Fatigue crack growth assessment method subject to model uncertainty

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Abstract

Fatigue crack growth (FCG) is an important degradation process of many critical mechanical equipment. Probabilistic FCG models are often used to account for the variability among FCG process conditions. In the well-known model of Yang and Manning, a deterministic FCG model is randomized by multiplying the crack growth rate with a random multiplier assumed to obey a lognormal distribution and unknown parameters are jointly estimated through Maximum likelihood estimation. By so doing, the modeling error due to inappropriate choice of the deterministic FCG model and that due to unsuitable assignment of the probability distribution of the random multiplier cannot be distinguished. Besides, the model uncertainty of the random multiplier is not explicitly considered. In this paper, a two-step least-square estimation method is proposed, which estimates the unknown parameters in the deterministic FCG model at first, and generates a sample set for the estimation of the random multiplier considering model uncertainty by way of Bayesian model selection. In Bayesian model selection, three types of Bayes factor are considered to select the appropriate candidate model and a simulation experiment is carried out to guide their selection. The effectiveness and feasibility of the proposed method are illustrated through two case studies using the real FCG datasets.

Keywords: Fatigue crack growth analysis; Model uncertainty; Yang and Manning’s probabilistic FCG model; Two-step least-square estimation; Bayesian model selection

1 Introduction

FCG between the emerging of a crack and the occurrence of fatigue fracture is main degradation process leading to failure of many critical mechanical equipment. In order to predict equipment lifetime, the crack growth process needs to be modeled. Deterministic FCG models have been built based on the physical mechanism analysis of fatigue crack [1,2]. In these models, the mechanic properties of metallic material are considered homogeneous and constant, but a large amount of experimental data has indicated the variability among FCG processes even if the same cyclic loading are applied [3]. Therefore, over the last few decades, probabilistic models have been applied to incorporate randomness in the FCG process [4].

In order to make use of the existing knowledge of crack growth mechanism, many probabilistic FCG models have been developed by inserting random factors into the original deterministic FCG models [3,5–7]. By this way, a deterministic FCG model was randomized to account for the randomness in the FCG processes. One of the most well-known methods has been proposed by Yang and Manning [5], which randomized a deterministic FCG model by multiplying the growth rate with a random factor (multiplier). Yang and Manning proposed to use a lognormal distribution to describe the multiplier variability and the maximum likelihood estimation (MLE) method to estimate the unknown parameters in the deterministic FCG model and in the lognormal distribution of the random multiplier [5].

Two problems are raised when applying Yang and Manning’s probabilistic FCG model. Firstly, the unknown parameters in the deterministic FCG model and lognormal distribution are simultaneously estimated by the MLE method; their estimates jointly maximize the likelihood function. Then, the modeling error due to inappropriate choice of deterministic FCG model and that due to the unsuitable assignment of the probability distribution of the random multiplier cannot be distinguished. In this situation, prediction accuracy is difficult to improve because the dominant source of error cannot be identified. Secondly, in most of the research works the random multiplier is assumed to obey a lognormal distribution [8]. This assumption is validated by the classical Kolmogorov-Smirnov (K-S) hypothesis test, wherein the lognormal distribution is found suitable at a certain level of confidence and without comparison with other candidate distribution. Therefore, the uncertainty in determining the best distribution model should be considered.

Model uncertainty has received increased attention in reliability engineering. Yu and Chang [9] considered model uncertainty in accelerated life data analysis by using the Bayesian model averaging method. The Bayesian model averaging method was also utilized to assess the computational fracture models by Hamdia et al. [10]. Zhang and Mahadevan [11] utilized a Bayesian updating approach to deal with the model uncertainty between the physical reliability models and statistical reliability models. Zio and Apostolakis [12] proposed the Adjustment Factor method to quantify the model uncertainty based on expert judgment for a nuclear safety problem. The method was, then, extended to establish a confidence band for the reliability of model prediction in [13]. Instead, to the best knowledge of the authors, the lognormal assumption was always applied without consideration of model uncertainty in the application of Yang and Manning's model [3,14,15].

In this paper, a two-step least-square (TSLS) estimation method is proposed based on Yang and Manning's probabilistic FCG model. The proposed method can estimate the unknown parameters in the deterministic FCG model at first, and assess the fitness of the deterministic FCG model by classical goodness-of-fit methods such as adjusted- R^2 . Then, samples for the estimation of the random multiplier can be generated and the model uncertainty with regard to different probability distributions (e.g. lognormal distribution, gamma distribution, inverse Gaussian distribution) is taken into account. Here, model uncertainty refers to the uncertainty involved in the selection of the best model from a set of candidates [13], and different methods such as information criteria [16] and Bayesian model selection [17] have been developed to deal with it. The Akaike information criteria (AIC) measures the difference between candidate models and true data-generating process from the perspective of Kullback–Leibler divergence [16]. The Bayesian model selection method compares candidate models in pairs by calculating the ratio of the model posterior probabilities, and selects the candidate model with the highest posterior probability as reference. In this paper, Bayesian model selection is utilized because of its mathematical stability and convenience of updating inferences using real-time data. In Bayes model selection, the better a model fits the data, the larger its posterior probability is. As the data volume approaches infinite, the ratio of the posterior probability of the right model to that of the wrong model will approach infinite. In engineering, the volume of obtained data may not be large enough to make this method select the right model. Under these circumstances, averaging the inferences of all the competing models can give more comprehensive results. In [18], the Bayesian model selection method is able to achieve better simulating results compared with AIC for degradation data analysis.

In Bayesian model selection, Bayes factor is the ratio of marginal likelihoods corresponding to two different models [19]. Note that model posterior probability is calculated by multiplying the model prior probability with the marginal likelihood of the obtained data. The ratio of two model posterior probabilities is equal to the Bayes factor when the model prior probabilities are set to be uniformly distributed. To select the appropriate candidate model, three types of Bayes factor are considered in this paper, which are: (1) Bayes factor with uniform parameter prior (UPP), (2) posterior Bayes factor and (3) Bayes factor based on cross validation (BFCV). Each has its own reasonability. A simulation experiment is carried out to compare their performance. According to the results, the BFCV is recommended.

Two improvements are achieved in the proposed method compared with the method of Yang and Manning. Firstly, by utilizing the adjusted- R^2 statistics, different kinds of deterministic FCG models are assessed and the most appropriate one can be selected instead of choosing one arbitrarily. Secondly, the model uncertainty with respect to the distribution of multiplier is considered and dealt with by using Bayesian model selection. Based on the Bayes factor, three candidate distributions (~~G~~gamma, ~~I~~inverse Gaussian and ~~L~~lognormal) can compete with each other in data fitting. With the model prior probabilities set to be uniformly distributed, larger normalized Bayes factor of a model indicates stronger evidence of the model conforming to the true data-generating distribution. The appropriate model can be built according to the inferences from all the candidates based on the calculated normalized Bayes factors, which are defined as model weights [20,21]. In this way, the assumption of the lognormal distribution in the method of Yang and Manning can be relaxed.

The rest of this paper is organized as follows. In Section 2, Yang and Manning's probabilistic FCG model and the proposed TSLS method are introduced in consideration of four types of deterministic FCG models [22]. In Section 3, Bayesian model selection dealing with the model uncertainty of the random multiplier in Yang and Manning's probabilistic FCG model is explained, and three types of Bayes factor are presented. In Section 4, a simulation experiment is carried out to compare the performance of the three types of Bayes factor. In Section 5, the effectiveness and feasibility of the proposed method are illustrated through two case studies on the datasets of literature [3] and [4]. Section 6 concludes the work.

