Introduction to Monte Carlo Simulation
The experimental view
Enrico Zio
CONTENTS

➢ Sampling Random Numbers

➢ Simulation of system transport

➢ Simulation for reliability/availability analysis of a component

➢ Examples
The History of Monte Carlo Simulation

- Buffon
- Kelvin
- Gosset (Student)
- Fermi, von Neumann, Ulam
- Neutron transport
- System Transport (RAMS)

Timeline:
- 1707-88
- 1824-1907
- 1908
- 1930-40's
- 1950's
- 1990's
SAMPLING RANDOM NUMBERS
Example: Exponential Distribution

Probability density function:

\[ f_T(t) = \lambda e^{-\lambda t} \quad t \geq 0 \]
\[ = 0 \quad t < 0 \]

Expected value and variance:

\[ E[T] = \frac{1}{\lambda} \]
\[ Var[T] = \frac{1}{\lambda^2} \]
Sampling Random Numbers from $F_X(x)$

Sample $R$ from $U_R(r)$ and find $X$:

$$X = F_X^{-1}(R)$$

Example: Exponential distribution

$$F_X(x) = 1 - e^{-\lambda x}$$

$$R = F_X(x) = 1 - e^{-\lambda x}$$

$$X = F_X^{-1}(R) = -\frac{1}{\lambda} \ln(1 - R)$$
Sampling from discrete distributions

\[ \Omega = \{ x_0, x_1, \ldots, x_k, \ldots \} \]

\[ F_k = P \{ X \leq x_k \} = \sum_{i=0}^{k} P \{ X = x_i \} \]

sample an \( R \sim U[0,1) \)

Graphically:

\[ F_\theta = f_0 \]
\[ F_1 = f_1 + f_0 \]
\[ F_2 = f_2 + f_1 + f_0 \]
1- Calculate the analytic solution for the failure probability of the network, i.e., the probability of no connection between nodes S and T

2- Repeat the calculation with Monte Carlo simulation

<table>
<thead>
<tr>
<th>Arc number $i$</th>
<th>Failure probability $P_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.050</td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
</tr>
<tr>
<td>3</td>
<td>0.050</td>
</tr>
<tr>
<td>4</td>
<td>0.020</td>
</tr>
<tr>
<td>5</td>
<td>0.075</td>
</tr>
</tbody>
</table>
SIMULATION OF SYSTEM TRANSPORT
PLANT = system of $Nc$ suitably connected components.

COMPONENT = a subsystem of the plant (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,... ). Stochastic transitions from state-to-state occur at stochastic times.

STATE of the PLANT at $t = \{n\text{-th state of the plant at time } t\}$. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.

PLANT TRANSITION = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the $n$-th transition is called $t_n$, and the plant state thereby entered is called $k_n$.

PLANT LIFE = stochastic process.
Stochastic Transitions: Governing Probabilities

- $T(t \mid t'; k')dt = \text{conditional probability of a transition at } t \in dt, \text{ given that the preceding transition occurred at } t' \text{ and that the state thereby entered was } k'.$

- $C(k \mid k'; t) = \text{conditional probability that the plant enters state } k, \text{ given that a transition occurred at time } t \text{ when the system was in state } k'.$ Both these probabilities form the "trasport kernel":

$$K(t; k \mid t'; k')dt = T(t \mid t'; k')dt \cdot C(k \mid k'; t)$$

- $\psi(t; k) = \text{ingoing transition density or probability density function (pdf) of a system transition at } t, \text{ resulting in the entrance in state } k$
Random walk = realization of the system life generated by the underlying state-transition stochastic process.
Example: System Reliability Estimation

\[ C^R(t) = C^R(t) \quad t \in [0, T_M] \]

\[ C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M] \]

\[ C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M] \]

\[ C^R(t) = C^R(t) \quad t \in [0, T_M] \]

\[ \hat{F}_T(t) = \frac{C^R(t)}{M} \]
Example: System Reliability Estimation

Events at components level, which do not entail system failure

\[ C^R(t) = C^R(t) \quad t \in [0, T_M] \]

\[ C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M] \]

\[ \hat{F}_T(t) = \frac{C^R(t)}{M} \]
SIMULATION FOR RELIABILITY/AVAILABILITY ANALYSIS OF A COMPONENT
Components’ times of transition between states are exponentially distributed
(\( \lambda_{j_i \rightarrow m_i}^i = \text{rate of transition of component } i \text{ going from its state } j_i \text{ to the state } m_i \))

