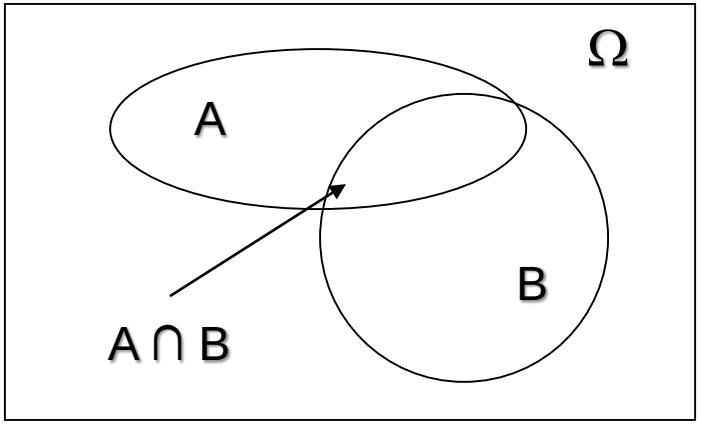
$$P(E_{\cup}) = \sum_{i=1}^{n} P(E_i) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P\left(E_i \cap E_j\right) + \cdots + (-1)^{n+1} P(E_1 \cap E_2 \cap \cdots \cap E_n)$$

- Upper bound $P(E_U) \leq \sum_{j=1}^{n} P(E_j)$
- Lower bound $P(E_U) \geq \sum_{j=1}^{n} P(E_j) - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(E_i \cap E_j)$
- Rare event approximation: events are considered as mutually exclusive ($E_i \cap E_j = \emptyset, \forall i, j, i \neq j$) → $P(E_{\cup}) = \sum_{i=1}^{n} P(E_i)$

Probability laws (3)

- Conditional Probability of A given B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$



- Event A is said to be statistically independent from event B if:

$$P(A \mid B) = P(A)$$

- If A and B are statistically independent then:

$$P(A \cap B) = P(A)P(B)$$

Exercise 2

P_a and P_b are two pumps in parallel sharing a common load.

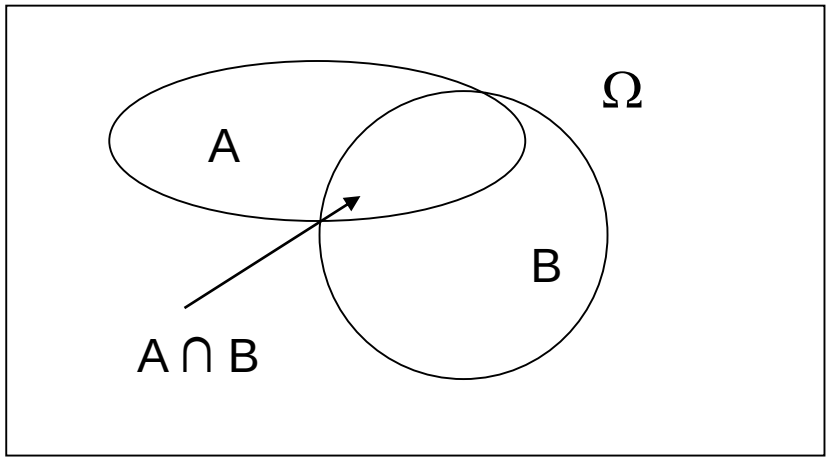
Let *A* denoting the event '*pump* P_a *is failed*' and B the event '*pump* P_b *is failed*' with *P*(*A*) = 0.040, *P*(*B*) = 0.075 and *P*(*A* U *B*) = 0.080

Questions:

1. What is the probability that both pumps are failed?
2. What is the probability tha pump P_a is also failed given that pump P_b is failed?
3. What is the probability tha pump P_b is also failed given that pump P_a is failed?