2 TSLS estimation method based on the Yang and Manning's probabilistic model

2.1 Yang and Manning's probabilistic model

To consider the randomness in the fatigue crack growth process, Yang and Manning [5] proposed a very general formula frame to randomize deterministic FCG models as follows:

$$\frac{da(t)}{dt} = G\left(\Delta K, R, K_{\max}, S, a(t)\right) \cdot X(a(t)) \tag{1}$$

where G indicates a general non-negative deterministic crack growth rate function which could be constructed as a surrogate model of experimental data [23], or determined based on the knowledge of the FCG mechanism; ΔK

is the stress intensity factor range; R is the stress ratio; K_{max} is the maximum stress intensity factor in a load cycle; S is the maximum stress level; $a(t)$ is the crack length at time t; $X(a(t))$ is the random part characterizing the randomness in the crack growth process. In this paper, Yang's assumption [8] that $X(a(t)) = X$ is a random multiplier obeying a certain probability distribution is applied because of its mathematical simplicity and conservative nature [24].

Let δ denote the parameters in the deterministic FCG model, $f(x|\theta)$ denote the probability density function (PDF) of the multiplier X, and θ denote the parameters in the probability distribution. Then, the Eq. (1) can be reformulated as:

$$\frac{da(t)}{dt} = G(a(t), \delta) \cdot X \tag{2}$$

$X f(x|\theta)$ $X \sim f(x|\theta)$

After the logarithm is taken on both sides of the Eq. (2), we can obtain that

$$log\left(\frac{da(t)}{dt}\right) = log(G(a(t), \delta)) + Z \tag{3}$$

where $Z = log(X)$.

In Yang's study [8], the random multiplier was assumed to follow a lognormal distribution with a mean value of 1 and an unknown variance σ . As a result, Z followed the normal distribution with a mean value of zero, $Z\, N(0,\sigma)$ $Z \sim N(0,\sigma)$, and

$$log\left(\frac{da(t)}{dt}\right) N(log(G(a(t), \delta)), \sigma) \quad log\left(\frac{da(t)}{dt}\right) \sim N(log(G(a(t), \delta)), \sigma) \tag{4}$$

The PDF of crack growth rate is

$$f_{normal}\left[log\left(\frac{da(t)}{dt}\right)\right] = \frac{1}{2\pi\sigma} \exp\left[-\frac{\left(log\left(\frac{da(t)}{dt}\right) - log(G(a(t), \delta))\right)^2}{2\sigma^2}\right] \tag{5}$$

where the unknown parameters can be estimated by the MLE method based on crack growth data. Supposing that the crack growth data set is $\mathbb{A} = \left\{\left(\frac{da(t)}{dt}\Big|_{a(t)=a_i^k}, a_i^k\right)\right\}$, where $k = 1, 2, \cdots, K$ indicates the k -th crack growth path which has I^k monitored crack size values, $a_i^k, i = 1, 2, \cdots, I^k$ indicates the i -th monitored crack size value of the k -th crack growth path and $\frac{da(t)}{dt}\Big|_{a(t)=a_i^k}$ indicates the corresponding crack growth rate of a_i^k . Based on Eq. (5), the maximum likelihood estimators can be obtained as

$$\left(\hat{\delta}, \hat{\sigma}\right) = \arg max\left(\prod_{k=1}^{K}\prod_{i=1}^{I^k} f_{normal}\left(log\left(\frac{da(t)}{dt}\Big|_{a(t)=a_i^k}\right)\right)\right) \tag{6}$$

Because of practicability and computational simplicity, the assumed probability distribution of the random multiplier and parameter estimation method have been widely applied [3,14,15]. In this situation, the estimates obtained in Eq. (6) are the same as the least-square (L-S) estimates. However, two problems arise. Firstly, the unknown parameters in the deterministic FCG model and that of lognormal distribution are simultaneously estimated by the MLE method. In this situation, their estimates jointly maximize the likelihood function and the modeling error due to the inappropriate choice of the deterministic FCG model and that due to the unsuitable assignment of the probability distribution of the multiplier cannot be distinguished. Prediction accuracy is, then, difficult to improve, because the source of error cannot be identified. Secondly, the random multiplier in most of the research works is assumed to obey a lognormal distribution [8]. But this assumption was only validated by classical Kolmogorov-Smirnov (K-S) hypothesis test, and the lognormal distribution is suitable only at a certain level of confidence with no comparison to other candidate distributions.

As a result, a TSLS estimation method is here proposed based on Yang and Manning's probabilistic FCG model. The proposed method can estimate the unknown parameters in the deterministic FCG model at first, and assess the deterministic FCG model by goodness-of-fit methods. Afterwards, the samples for the estimation of the random multiplier are generated and the model uncertainty for different probability distributions can be considered.

2.2 TSLS estimation method

The proposed TSLS method employs the L-S estimation twice. In the first step, the random multiplier is neglected and the unknown parameters in the deterministic FCG model are estimated from all the obtained data.

$$\hat{\delta}_{L-S} = \arg\min_{\delta} \left(\sum_{k=1}^{K} \sum_{i=1}^{I} \left(log \left(\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right) - log \left(G \left(a_i^k, \delta \right) \right) \right)^2 \right) \tag{7}$$

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In the second step, the random multiplier of each crack growth path is estimated as

$$\hat{z}_k = \arg\min_{z} \left(\sum_{i=1}^{I} \left(log \left(\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right) - log \left(G \left(a_i^k, \hat{\delta}_{L-S} \right) \right) - z \right)^2 \right), k = 1, 2, \cdots, K \tag{8}$$

$$\hat{x}_k = \exp \left(\hat{z}_k \right), k = 1, 2, \cdots, K \tag{9}$$

where the $\hat{x}_k$ is the estimate of the random multiplier for the k -th crack growth path.

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By so doing, all the obtained data are used to estimate the unknown parameters in the deterministic FCG model, and the goodness-of-fit methods, e.g. Pearson correlation coefficient, adjusted- R^2 , can be used to test whether the chosen deterministic FCG model is suitable. Besides, a sample set for the estimation of the random multiplier can be generated, with sample size equal to the number of FCG paths. If data of new crack growth paths are obtained, more estimates of random multiplier can be obtained for the different crack growth paths, and information reflecting unit-to-unit variation is available. If subsequent data of certain crack growth paths are later obtained, the estimation accuracy of the corresponding random multipliers can be further improved.

In this paper, four types of deterministic FCG models are considered: (a) power function; (b) polynomial function; (c) rational function; (d) function based on curve fitting technique, which have been reviewed in [22]. Most of widely-used crack growth models belong to these four categories, such as Paris law [25] and its improved forms [26] which are sorted into the power function, and Forman's model which describes the third stage of fatigue crack growth [27] can be sorted into the rational function. Their detailed formulas are as follows:

Power function

$$\frac{da(t)}{dt} = X \cdot Q[a(t)]^b \tag{10}$$

Polynomial function

$$\frac{da(t)}{dt} = X \cdot \left\{ p + q \cdot a(t) + r \cdot [a(t)]^2 \right\} \tag{11}$$

Rational function

$$\frac{da(t)}{dt} = X \cdot \frac{Q_1 [a(t)]^b}{Q_2 - Q_3 \sqrt{a(t)}} \tag{12}$$

Function based on curve fitting technique

$$\frac{da(t)}{dt} = X \cdot \frac{1}{Q_1 [a(t)]^b + Q_2} \tag{13}$$

The parameters which need to be estimated are Q, b in power function, p, q, r in polynomial function, Q_1, Q_2, Q_3, b in rational function, and Q_1, Q_2, b in function based on curve fitting technique. By replacing $L\left(a_i^k, \hat{\delta}\right)$ $G\left(a_i^k, \hat{\delta}\right)$ in Eq.