<table>
<thead>
<tr>
<th>Initial</th>
<th>Arrival</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>- (\lambda_{1 \rightarrow 2}^A(B)) (\lambda_{1 \rightarrow 3}^A(B))</td>
</tr>
<tr>
<td>2</td>
<td>(\lambda_{2 \rightarrow 1}^A(B)) - (\lambda_{2 \rightarrow 3}^A(B))</td>
</tr>
<tr>
<td>3</td>
<td>(\lambda_{3 \rightarrow 1}^A(B)) (\lambda_{3 \rightarrow 2}^A(B)) -</td>
</tr>
</tbody>
</table>
The components are initially ($t=0$) in their nominal states (1,1,1)

One minimal cut set of order 1 (C in state 4:(*,*,4)) and one minimal cut set of order 2 (A and B in 3: (3,3,*)).
The rate of transition of component A(B) out of its nominal state 1 is:

\[ \lambda_1^{A(B)} = \lambda_{1\rightarrow 2}^{A(B)} + \lambda_{1\rightarrow 3}^{A(B)} \]

- The rate of transition of component C out of its nominal state 1 is:

\[ \lambda_1^{C} = \lambda_{1\rightarrow 2}^{C} + \lambda_{1\rightarrow 3}^{C} + \lambda_{1\rightarrow 4}^{C} \]

- The rate of transition of the system out of its current configuration (1, 1, 1) is:

\[ \lambda^{(1,1,1)} = \lambda_1^{A} + \lambda_1^{B} + \lambda_1^{C} \]

- We are now in the position of sampling the first system transition time \( t_1 \), by applying the inverse transform method:

\[ t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}} \ln(1 - R_t) \]

where \( R_t \sim U[0,1] \)
Sampling the Kind of Transition (1)

- Assuming that $t_1 < T_M$ (otherwise we would proceed to the successive trial), we now need to determine which transition has occurred, i.e. which component has undergone the transition and to which arrival state.

- The probabilities of components A, B, C undergoing a transition out of their initial nominal states 1, given that a transition occurs at time $t_1$, are:

  \[
  \frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}}
  \]

- Thus, we can apply the inverse transform method to the discrete distribution

\[
\begin{align*}
0 & \quad \frac{\lambda_1^A}{\lambda^{(1,1,1)}} & \quad R & \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}} & \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}} & \quad 1
\end{align*}
\]
Sampling the Kind of Transition (2)

- Given that at \( t_1 \) component B undergoes a transition, its arrival state can be sampled by applying the inverse transform method to the set of discrete probabilities

\[
\begin{pmatrix}
\frac{\lambda_{1\rightarrow 2}^B}{\lambda_1^B}, & \frac{\lambda_{1\rightarrow 3}^B}{\lambda_1^B}
\end{pmatrix}
\]

of the mutually exclusive and exhaustive arrival states

- As a result of this first transition, at \( t_1 \) the system is operating in configuration \((1,3,1)\).

- The simulation now proceeds to sampling the next transition time \( t_2 \) with the updated transition rate

\[
\lambda^{(1,3,1)} = \lambda_1^A + \lambda_3^B + \lambda_1^C
\]
Sampling the Next Transition

• The next transition, then, occurs at

\[ t_2 = t_1 - \frac{1}{\lambda^{(1,3,1)}} \ln(1 - R_t) \]

where \( R_t \sim U[0,1) \).

• Assuming again that \( t_2 < T_M \), the component undergoing the transition and its final state are sampled as before by application of the inverse transform method to the appropriate discrete probabilities.