Ex. 2 - Question 1 - Solution

- The probability that both pumps are failed:

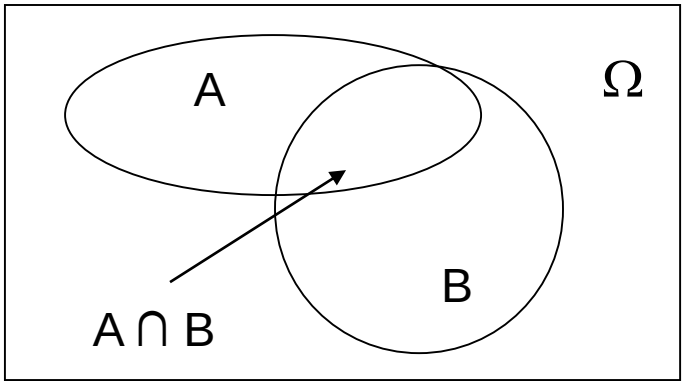


$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.040 + 0.075 - 0.080 \\ &= 0.035 \end{aligned}$$

Ex. 2: *Other questions - Solutions*

- The probability that pump P_a is also failed given that pump P_b is failed:



$$P(A|B)=P(A \cap B)/P(B)=$$
$$=0.035/0.080= 0.46$$

- The probability that pump P_b is also failed given that pump P_a is failed:

$$P(B|A) = P(A \cap B)/P(A) =$$
$$= 0.035/0.040 = 0.875$$

Theorem of Total Probability

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events. In terms of Boolean events:

$$E_i \cap E_j = 0 \quad \forall i \neq j \qquad \bigcup_{j=1}^{n} E_j = \Omega$$

Ω

E_1	E_2	E_3
E_4	E_5	E_6

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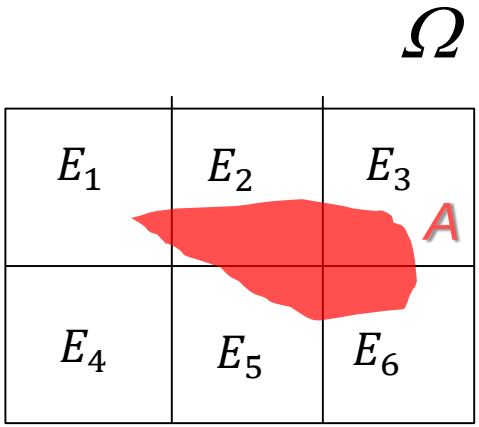


- Given any event A in Ω, its probability can be computed in terms of the partitioning events and the conditional probabilities of A on these events: $A = \cup_j (A \cap E_j) \rightarrow P(A) = \sum_j P(A \cap E_j)$

$$P(A) = P(A|E_1)P(E_1) + P(A|E_2)P(E_2) + \ldots + P(A|E_n)P(E_n)$$

Exercise 3

A motor operated valve opens and closes intermittently on demand to control the coolant level in an industrial process. An auxiliary battery pack is used to provide power for approximately the 0.5% of the time when there are plant power outages. The demand failure probability of the valve is $3 \cdot 10^{-5}$ when operated from the plant power and $9 \cdot 10^{-4}$ when operated from the battery pack. You are requested to:

- find the demand failure probability assuming that the number of demands is independent of the power source. Is the increase due to the battery pack operation significant?

Exercise 3: Solution

A motor operated valve opens and closes intermittently on demand to control the coolant level in an industrial process. An auxiliary battery pack is used to provide power for approximately the 0.5% of the time when there are plant power outages. The demand failure probability of the valve is $3 \cdot 10^{-5}$ when operated from the plant power and $9 \cdot 10^{-4}$ when operated from the battery pack. You are requested to:

- find the demand failure probability assuming that the number of demands is independent of the power source. Is the increase due to the battery pack operation significant?

$X =$ {power outage} → $P(X) = 0.005$,

$P(\bar{X}) = 1 - P(X) = 0.995$

$Y =$ {valve failure}

$P(Y|\bar{X}) = 3 \cdot 10^{-5}$

$P(Y|X) = 9 \cdot 10^{-4}$

Theorem of Total Probability:

$$P(Y) = P(Y|X)P(X) + P(Y|\bar{X})P(\bar{X}) = 0.995 \cdot 3 \cdot 10^{-5} + 0.005 \cdot 9 \cdot 10^{-4} = 3.43 \cdot 10^{-5}$$

The battery pack operation is causing a 15% increase of the valve failure probability, even if the battery is used only for the 0,5% of the time

Exercise 4

The air pollution in a city is caused mainly by industrial (I) and automobile (A) exhausts. In the next 5 years, the chances of successfully controlling these two sources of pollution are, respectively, 75% and 60%. Assume that if only one of the two sources is successfully controlled, the probability of bringing the pollution below acceptable level would be 80%.