(7) by the above four functions, the TSLS estimates of the unknown parameters and the random multiplier can be obtained as

Power function:

$$\left(\hat{Q}_{L-S}, \hat{b}_{L-S}\right) = \arg\min_{(Q,b)} \left(\sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - b \cdot log\left(a_i^j\right) + log(Q) \right\}^2 \right) \quad \left(\hat{Q}_{L-S}, \hat{b}_{L-S}\right) = \arg\min_{(Q,b)} \left(\sum_{k=1}^{K} \sum_{i=1}^{I^k} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - b \cdot log\left(a_i^k\right) - log(Q) \right\}^2 \right) \tag{14}$$

$$\hat{z}_k = \arg\min_{z} \left(\sum_{i=1}^{I} \left\{ \log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - \hat{b}_{L-S} \cdot log\left(a_i^j\right) + log\left(\hat{Q}_{L-S}\right) - z \right\}^2 \right), k = 1, 2, \cdots, K \quad \hat{z}_k = \arg\min_{z} \left(\sum_{i=1}^{I^k} \left\{ \log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - \hat{b}_{L-S} \cdot log\left(a_i^k\right) - log\left(\hat{Q}_{L-S}\right) - z \right\}^2 \right), k = 1, 2, \cdots, K \tag{15}$$

Polynomial function:

$$\left(\hat{p}_{L-S}, \hat{q}_{L-S}, \hat{r}_{L-S}\right) = \arg\min_{(p,q,r)} \left(\sum_{k=1}^{K} \sum_{i=1}^{I} \text{A3} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - \log \left[p + q \cdot a_i^j + \text{A1} \left(a_i^j\right)^2 \text{A2} \right] \right\}^2 \right) \tag{16}$$

$$\hat{z}_k = \arg\min_{z} \left(\sum_{i=1}^{I} \text{A4} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] - \log \left(\hat{p}_{L-S} + \hat{q}_{L-S} \cdot a_i^j + \text{A5}_{L-S} \cdot \left[a_i^j\right]^2 \text{A6} \right) - z \right\}^2 \right), k = 1, 2, \cdots, K \tag{17}$$

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Rational function:

$$\left(\widehat{Q}_{1L-S}, \widehat{Q}_{2L-S}, \widehat{Q}_{3L-S}, \hat{b}_{L-S}\right) = \arg\min_{(Q_1,Q_2,Q_3,b)} \left(\sum_{k=1}^{K} \sum_{i=1}^{I} \text{A3} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] + \log \left(Q_2 - Q_3 \sqrt{a_i^j} \right) \text{A2} \, b \cdot \log\left(a_i^j\right) \text{A1} \log\left(Q_1\right) \right\}^2 \right) \tag{18}$$

$$\hat{z}_k = \arg\min_{z} \left(\sum_{i=1}^{I} \text{A4} \left\{ log \left[\frac{da(t)}{dt} \bigg|_{a(t)=a_i^k} \right] + \log \left(\widehat{Q}_{2L-S} - \widehat{Q}_{3L-S} \sqrt{a_i^j} \right) \text{A5} \, \hat{b}_{L-S} \cdot \log\left(a_i^j\right) \text{A6} \log\left(\widehat{Q}_{1L-S}\right) - z \right\}^2 \right), k = 1, 2, \cdots, K \tag{19}$$

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Function based on curve fitting technique:

$$\left(\widehat{Q}_{1L-S}, \widehat{Q}_{2L-S}, \widehat{b}_{L-S}\right) = \arg\min_{(Q_1,Q_2,b)} \left(\sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] + \log \left[Q_1 \left(a_i^k\right)^b + Q_2 \right] \right\}^2 \right) \quad (20)$$

$$\widehat{z}_k = \arg\min_z \left(\sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] + \log \left[\widehat{Q}_{1L-S} \left(a_i^k\right)^{\widehat{b}_{L-S}} + \widehat{Q}_{2L-S} \right] - z \right\}^2 \right), k = 1, 2, \cdots, K \quad (21)$$

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In this paper, adjusted- R^2 [28] is chosen to assess the deterministic FCG models. The larger the value of the adjusted- R^2 is, the better the deterministic FCG model is. Adjusted- R^2 is a correction of R^2 attempting to prevent the automatically and spuriously increasing of R^2 when extra parameters are added into the model. The adjusted- R^2 can be obtained as

$$adjusted - R^2 = 1 - \frac{\left(\sum_{k=1}^{K} I^k\right) - 1}{\left(\sum_{k=1}^{K} I^k\right) - p - 1} \cdot \frac{SS_{res}}{SS_{tot}} \quad (22)$$

where $\sum_{k=1}^{K} I^k$ is the total size of the obtained dataset $\mathbb{A}$, p indicates the number of unknown parameters in the deterministic FCG model (2 in power function, 3 in polynomial function, 4 in rational function, and 3 in function based on curve fitting technique), SS_{tot} denotes the total sum of squares which is obtained as

$$SS_{tot} = \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] - \frac{1}{\sum_{k=1}^{K} I^k} \sum_{k=1}^{K} \sum_{i=1}^{I^k} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] \right\} \right\}^2 \quad (23)$$

and SS_{res} denotes the residual sum of squares which is obtained as

$$SS_{res} = \begin{cases} \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] - b \cdot log\left(a_i^j\right) \cdot log(Q) \right\}^2 & Power \\ \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] - \log\left(p + q \cdot a_i^j + \left[a_i^j\right]^2\right) \right\}^2 & Polynomial \\ \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] + \log\left(Q_2 - Q_3 \sqrt{a_i^j}\right) \cdot b \cdot \log\left(a_i^j\right) \log\left(Q_1\right) \right\}^2 & Rational \\ \sum_{k=1}^{K} \sum_{i=1}^{I} \left\{ log \left[\frac{da(t)}{dt} \Big|_{a(t)=a_i^k} \right] + \log\left(Q_1 \left[a_i^k\right]^b + Q_2\right) \right\}^2 & Curvefitting \end{cases} \quad (24)$$

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After that, a set of random multiplier estimates can be obtained based on the chosen deterministic FCG model with the largest value of adjusted- R^2 , and some methods, e.g. AIC or Bayesian model selection, could be applied to consider the model uncertainty rather than utilizing one type of distribution. Bayesian model selection is chosen because of its mathematical stability and convenience of updating inferences using real-time data. The detailed procedure of Bayesian model selection is introduced in Section 3.

3 Introduction and application of Bayesian model selection

In Section 2, the TSLS estimation method has been proposed to generate the samples for the estimation of the random multiplier. Based on the generated samples, Bayesian model selection is applied to deal with the model uncertainty associated with the random multiplier. In this section, the theory and implementation of Bayesian model selection are introduced. Besides, to select the appropriate candidate model, three types of Bayes factor are considered.