• The trial simulation then proceeds through the various transitions from one system configuration to another up to the mission time \( T_M \).
• When the system enters a failed configuration (*,*,4) or (3,3,*), where the * denotes any state of the component, tallies are appropriately collected for the unreliability and instantaneous unavailability estimates (at discrete times $t_j \in [0, T_M]$);

• After performing a large number of trials $M$, we can obtain estimates of the system unreliability and instantaneous unavailability by simply dividing by $M$, the accumulated contents of $C^R(t_j)$ and $C_A(t_j)$, $t_j \in [0,T_M]$.
For any arbitrary trial, starting at $t=0$ with the system in nominal configuration $(1,1,1)$ we would sample all the transition times:

$$t^i_{1\rightarrow m_i} = t_0 - \frac{1}{\lambda^i_{1\rightarrow m_i}} \ln(1 - R^i_{t,1\rightarrow m_i})$$

where $R^i_{t,1\rightarrow m_i} \sim U[0,1)$

These transition times would then be ordered in ascending order from $t_{\text{min}}$ to $t_{\text{max}} \leq T_M$. Let us assume that $t_{\text{min}}$ corresponds to the transition of component $A$ to state $3$ of failure. The current time is moved to $t_1 = t_{\text{min}}$ in correspondence of which the system configuration changes, due to the occurring transition, to $(3,1,1)$ still operational.
These transition times would then be ordered in ascending order from $t_{\text{min}}$ to $t_{\text{max}} \leq T_M$.

Let us assume that $t_{\text{min}}$ corresponds to the transition of component A to state 3 of failure. The current time is moved to $t_1 = t_{\text{min}}$ in correspondence of which the system configuration changes, due to the occurring transition, to $(3,1,1)$ still operational.
Example (1)

Diagram showing timelines and transitions for systems A, B, and C, with labels such as $t^A_{1\rightarrow 3}$, $t^B_{1\rightarrow 3}$, $t^C_{1\rightarrow 4}$, and $t^{Sys}_{(1,1,1)\rightarrow (3,1,1)}$. The timeline for the system $(A,B,C)$ is also depicted, with $T_M$ as the total time frame.
Example (2)

The new transition times of component A are then sampled

\[ t_{3\rightarrow m_A}^A = t_1 - \frac{1}{\lambda_{3\rightarrow m_A}^A} \ln(1 - R_{t,3\rightarrow m_A}^A) \]

\[ k = 1,2 \]

\[ R_{t,3\rightarrow m_A}^A \sim U[0,1] \]

and placed at the proper position in the timeline of the succession of occurring transitions

- The simulation then proceeds to the successive times in the list, in correspondence of which a system transition occurs.
- After each transition, the timeline is updated with the times of the transitions that the component which has undergone the last transition can do from its new state.
- During the trial, each time the system enters a failed configuration, tallies are collected and in the end, after M trials, the unreliability and unavailability estimates are computed.
One component with exponential distribution of the failure time

State $X=1 \rightarrow \text{ON}$
State $X=2 \rightarrow \text{OFF}$
One component with exponential distribution of the failure time

State $X=1 \rightarrow ON$

State $X=2 \rightarrow OFF$
One component with exponential distribution of the failure time

\[ \lambda \]

1 \hspace{2cm} 2

\[ \mu \]

State 1

State 2

State \(X=1\) \(\rightarrow\) ON

State \(X=2\) \(\rightarrow\) OFF
One component with exponential distribution of the failure time

\begin{align*}
\lambda & \quad 3 \cdot 10^{-3} \text{ h}^{-1} \\
\mu & \quad 25 \cdot 10^{-3} \text{ h}^{-1}
\end{align*}
PRODUCTION AVAILABILITY EVALUATION OF AN OFFSHORE INSTALLATION
A real example of Indirect Simulation
System description: basic scheme