- What is the probability of successfully controlling air pollution in the next 5 years?
- If, in the next 5 years, the pollution level is not sufficiently controlled, what is the probability that is entirely caused by the failure to control automobile exhaust?
- If pollution is not controlled, what is the probability that control of automobile exhaust was not successful?

Exercise 4 - *Solution*

- Events:

A = event of successful control of the automobile exhausts

I = event of successful control of the industrial exhausts

E = event of bringing the pollution below the acceptable level

- From the problem statement we have:

$$P(I) = 0.75$$
$$P(A) = 0.60$$

$$P\left(E \middle| \bar{A} I\right) = P\left(E \middle| A \bar{I}\right) = 0.8$$
$$P\left(E \middle| \bar{A} \bar{I}\right) = 0$$
$$P\left(E \middle| AI\right) = 1$$

Exercise 4 - *Solution*

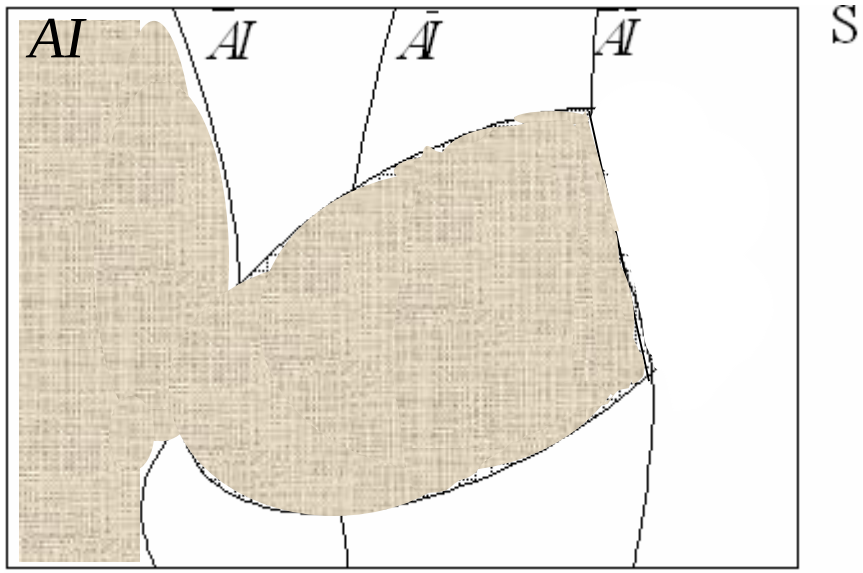
Probability of controlling air pollution in the next 5 years:

- The possible combinations of the two pollution events are:

$$AI, A\bar{I}, \bar{A}I, \bar{A}\bar{I}$$

- We assume statistical independence between I and A:

$$P(AI) = 0.60 \cdot 0.75 = 0.45$$
$$P(A\bar{I}) = 0.60 \cdot 0.25 = 0.15$$
$$P(\bar{A}I) = 0.40 \cdot 0.75 = 0.30$$
$$P(\bar{A}\bar{I}) = 0.40 \cdot 0.25 = 0.10$$



- We can use the Theorem of total probability:

$$P(E) = P(E \mid AI)P(AI) + P(E \mid A\bar{I})P(A\bar{I}) + P(E \mid \bar{A}I)P(\bar{A}I) + P(E \mid \bar{A}\bar{I})P(\bar{A}\bar{I})$$
$$= 0.81$$