3.1 The Bayesian model selection method

In Bayesian model selection, a model set consisting of different candidate probability distributions is constructed, which is assumed to contain the true data-generating process. Bayesian model selection compares candidate models in pairs by calculating the ratio of the model posterior probabilities, and selects the candidate model with the highest posterior probability as reference. In Bayes model selection, the better a model fits the data, the larger its posterior probability is. As the data volume approaches infinite, the ratio of the posterior probability of the right model to that of the wrong model will approach infinite. If the true data-generating process is not in the model set, the most similar one will be selected. Supposing that $X = \left\{\hat{x}_k, k = 1, 2, \cdots, K\right\}$ is the samples for the estimation of the random multiplier generated by using the TSLS estimation, $\mathbb{M} = \left\{M_c, c \in I_c\right\}$ is the model set, then, different models compete with each other by

$$\frac{posterior\left(M_p\right)}{posterior\left(M_q\right)} = \frac{pr\left(X\middle|M_p\right)\cdot prior\left(M_p\right)}{pr\left(X\middle|M_q\right)\cdot prior\left(M_q\right)}, p, q \in I_c \tag{25}$$

In Eq. (25), $prior(M_p), p\in I_c$ is the model prior representing the prior evidence of the model M_p before the data are obtained; $pr\left(X\middle|M_p\right), p \in I_c$ is the marginal likelihood calculated by

$$pr\left(X\middle|M_p\right) = \int L\left(X\middle|\theta_p, M_p\right) prior\left(\theta_p\right) d\theta_p \tag{26}$$

where $prior(\theta_c)$ is the prior distribution of the parameter vector θ_c , $L\left(X\middle|\theta_c, M_c\right)$ is the likelihood function calculated by

$$L\left(X\middle|\theta_p, M_p\right) = \prod_{\hat{x}_k\in X} f\left(\hat{x}_k\middle|\theta_p, M_p\right) \tag{27}$$

As a ratio, $\frac{posterior\left(M_p\right)}{posterior\left(M_q\right)}>1$ indicates stronger evidence of the model M_p conforming to the true data-generating distribution. The inference of Bayesian model selection is given based on both the prior knowledge and the obtained data. When new data are obtained, by setting the model prior probabilities as the obtained model posterior probabilities during the last Bayesian model selection analysis, the new inference of Bayesian model selection can be calculated based on both the historical and the new data. In this way, the inference can be revised and updated iteratively. In Bayesian model selection, Bayes factor is the ratio of marginal likelihoods corresponding to two different models [19]:

$$BF_{p,q} = \frac{pr\left(X\middle|M_p\right)}{pr\left(X\middle|M_q\right)}, \quad p, q \in I_c \tag{28}$$

Note that model posterior probability is calculated by multiplying the model prior probability with the marginal likelihood of the obtained data. The ratio of two model posterior probabilities is equal to the Bayes factor when the model prior probabilities are set to be uniformly distributed. Based on the Bayes factor, the candidate models can compete with each other in data fitting. The model is built according to the inferences from all the candidates based on the calculated Bayes factors. The evidence of candidate model in terms of Bayes factor is discussed by Jeffreys [20], which is shown in Table 1. If the evidence of a model is substantial, it will be selected as reference, otherwise, averaging the inferences of the competing models can give more comprehensive results [21].

Table 1 Model building based on Bayes factor.

$BF_{p,q}<\frac{1}{3.162}$ $BF_{p,q}<\frac{1}{3.162}$	Substantial evidence against the model M_p M_p	M_p M_p is abandoned
$\frac{1}{3.162}<BF_{p,q}<3.162$ $\frac{1}{3.162}<BF_{p,q}<3.162$	insufficient evidence	Model M_q M_q and M_p M_p are averaged
$BF_{p,q} > 3.162$ $BF_{p,q}>3.162$	Substantial evidence of the model M_p M_p	M_p M_p is selected

To select the appropriate candidate model, three types of Bayes factor are considered in this paper. The best Bayes factor is considered as that which enables the Bayesian model selection method to select the right model.

3.2 Three types of Bayes factor

The considered three types of Bayes factor are: (1). Bayes factor with UPP; (2). Posterior Bayes factor and (3). BFCV. These three types of Bayes factor are widely applied in Bayesian model selection [29–31], and each has its own reasonability.

3.2.1 Bayes factor with UPP

UPP is a natural choice under the *Principle of indifference*. In the *Principle of indifference*, if here is no evidence that some values of the parameter are more possible than the others, equal probability should be assigned to all the parameter values. Therefore, a uniform distribution is considered as an objective choice before any data is obtained.

$$prior_{uniform}(\theta) \propto 1 \quad \textbf{(29)}$$

and Eqs. (26) and (28) can be reformulated as

$$uni-pr\left(X \middle| M_p\right) = \int L\left(X \middle| \theta_p, M_p\right) prior_{uniform}\left(\theta_p\right) d\theta_p \quad \textbf{(30)}$$

A1

$$uni-BF_{p,q} = \frac{uni-pr\left(X \middle| M_p\right)}{uni-pr\left(X \middle| M_q\right)}, \quad p, q \in I_c \quad \textbf{(31)}$$

Annotations:

A1. please add a 'p∈I_c,' at the end of the formula

However, when the parameter space is unbounded, the setting of equal probability will lead to the infinite of total probability. Then, the closed-form PDF of UPP is not available. To overcome it, an approximation is utilized, which is introduced in Section 3.3.3.

3.2.2 Posterior Bayes factor

The posterior Bayes factor [29] utilizes the Bayes theorem and all the obtained data to update the UPP first; then, the updated parameter distribution is used for the calculation of the Bayes factor, which is obtained as

$$pos-BF_{p,q} = \frac{pos-pr\left(X \middle| M_p\right)}{pos-pr\left(X \middle| M_q\right)}, \quad p, q \in I_c \quad \textbf{(32)}$$

$$pos-pr\left(X \middle| M_p\right) = \int L\left(X \middle| \theta_p, M_p\right) \cdot posterior\left(\theta_p\right) d\theta_p \quad \textbf{(33)}$$

A1

where

$$posterior\left(\theta_p\right) \propto L\left(X \middle| \theta_p, M_p\right) \times prior_{uniform}\left(\theta_p\right) \quad \textbf{(34)}$$

and $L\left(X\middle|\theta_p, M_p\right)$ is the likelihood function of all the obtained data.

Annotations:

A1. please add a 'p∈I_c,' at the end of the formula

3.2.3 Bayes factor based on cross validation

The BFCV is calculated with the obtained data based on the cross validation method [32–34]. In the calculation of BFCV, the data are randomly split into H mutually exclusive subsets of approximately equal size, $X=\left\{\mathcal{X}_h, h=1,2,\cdots,H\right\}$. One subset is used to calculate the $L\left(X\middle|\theta_p, M_p\right)$ in Eq. (27) and the others are used to update the UPP based on the Bayes theorem. This process is repeated H times, with each of the H subsets used exactly once to calculate the $L\left(X\middle|\theta_p, M_p\right)$ in Eq. (27). The average of the H is the corresponding marginal likelihood. Let $X/\mathcal{X}_h$ denote all the data excluding $\mathcal{X}_h$. Then, Eqs. (28) and (26) can be reformulated as

$$cv-BF_{p,q}=\frac{cv-pr\left(X\middle|M_p\right)}{cv-pr\left(X\middle|M_q\right)},\quad p,q\in I_c \tag{35}$$

$$cv-pr\left(X\middle|M_p\right)=\frac{1}{H}\sum_{h=1}^{H}\int L\left(\mathcal{X}_h\middle|\theta_p, M_p\right) CVposterior\left(\theta_p\middle|X/\mathcal{X}_h, M_p\right) d\theta_p \tag{36}$$

A1

where

$$CVposterior\left(\theta_p\middle|X/\mathcal{X}_h, M_p\right)\propto L\left(X/\mathcal{X}_h\middle|\theta_p, M_p\right)\times prior_{uniform}\left(\theta_p\right) \tag{37}$$

A2

and H is called the fold of cross validation.