- Wells
- Separation
- Oil Trt°
- Wat. Trt°
- Sea
- Water Inj.
- Flare
- Gas Export 3.0 mSm3/d, 60b
- Oil export

TC 50%
TEG
TC 50%
System description: gas-lift

First loop

<table>
<thead>
<tr>
<th>Gas-lift pressure</th>
<th>Production of the Well</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100%</td>
</tr>
<tr>
<td>60</td>
<td>80%</td>
</tr>
<tr>
<td>0</td>
<td>60%</td>
</tr>
</tbody>
</table>
System description: fuel gas generation and distribution

Second loop

Wells → Separation

TC → TC → TEG → Gas Export 3.0 MSm³/d, 60b

0.1 MSm³/d

0.1 MSm³/d

0.1 MSm³/d

Fuel Gas 25 b

0.4 MSm³/d
System description: electricity power production and distribution
The offshore production plant

Production:
- Gas: 5 MSm$^3$/d
- Oil: 26500 Sm$^3$/d
- Water: 8000 Sm$^3$/d

Gas Export: 3.0 MSm$^3$/d, 60b

Flare

Fuel Gas 25 b

Oil Export

Water Inj

Sea

Back
Component failures and repairs: TCs and TGs

State 0 = as good as new
State 1 = degraded (no function lost, greater failure rate value)
State 2 = critical (function is lost)

<table>
<thead>
<tr>
<th></th>
<th>TC</th>
<th>TG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{01}$</td>
<td>$0.89 \times 10^{-3}$ h$^{-1}$</td>
<td>$0.67 \times 10^{-3}$ h$^{-1}$</td>
</tr>
<tr>
<td>$\lambda_{02}$</td>
<td>$0.77 \times 10^{-3}$ h$^{-1}$</td>
<td>$0.74 \times 10^{-3}$ h$^{-1}$</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
<td>$1.86 \times 10^{-3}$ h$^{-1}$</td>
<td>$2.12 \times 10^{-3}$ h$^{-1}$</td>
</tr>
<tr>
<td>$\mu_{10}$</td>
<td>$0.033$ h$^{-1}$</td>
<td>$0.032$ h$^{-1}$</td>
</tr>
<tr>
<td>$\mu_{20}$</td>
<td>$0.048$ h$^{-1}$</td>
<td>$0.038$ h$^{-1}$</td>
</tr>
</tbody>
</table>
Component failures and repairs: EC and TEG

State 0 = as good as new
State 2 = critical (function is lost)

<table>
<thead>
<tr>
<th></th>
<th>EC</th>
<th>TEG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
<td>$0.17 \cdot 10^{-3} \text{ h}^{-1}$</td>
<td>$5.7 \cdot 10^{-5} \text{ h}^{-1}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$0.032 \text{ h}^{-1}$</td>
<td>$0.333 \text{ h}^{-1}$</td>
</tr>
</tbody>
</table>
Production priority

When a failure occurs, the system is reconfigured to minimise (in order):

- the impact on the export oil production
- the impact on export gas production

➤ The impact on water injection does not matter
Production priority: example

Production:
Gas: 5 MSm³/d
Oil: 26500 Sm³/d
Water: 8000 Sm³/d

Fuel Gas 25 b
0.1 MSm³
0.1 MSm³
2.2 MSm³/d
2.2 MSm³/d
4.4 MSm³/d

Gas Lift 60b
Gas Lift 100b
Gas Export 3.0 MSm³/d, 60b

Flare
Gas Export

Wells
Production
Gas: 5 MSm³/d
Oil: 26500 Sm³/d
Water: 8000 Sm³/d

Separation
23300 Sm³/d
7000 Sm³/d
0.1 MSm³

Oil Trt°
13 MW
7 MW

Wat. Trt°

TG
13 MW
50%

Water Inj
Maintenance policy: reparation

Only 1 repair team

Priority levels of failures:
1. Failures leading to total loss of export oil (both TG’s or both TC’s or TEG)
2. Failures leading to partial loss of export oil (single TG or EC)
3. Failures leading to no loss of export oil (single TC failure)
Maintenance policy: preventive maintenance