Exercise 4 - *Solution*

Pollution level is not sufficiently controlled, what is the probability that is entirely caused by the failure to control automobile exhaust :

$$P(\bar{A}I \mid \bar{E}) = \frac{P(\bar{E} \mid \bar{A}I)P(\bar{A}I)}{P(\bar{E})} = \frac{[1 - P(E \mid \bar{A}I)]P(\bar{A}I)}{P(\bar{E})} = 0.32$$

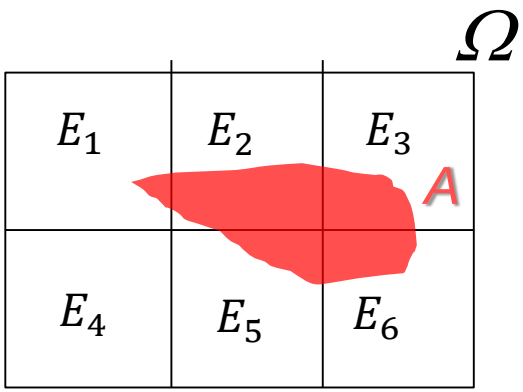
Probability automobile that exhaust is not controlled given that pollution is not controlled

$$P(\bar{A} \mid \bar{E}) = P\left(\bar{A}I \cup \overline{AI} \mid \bar{E}\right) = P\left(\bar{A}I \mid \bar{E}\right) + P\left(\overline{AI} \mid \bar{E}\right) = \frac{P(\bar{E} \mid \bar{A}I)P(\bar{A}I)}{P(\bar{E})} + \frac{P(\bar{E} \mid \overline{AI})P(\overline{AI})}{P(\bar{E})}$$

$$= \frac{0.2 \cdot 0.3}{0.19} + \frac{1 \cdot 0.1}{0.19} = 0.84$$

Bayes Theorem

- Let us consider a partition of the sample space Ω into n mutually exclusive and exhaustive events E_j. We know
- Event A has occurred



Can I use this information to update the probability of $P(E_j)$?

$$P(E_i \mid A) = \frac{P(E_i A)}{P(A)} = \frac{P(A \mid E_i)P(E_i)}{\sum_{j=1}^{n} P(A \mid E_j)P(E_j)}$$

theorem of total probability

The Bayesian Subjective Probability Framework

P(E|K) is the **degree of belief** of the **assigner** with regard to the occurrence of E (numerical encoding of the **state of knowledge** – K - of the assessor)

Bayes Theorem to update the probability assignment in light of new information

Updated

Old

$$P(E_i|A,K) = \frac{P(A|E_i,K) \cdot P(E_i|K)}{\sum_{j=1}^{n} P(A|E_j,K) \cdot P\left(E_j|K\right)}$$

new information

Example: *Problem statement*

Consider a pile foundation, in which pile groups are used to support the individual column footings. Each of the pile group is designed to support a load of 200 tons. Under normal condition, this is quite safe. However, on rare occasions the load may reach as high as 300 tons. The foundation engineer wishes to know the probability that a pile group can carry this extreme load of up to 300 tons.

Based on experience with similar pile foundations, supplemented with blow counts and soil tests, the engineer estimated a probability of 0.70 that any pile group can support a 300-ton load. Also, among those that have capacity less than 300 tons, 50% failed at loads less than 280 tons.

To improve the estimated probability, the foundation engineer ordered one pile group to be proof-loaded to 280 tons.

If the pile group survives the specified proof load, what is the probability that the pile group can support a load of 300 tons?

Example: *Solution*

- B = event that the capacity of pile group ≥ 300 tons
- T = event of a successful proof load to 280 tons.

$$P(B) = 0{,}7$$
$$P(\bar{T}|\bar{B}) = 0.5$$
$$P(T|B) = 1$$

- Bayes' theorem then gives

$$P(B|T) = \frac{P(T|B)P(B)}{\sum_C P(T|C)P(C)} = \frac{P(T|B)P(B)}{P(T|B)P(B) + P(T|\bar{B})P(\bar{B})} = \frac{1 \cdot 0.7}{1 \cdot 0{,}7 + (1 - 0.5) \cdot 0.3}$$

- If the proof test is successful, the required probability is increased from 0.70 to 0.824.