Annotations:

A1. please add a 'p∈I_c,' at the end of the formula

A2. please add a 'p∈I_c,' at the end of the formula

The BFCV has similar behavior with the posterior Bayes factor. By averaging multiple results, it performs more stably at the expense of more computing time. The computing time and stability of the BFCV will increase with the increase of H [35]. When H is equal to the sample size, the BFCV can be called as the Bayes factor based on leave-one-out cross validation (BFLOOCV). Note that the number of FCG paths is small in most datasets; the BFLOOCV is employed in this paper to extract stable results. In addition, an easy-to-compute approximation [30] of the BFLOOCV can be utilized when the sample size is large.

3.3 The implementation of the Bayesian model selection method

3.3.1 Candidate models

In this paper, three candidate models for the random multiplier distribution are considered: (1) gamma distribution; (2) inverse Gaussian (IG) distribution and (3) lognormal distribution. The lognormal distribution has been widely applied [3,14,15]. The gamma distribution and the IG distribution are considered because of the following reasons. Firstly, the gamma and IG distributions have been widely applied in degradation data analysis [36,37]. Secondly, the support sets of the gamma and IG distributions lead the random multiplier to be greater than zero, which coincides with Yang and Manning’s assumption that the FCG is monotonous. Thirdly, the similarity between the gamma, IG and lognormal distributions suggests that the gamma and IG distributions are good potential alternatives for the lognormal [37].

Let M_1, M_2 and M_3 denote the gamma, IG and lognormal distributions, respectively. Their detailed PDFs are as follows:

M_1

$$f_{M_1}\left(x\middle|\alpha,\beta\right)=\frac{1}{\Gamma(\alpha)\cdot\beta^{\alpha}}x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right), \alpha>0, \beta>0 \tag{38}$$

M_2

$$f_{M_2}(x|\eta,\rho) = \left[\frac{\eta}{2\pi x^3}\right]^{\frac{1}{2}} \exp\left[\frac{-\eta(x-\rho)^2}{2\rho^2 x}\right], \eta>0, \rho>0 \tag{39}$$

M_3

$$f_{M_3}(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right], \mu \in (-\infty,+\infty), \sigma>0 \tag{40}$$

3.3.2 Model prior $p(M_c)$

The model prior probabilities can be elicited from expert judgment or derived from the last Bayesian model selection analysis using historical data. When no information is available, uniform prior can be applied.

3.3.3 Calculation of the Bayes factor

A difficulty in calculating the Bayes factor is the calculation of the marginal likelihood $\int L\left(X\middle|\theta_p, M_p\right) prior\left(\theta_p\right) d\theta_p$ in Eq. (26). As for the candidate models, closed-form solutions for the above integration are not easy to obtain. In this paper, the Markov Chain Monte Carlo (MCMC) method is utilized to calculate the marginal likelihood. MCMC is a kind of sampling algorithm in which a specific Markov chain is constructed. The stationary distribution of the Markov Chain can be regarded as the desired Bayesian posterior distribution. After the burn-in period, the state values of the constructed Markov Chain are sampled and the Monte Carlo method is carried out to obtain Bayesian inference. In the calculation of the Bayes factor with UPP, the MCMC samples are generated from the parameter posterior distribution, which is updated based on all the obtained data. Let $\theta^s_{c,uniform}, s$ A1 $1, 2, \cdots, S$ denote the samples. Then, based on the harmonic mean method proposed by Newton and Raftery [38], the corresponding marginal likelihood can be approximated as

$$pr\left(X\middle|M_c\right) \approx \left[\frac{1}{S}\sum_{s=1}^{S}\frac{1}{L\left(X\middle|\theta^s_{c,uniform}, M_c\right)}\right]^{-1} \quad pr\left(X\middle|M_c\right) \approx \left[\frac{1}{S}\sum_{s=1}^{S}\frac{1}{L\left(X\middle|\theta^s_{c,uniform}, M_c\right)}\right]^{-1}, c \in I_c \tag{41}$$

Annotations:

A1. please add a formula 'c∈I_c,' between the comma and the letter 's'

In the calculation of the posterior Bayes factor, the MCMC samples are generated from the parameter posterior distribution, which is also updated based on all the obtained data. Let $\theta^s_{c,posprior}, s$ A1 $1, 2, \cdots, S$ denote the samples. The corresponding marginal likelihood can be approximated as

$$pr\left(X\middle|M_c\right) \approx \frac{1}{S}\sum_{s=1}^{S} L\left(X\middle|\theta^s_{c,posprior}, M_c\right) \quad pr\left(X\middle|M_c\right) \approx \frac{1}{S}\sum_{s=1}^{S} L\left(X\middle|\theta^s_{c,posprior}, M_c\right), c \in I_c \tag{42}$$

Annotations:

A1. please add a formula 'c∈I_c,' between the comma and the letter 's'

In the calculation of the BFLOOCV, H sets of MCMC samples are generated. The h -th set of the MCMC samples are generated from the parameter posterior distribution, which is updated based on the data $X/\mathcal{X}_h$. Let $\Theta_{c,h-CVprior} = \left\{\theta^s_{c,h-CVprior}, s = 1, 2, \cdots, S\right\}$ A1 denote the h -th set of the MCMC samples. The corresponding marginal likelihood of the BFLOOCV can be approximated as

$$pr\left(X\middle|M_c\right) = \frac{1}{H}\sum_{h=1}^{H} pr\left(\mathcal{X}_h\middle|M_c, X/\mathcal{X}_h\right) \tag{43}$$

where

$$pr\left(\mathcal{X}_h\middle|M_c, X/\mathcal{X}_h\right) \approx \frac{1}{S}\sum_{s=1}^{S} L\left(\mathcal{X}_h\middle|\theta^s_{c,h-CVprior}, M_c\right) \tag{44}$$

A2

Annotations:

A1. please add a formula 'c∈I_c,' between the comma and the letter 's'

A2. please add a formula 'c∈I_c,' on the right side of right parenthesis ')'

In this paper, the software *WinBUGS* executed in the Windows is utilized to carry out the MCMC algorithm. After discarding the initial 5000 iterations for the convergence, 5000 MCMC samples are generated to calculate the marginal likelihood based on Eqs. (41)-(44). When the parameter space is unbounded, a normal distribution with a large variance (10^6 in WinBUGS) is set to approximate the corresponding UPP as

$$prior_{uniform}(\theta) \approx f_{normal}\left(\theta \mid mean = 0, variance = 1 \times 10^6\right) \tag{45}$$

The flowchart of the whole proposed method is shown in Fig. 1.