➢ Only 1 preventive maintenance team

➢ The preventive maintenance takes place only if the system is in perfect state of operation

<table>
<thead>
<tr>
<th>Type of maintenance</th>
<th>Frequency [hours]</th>
<th>Duration [hours]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbo-Generator and Turbo-Compressors</td>
<td>Type 1</td>
<td>2160 (90 days)</td>
</tr>
<tr>
<td></td>
<td>Type 2</td>
<td>8760 (1 year)</td>
</tr>
<tr>
<td></td>
<td>Type 3</td>
<td>43800 (5 years)</td>
</tr>
<tr>
<td>Electro Compressor</td>
<td>Type 4</td>
<td>2666</td>
</tr>
</tbody>
</table>
MARKOV APPROACH

\[
\begin{align*}
\text{Number of components} &= 6 \\
\text{Number of states for component} &= 2 \text{ or } 3 \\
2^2 \cdot 3^4 &= 324 \text{ plant states}
\end{align*}
\]

\[
\begin{align*}
\text{1 repair team} & \rightarrow 129 \text{ new plant states} \\
\text{1 maintenance team} & \rightarrow \text{Non homogeneous Markov chain}
\end{align*}
\]

Markov approach too complex

MONTE CARLO APPROACH
Associate a production level to each of the 453 plant states

too long, error prone
A systematic procedure

<table>
<thead>
<tr>
<th>Production Level</th>
<th>Gas [kSm³/d]</th>
<th>Oil [k m³/d]</th>
<th>Water [m³/d]</th>
<th>mcs</th>
<th>MCS</th>
</tr>
</thead>
<tbody>
<tr>
<td>0=(100%)</td>
<td>3000</td>
<td>23.3</td>
<td>7000</td>
<td>X5, X6</td>
<td>X5, X6</td>
</tr>
<tr>
<td>1</td>
<td>900</td>
<td>23.3</td>
<td>7000</td>
<td>X5, X6</td>
<td>X5, X6</td>
</tr>
<tr>
<td>2</td>
<td>2700</td>
<td>21.2</td>
<td>0</td>
<td>X3, X4</td>
<td>X2X3, X2X4</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>21.2</td>
<td>0</td>
<td>X3X5, X3X6, X4X5, X4X6</td>
<td>X2X3X5, X2X3X6, X2X4X5, X2X4X6</td>
</tr>
<tr>
<td>4</td>
<td>2600</td>
<td>21.2</td>
<td>6400</td>
<td>X2</td>
<td>X2</td>
</tr>
<tr>
<td>5</td>
<td>900</td>
<td>21.2</td>
<td>6400</td>
<td>X2X5, X2X6</td>
<td>X2X5, X2X6</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>X1, X3X4, X5X6</td>
<td>X1X2X3X4X5X6</td>
</tr>
</tbody>
</table>

7 different production levels
6 different system faults
6 fault trees
6 families of mcs
Numerical results

Case A: corrective maintenance and no preventive maintenance ($T_{\text{miss}} = 1 \cdot 10^3$ hours, trials=$10^6$)

CPU time $\approx 15$ min

Case B: perfect system (no failures) and preventive maintenance ($T_{\text{miss}} = 10^4$ hours, trials=$10^5$)

CPU time $\approx 12$ min

Case C: corrective and preventive maintenance

($T_{\text{miss}} = 5 \cdot 10^5$ hours, trials=$10^5$)

CPU time $\approx 20$ h
Case A: no preventive maintenances

<table>
<thead>
<tr>
<th>Production level</th>
<th>Average availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.23E-1</td>
</tr>
<tr>
<td>1</td>
<td>3.13E-2</td>
</tr>
<tr>
<td>2</td>
<td>3.67E-2</td>
</tr>
<tr>
<td>3</td>
<td>2.47E-3</td>
</tr>
<tr>
<td>4</td>
<td>4.88E-3</td>
</tr>
<tr>
<td>5</td>
<td>3.50E-4</td>
</tr>
<tr>
<td>6</td>
<td>1.79E-3</td>
</tr>
</tbody>
</table>
Case A: no preventive maintenances

<table>
<thead>
<tr>
<th></th>
<th>Asymptotic values</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Oil</strong></td>
<td>[k m³/d]</td>
<td>23.24</td>
</tr>
<tr>
<td><strong>Gas</strong></td>
<td>[k Sm³/d]</td>
<td>2918</td>
</tr>
<tr>
<td><strong>Water</strong></td>
<td>[k m³/d]</td>
<td>6.703</td>
</tr>
</tbody>
</table>
### Case B: perfect system and preventive maintenances

<table>
<thead>
<tr>
<th>Production level</th>
<th>Average availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.12E-1</td>
</tr>
<tr>
<td>1</td>
<td>2.73E-2</td>
</tr>
<tr>
<td>2</td>
<td>2.72E-2</td>
</tr>
<tr>
<td>3</td>
<td>0.00</td>
</tr>
<tr>
<td>4</td>
<td>3.40E-2</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
</tr>
<tr>
<td>6</td>
<td>0.00</td>
</tr>
</tbody>
</table>
Case B: perfect system and preventive maintenances

- P.Maintenance Type 1 (TC, TG)
- P.Maintenance Type 2 (EC)
- P.Maintenance Type 3 (TC, TG)

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil [k m³/d]</td>
<td>23.230</td>
<td>0.263</td>
</tr>
<tr>
<td>Gas [k Sm³/d]</td>
<td>2929</td>
<td>194.0</td>
</tr>
<tr>
<td>Water [k m³/d]</td>
<td>6.811</td>
<td>0.883</td>
</tr>
</tbody>
</table>
**Case C: real system with preventive maintenances**

<table>
<thead>
<tr>
<th>Production level</th>
<th>Average availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.13E-1</td>
</tr>
<tr>
<td>1</td>
<td>5.68E-2</td>
</tr>
<tr>
<td>2</td>
<td>6.58E-2</td>
</tr>
<tr>
<td>3</td>
<td>1.19E-2</td>
</tr>
<tr>
<td>4</td>
<td>3.55E-2</td>
</tr>
<tr>
<td>5</td>
<td>2.34E-3</td>
</tr>
<tr>
<td>6</td>
<td>1.50E-2</td>
</tr>
</tbody>
</table>

![Graph showing average availability for different production levels]
Case C: real system with preventive maintenances

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil</td>
<td>22.60</td>
<td>0.42</td>
</tr>
<tr>
<td>[k m³/d]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gas</td>
<td>2687</td>
<td>194.3</td>
</tr>
<tr>
<td>[k Sm³/d]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>6.04</td>
<td>0.76</td>
</tr>
<tr>
<td>[k m³/d]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusions

- Complex multi-state system with maintenance and operational loops
  - MC simulation

- Systematic procedure to assign a production level to each configuration

- Investigation of effects maintenance on production
The Theoretical View
Sampling
Evaluation of definite integrals
Simulation of system transport
Simulation for reliability/availability analysis
SAMPLING
Buffon considered a set of parallel straight lines a distance $D$ apart onto a plane and computed the probability $P$ that a needle of length $L < D$ randomly positioned on the plane would intersect one of these lines.

$$P = P\{Y \leq L \sin \emptyset\}$$

$$f_Y(y) = \frac{1}{D} \quad y \in [0, D]$$

$$f_{\emptyset}(\varphi) = \frac{1}{\pi} \quad \varphi \in [0, \pi]$$

$$P = \iint_A \frac{dy}{D} \cdot \frac{d\varphi}{\pi} = \frac{L}{D} / \frac{\pi}{2}$$
Sampling (pseudo) Random Numbers Uniform Distribution

cdf: \( U_R(r) = P\{R \leq r\} = r \)

pdf: \( u_R(r) = \frac{dU_R(r)}{dr} = 1 \)
Sampling (pseudo) Random Numbers Uniform Distribution

\[ R \sim U[0, 1) \]

\[ x_i = (ax_{i-1} + c) \mod m \]

where \( a, c \in [0, m - 1] \)
\[ m \neq 1 \]

\[ r_i = \frac{x_i}{m} \]