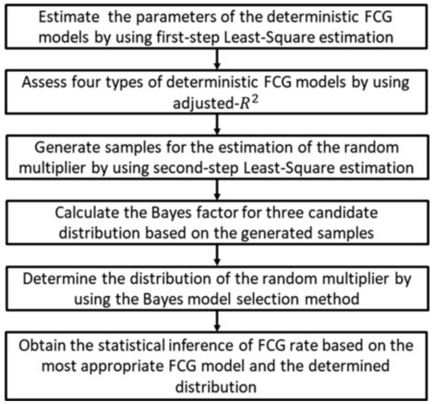

Fig. 1 Flowchart of the proposed method.

4 Simulation experiment

Three types of Bayes factor are considered for the Bayesian model selection in Section 3.2. In this section, a simulation experiment is carried out to guide their selection. In one simulation trial, data are generated from the candidate models in Section 3.3.1 and the Bayesian model selection method is carried out based on the three types of Bayes factor. Then, the results of multiple simulation trials are analyzed. Under the condition that the model prior is set the same, the ratio of two model posterior probabilities is equal to the Bayes factor. A better Bayes factor is considered as the one that enables Bayesian model selection to select the right model. The case that simulating data are generated from a mixed distribution of candidate models is not studied in this paper.

The simulation experiment consists of three groups for which the data-generating distributions are gamma, IG and lognormal, respectively. In each group, four sample sizes, 10, 20, 30, 40, are considered, which cover the path numbers in most FCG datasets. The number of simulation trials for each case is set as 400 with tests on different values to ensure that the absolute relative difference of the average results obtained in the different times is less than 5% for each group. The simulation setting is shown in Table 2.

Table 2 Simulation setting.

	Data-generating distribution	Sample size			
Group 1	gamma	10	20	30	40
Group 2	IG	10	20	30	40
Group 3	lognormal	10	20	30	40

According to Yang’s assumption [8], the mean of the random multiplier is equal to one and to make the shapes of data-generating distributions similar to each other, their variances are assumed to be the same. The specific parameter values are shown in Table 3.

Table 3 Parameter values of data-generating distributions.

	Data-generating distribution	Parameter value	
Group 1	gamma	α	1
		β	1
Group 2	IG	η	1
		ρ	1
Group 3	lognormal	μ	$-\frac{\ln 2}{2}$
		σ	$\sqrt{\ln 2}$

Three indicators are used to assess the performance of the three types of Bayes factor, which are the mean error (ME), the mean square error (MSE) and the proportion of the best (PB). The ME is defined as

$$ME = \frac{1}{400}\sum_{t=1}^{400}\left(\frac{\sum_{p\in I_c, p\neq right} BF^t_{p,right}}{\sum_{p\in I_c} BF^t_{p,right}}\right) \quad \textbf{(46)}$$

where $BF^t_{p,right}$ is the Bayes factor for the model M_p versus the data-generating model, which is obtained in the t -th simulation trial. $\sum_{p\in I_c} BF^t_{p,right}$ is the normalization constant. A Bayes factor with smaller ME will behave better in average sense.

The MSE is defined as

$$MSE = \frac{1}{400}\sum_{t=1}^{400}\left(\frac{\sum_{p\in I_c, p\neq right} BF^t_{p,right}}{\sum_{p\in I_c} BF^t_{p,right}}\right)^2 \quad \textbf{(47)}$$

MSE is a comprehensive measure of bias and variance. A smaller MSE indicates better and more stable performance.

The PB is defined from the perspective of frequency. When a single simulation trial is completed, the errors of the three types of Bayes factor are calculated and ordered from small to large; then, their rankings (the best, the medium and the worst) are recorded. After 400 simulation trials, the proportion that a type of Bayes factor is the best, the medium and the worst can be obtained. If a certain type of Bayes factor is always the best, it will be more likely to be the best in the future.

The ME, MSE and PB for each group are shown in Figs. 2-4.

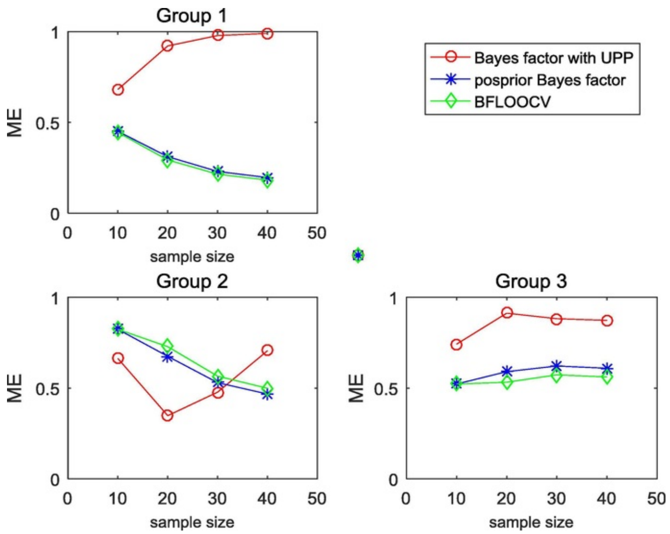

Fig. 2 ME of the three types of Bayes factor.

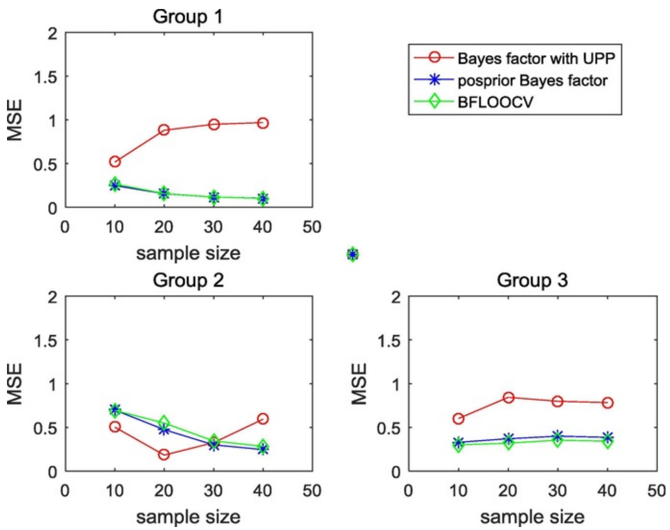

Fig. 3 MSE of the three types of Bayes factor.

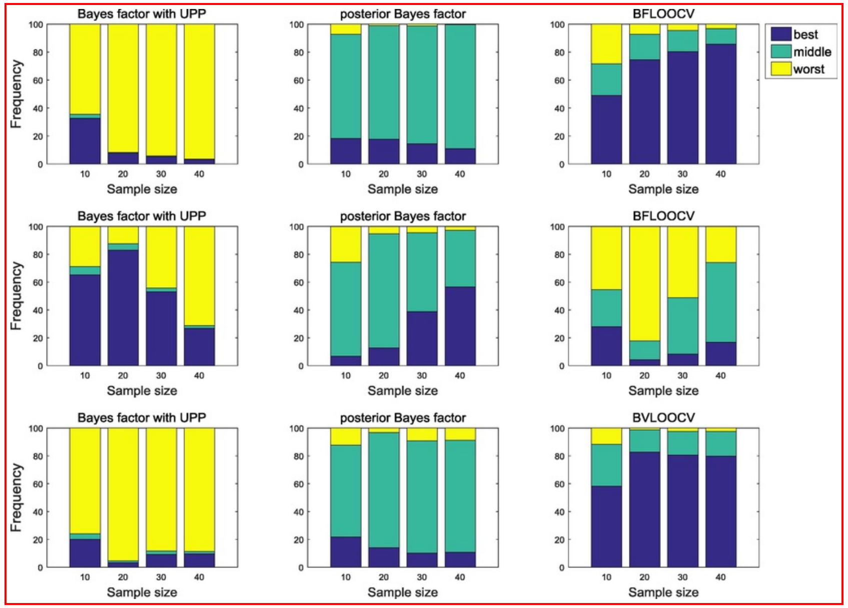

Fig. 4 PB of the three types of Bayes factor.