Example: \( a = 5, c = 1, m = 16 \)

\[ x_0 = 2 \Rightarrow r_0 = \frac{2}{16} \]
\[ x_1 = (5 \cdot 2 + 1) \mod 16 = 11 \Rightarrow r_1 = \frac{11}{16} \]

\[ r_{i5} = 13 \Rightarrow r_{i5} = \frac{13}{16} \]
\[ x_{16} = 2 \]
Sample $R$ from $U_R(r)$ and find $X$:

$$X = F_X^{-1}(R)$$

**Question:** which distribution does $X$ obey?

$$P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}$$

Application of the operator $F_X$ to the argument of $P$ above yields

$$P\{X \leq x\} = P\{R \leq F_X(x)\} = F_X(x)$$

**Summary:**

From an $R \sim U_R(r)$ we obtain an $X \sim F_X(x)$
Example: Exponential Distribution

- Markovian system with two states (good, failed)
- hazard rate, $\lambda = \text{constant}$
  \[ F_T(t) = P\{T \leq t\} = 1 - e^{-\lambda t} \]
- cdf
  \[ f_T(t) \cdot dt = P\{t \leq T < t + dt\} = \lambda e^{-\lambda t} \cdot dt \]
- pdf
  \[ R \equiv F_R(r) = F_T(t) = 1 - e^{-\lambda t} \]
- Sampling a failure time $T$
  \[ T = F_T^{-1}(R) = -\frac{1}{\lambda} \ln(1 - R) \]
Example: Weibull Distribution

- hazard rate, $\lambda = \text{constant}$

- cdf
  \[ F_T(t) = P\{T \leq t\} = 1 - e^{-\beta t^\alpha} \]

- pdf
  \[ f_T(t) \cdot dt = P\{t \leq T < t + dt\} = \alpha \beta t^{\alpha-1} e^{-\beta t^\alpha} \cdot dt \]

- Sampling a failure time $T$
  \[ R \equiv F_R(r) = F_T(t) = 1 - e^{-\lambda t^\alpha} \]

  \[ T = F_T^{-1}(R) = \left(-\frac{1}{\beta} \ln\left(1 - R\right)\right)^{\frac{1}{\alpha}} \]
Sampling by the Inverse Transform Method: Discrete Distributions

\[ \Omega = \{ x_0, x_1, ..., x_k, \ldots \} \]

\[ F_k = P \{ X \leq x_k \} = \sum_{i=0}^{k} P \{ X = x_i \} \]

sample an \( R \sim U[0,1) \)

\[ P \left[ F_{k-1} < R \leq F_k \right] = F_R (F_k) - F_R (F_{k-1}) \]

\( R \sim U[0,1) \) and \( F_R (r) = r \)

\[ \Rightarrow P \left[ F_{k-1} < R \leq F_k \right] = F_k - F_{k-1} = f_k = P \{ X = x_k \} \]

Graphically:

**Diagram:**
- \( F_0 = f_0 \)
- \( F_1 = f_1 + f_0 \)
- \( F_2 = f_2 + f_1 + f_0 \)
- \( 0 \rightarrow x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow 1 \)

**Equation:**
- \( F_k = \sum_{i=0}^{k} P \{ X = x_i \} \)

**Inequality:**
- \( P \left[ F_{k-1} < R \leq F_k \right] = F_k - F_{k-1} = f_k = P \{ X = x_k \} \)
Sampling by the Rejection Method: von Neumann Algorithm

- Given a pdf \( f_X(x) \) limited in \((a,b)\), let
  \[
  h(x) = \frac{f_X(x)}{f_M}
  \]
  so that \( 0 \leq h(x) \leq 1, \forall x \in (a,b) \)