Based on the above results, some conclusions can be drawn as follows:

(1) The performance of the posterior Bayes factor and the BFLOOCV are similar from the perspective of ME and MSE, no matter what the data-generating distribution and sample size are.

(2) The errors of the posterior Bayes factor and the BFLOOCV decrease with the increase of sample size, which indicates a good large sample property of them. However, the Bayes factor with UPP does not possess this property.

(3) In most instances, the BFLOOCV has the highest PB, which indicates a large possibility that the BFLOOCV behaves better than the others. Besides, the posterior Bayes factor has the highest proportion of the medium and the Bayes factor with UPP has the highest proportion of the worst.

As a result, the BFLOOCV is selected in this paper to carry out the Bayesian model selection method.

5 Case study

The software *Matlab* is utilized to calculate the statistic inference of proposed probabilistic FCG model. In order to show the feasibility of the proposed method, two case studies is carried out by using the FCG data from [3,4]. As for the FCG data, there are three main aspects that can affect the data quality: unit-to-unit variation, measurement error and sample size.

The unit-to-unit variation can be considered and characterized in our model by the proposed TSLS estimation method. Based on the TSLS estimation method, samples of different FCG paths for the estimation of the random multiplier can be generated. Then, the random multiplier is modeled by a probabilistic distribution in consideration of unit-to-unit variation.

Smaller measurement error can lead to better analysis results. In this paper, the effects of measurement error can also be considered in our model when dealing with model uncertainty. By using the Bayesian model selection method, the appropriate probability distribution of the random multiplier can be obtained taking into account the true data-generating distribution and the distribution of measurement error. If the distribution of measurement error is available, Bayesian filtering method [39] can be employed to estimate the true crack length, which hasn't been explored in this paper. We plan to investigate this issue in future work.

As for the sample size, the larger it is, the more reliable statistical results can be obtained. In this paper, the Bayes factor based on leave-one-out cross validation is used in Bayesian model selection, which can extract stable results and alleviate the influences of small sample size to a certain extent as illustrated in the simulation experiment. However, small sample size cannot lead to satisfactory results especially when the data happen to locate at the tail of the distribution. In fact, small sample size is a universal challenge and appropriate experimental sample size is problem-specific. The details of case studies are as follows:

5.1 Case 1: FCG dataset from wu and Ni's work [3]

In [3], a group of 30 2024-T351 aluminum alloy plates, which were loaded by sinusoidal signals with maximum of 4.5 kN, minimum of 0.9 kN, and frequency of 15 Hz, are measured. The detailed observed FCG processes are shown in Fig. 5.

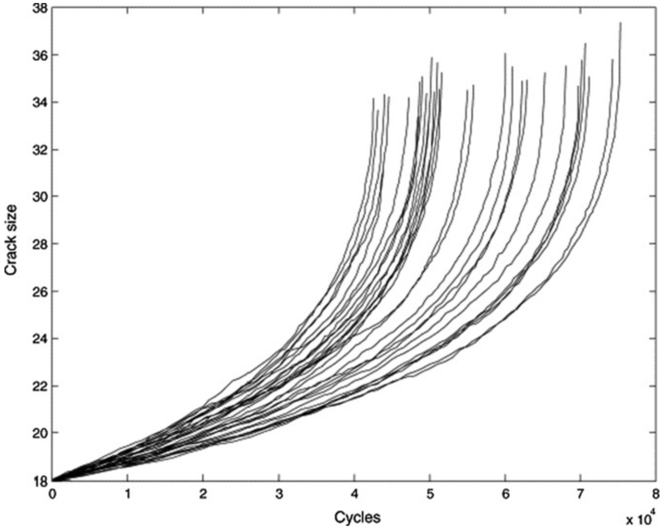

Fig. 5 FCG dataset from [3].

The crack growth rates of each FCG process are calculated by using the 5-points Incremental Polynomial Method given in the ASTM standard E647-15e1 [40], for which 12 data points of each FCG path are utilized at the crack size of $22\ \mathrm{mm}, 23\ \mathrm{mm}, \cdots, 33\ \mathrm{mm}$. By this way, a dataset $\mathbb{A} = \left\{ \left(\frac{da(t)}{dt}\Big|_{a(t)=a_i^k}, a_i^k \right), i = 1, 2, \cdots, 12, k = 1, 2, \cdots, 30 \right\}$ is obtained and the proposed TSLS estimation method is applied. In the first step, the parameters of deterministic FCG models are estimated, which are listed in Table 4. The corresponding fitting curves are plotted in Fig. 6.

Table 4 Parameter estimates of deterministic FCG models.

Power function	Q	5.30e−14
	b	7.04
polynomial function	p	1.17e−2
	q	−1.01e−3
	r	0.02e−3
rational function	Q_1	6.20e−8
	b	2.60
	Q_2	4.80

	Q_3	0.81
function based on curve fitting technique	Q_1	244.26e3
	b	−0.99
	Q_2	−7.56e3

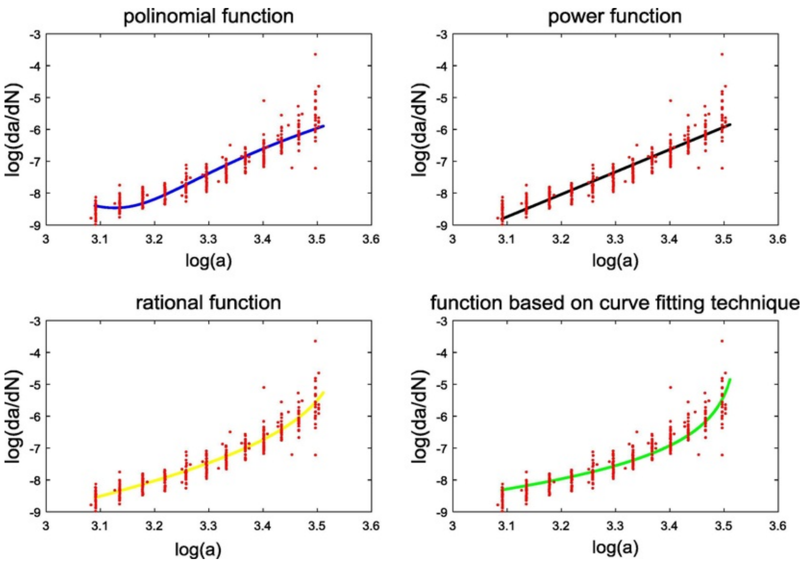

Fig. 6 Data points and corresponding fitting curves.

Then, the adjusted- R^2 is calculated to assess the accuracies of four types of deterministic FCG models. According to the results listed in Table 5, the function based on curve fitting technique is the most appropriate deterministic FCG model, which should be employed in the second step of the TSLS estimation.

Table 5 Adjusted- R^2 of four types of deterministic FCG models.

Deterministic FCG model	Power function	Polynomial function	Rational function	Function based on curve fitting technique
Adjusted- R^2	0.8710	0.8758	0.8924	**0.8925**

In the second step, by solving Eqs. (9) and (21), 12 sets of samples for the estimation of the random multiplier $X^i = \{x_1^i, x_2^i, \cdots, x_{30}^i\}, i = 1, 2, \cdots, 12$ are generated based on the FCG data of crack lengths $22\ \text{mm}, 23\ \text{mm}, \cdots, 33\ \text{mm}$. Based on the generated samples, model uncertainty with regard to the distribution of the random multiplier can, then, be considered by the Bayesian model selection method. The model prior probabilities is set to be uniformly distributed and the calculated Bayes factors of each candidate model are normalized and plotted in Fig. 7.

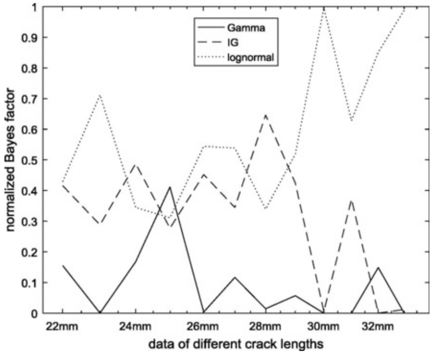

Fig. 7 The normalized Bayes factor for data of different crack lengths.

As shown in Fig. 7, three candidate distributions (Ggamma, Iinverse Gaussian and Llognormal) compete with each other in data fitting. Note that larger Bayes factor of a model indicates stronger evidence of the model conforming to the true data-generating distribution. The performance of lognormal distribution is not the best for the data of crack lengths $24\ \text{mm}, 25\ \text{mm}$ and $28\ \text{mm}$, which demonstrates the necessity of considering model uncertainty. In Bayesian model selection, the model of random multiplier is built according to the inferences from all the candidates based on the calculated normalized Bayes factors. According to the Jeffreys' scale of evidence for model selection [20], if both $BF_{Lognormal,Gamma}$ and $BF_{Lognormal,IG}$ are greater than 3.162, the lognormal distribution will be selected as reference, otherwise, averaging the inferences of all the competing models can give more comprehensive result [21]. Kolmogorov-Smirnov statistics are employed to evaluate the goodness of fit of the built model. Smaller Kolmogorov-Smirnov statistics indicates better goodness of fit. A comparison between the proposed model and the Yang and Manning’s model is provided in Fig. 8. It can be observed that the proposed model fit the data better than the Yang and Manning’s model.

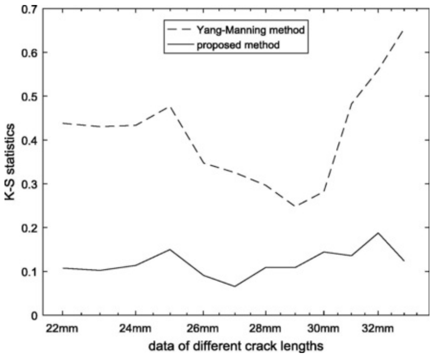

Fig. 8 The comparison of two methods for data of different crack lengths.

5.2 Case 2: FCG dataset from Virker’s work [4]

In [4], a group of 2.54 mm thick, 588.8 mm long and 152.4 mm wide panels of 2024-T3 aluminum alloy were tested to obtained the FCG dataset, for which the detail of specimen geometry have been given in [4]. The crack growth rates of each FCG process are calculated by using the 7-points Incremental Polynomial Method. 39 data points of each FCG path are utilized at the crack size uniformly distributed on the logarithmic axis, and the proposed TSLS estimation is applied.

In the first step, the parameters of deterministic FCG models are estimated, which are listed in Table 6. Then, the adjusted- R^2 is calculated to assess the accuracies of four types of deterministic FCG models. According to the results listed in Table 7, the Polynomial function is the most appropriate deterministic FCG model, which should be employed in the second step of the TSLS estimation.

Table 6 Parameter estimates of deterministic FCG models.

Power function	Q	0.0471

	b	2.8699
polynomial function	p	−38.8009
	q	3.4177
	r	0.4217
rational function	Q_1	0.0831
	b	3.1214
	Q_2	1.6165
	Q_3	−0.4832
function based on curve fitting technique	Q_1	253.9450
	b	−4.0506
	Q_2	0.0049

Table 7 Adjusted- R^2 of four types of deterministic FCG models.

Deterministic FCG model	Power function	Polynomial function	Rational function	Function based on curve fitting technique
Adjusted- R^2	0.9064	**0.9188**	0.9067	0.9165

In the second step, by solving Eqs. (9) and (17), 39 sets of samples for the estimation of the random multiplier are generated and model uncertainty with regard to the distribution of the random multiplier can, then, be considered by the Bayesian model selection method. The model prior probabilities are set to be uniformly distributed and the calculated Bayes factors of each candidate model are normalized and plotted in Fig. 9.

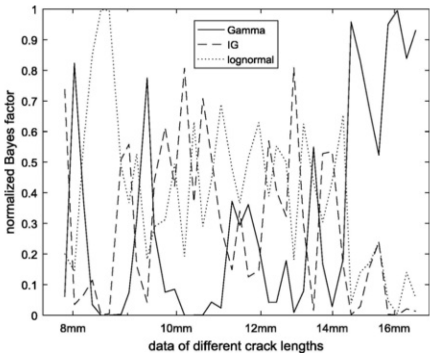

Fig. 9 The normalized Bayes factor for data of different crack lengths.

As shown in Fig. 9, the performance of lognormal distribution is not always the best, which demonstrates the necessity of considering model uncertainty. Similar to Case 1, the model of random multiplier is built according to the inferences from all the candidates based on the calculated normalized Bayes factors. Based on the Kolmogorov-Smirnov statistics, a comparison between the proposed model and the Yang and Manning's model is provided in Fig. 10. It can be observed that the proposed model fit the data better than the Yang and Manning's model.

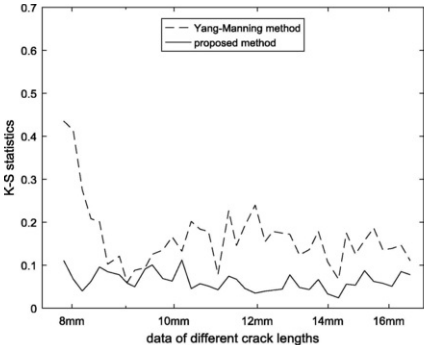

Fig. 10 The comparison of two methods for data of different crack lengths.

6 Conclusion

In this paper, the TSLS estimation method is proposed based on Yang and Manning's probabilistic FCG model. The method estimates the unknown parameters in the deterministic FCG model, and generates samples for the estimation of the random multiplier. By this way, the modeling error due to inappropriate choice of deterministic FCG models and that due to unsuitable assignment of probability distribution to the random multiplier can be distinguished. In the proposed method, the adjusted- R^2 is used to select the appropriate deterministic FCG model and Bayesian model selection is used to deal with the model uncertainty associated with the random multiplier. In the application of Bayesian model selection, three types of Bayes factor are taken into account and a simulation experiment is carried out to guide their selection. According to the simulation results, the BFLOOCV is suggested. Two case studies based on real FCG data are presented to illustrate the feasibility of the proposed method and the necessity of considering model uncertainty.

Acknowledgement

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