- The operative procedure to sample a realization of \( X \) from \( f_X(x) \):
  - sample \( X' \sim U(a,b) \), the tentative value for \( X \), and calculate \( h(X') \)
  - sample \( R \sim U[0,1) \). If \( R \leq h(X') \) the value \( X' \) is accepted; else start again.
Sampling by the Rejection Method: von Neumann Algorithm

• More generally:
  \[ X \sim f_X(x) = g_X(x) \cdot H(x) \]
  \[ B_H \colon \text{max}_x H(x) \]
  \[ h(x) = \frac{H(x)}{B_H}, \quad 0 \leq h(x) \leq 1 \]

• The operative procedure:
  • sample \( X' \sim g_X(x) \), and calculate \( h(X') \)
  • sample \( R \sim U[0,1) \). If \( R \leq h(X') \) the value \( X' \)
    is accepted; else start again.

• We show that the accepted value is actually a realization of \( X \) sampled from \( f_X(x) \)

1. \[ P[ X' \leq x \mid \text{accepted} ] = \frac{P[ X' \leq x \cap \text{accepted} ]}{P[\text{accepted}]} = \frac{P[ X' \leq x \cap R \leq h(X')]}{P[\text{accepted}]} \]
Sampling by the Rejection Method: von Neumann Algorithm

\[ P \left[ z \leq X' \leq z + dz \cap \text{accepted} \right] = P \left[ z \leq X' \leq z + dz \right] \cdot P \left[ R \leq h(z) \right] = g_{X'}(z)dz \cdot h(z) \]

\[ P \left[ X' \leq x \cap R \leq h(X') \right] = \int_{-\infty}^{x} g_{X'}(z)dz \cdot h(z) \]

\[ P [\text{accepted}] = \int_{-\infty}^{\infty} g_{X'}(z)dz \cdot h(z) = 4. \]

\[ = \frac{1}{B_{H}} \int_{-\infty}^{\infty} g_{X'}(z)dz \cdot H(z) = \frac{1}{B_{H}} \int_{-\infty}^{\infty} f_{X}(x)dx = \frac{1}{B_{H}} \]
Sampling by the Rejection Method: von Neumann Algorithm

The efficiency of the method is given by the probability of accepted:

\[ \varepsilon = P[\text{accepted}] = \int_{-\infty}^{\infty} g_{X'}(z) h(z) dz = \frac{1}{B_H} \]
Sampling by the Rejection Method: von Neumann Algorithm

Example

- Let the pdf:
  
  \[
  f_X(x) = \frac{2}{\pi} \cdot \frac{1}{(1 + x)\sqrt{x}} \quad 0 \leq x \leq 1
  \]

- consider \( X' = R^2, R \sim U[0,1] \)

- the cd of the r.v. \( X \) is:
  
  \[
  G_X(x) = P[X' \leq x] = P\left[ R^2 \leq x \right] = P\left[ R \leq \sqrt{x} \right] = \sqrt{x}
  \]

- the corresponding pdf:
  
  \[
  g_X(x) = \frac{dG_X(x)}{dx} = \frac{1}{2\sqrt{x}}
  \]

- \( f_X(x) = g_X(x) \cdot H(x) = \frac{1}{2\sqrt{x}} \cdot \left( \frac{4}{\pi} \cdot \frac{1}{(1+x)} \right) \)

\[
\Rightarrow B_H = \frac{4}{\pi} ; \quad h(x) = \frac{H(x)}{B_H} = \frac{1}{1+x} \quad 0 \leq x \leq 1
\]
Sampling by the Rejection Method: von Neumann Algorithm

Example

• The operative procedure:
  
  • sample $R_1 \sim U[0,1) \Rightarrow X' = R_1^2$ and $h(X') = \frac{1}{1 + R_1^2}$
  
  • sample $R_2 \sim U[0,1)$. If $R_2 \leq h(X')$ accept $X = X'$; else start again

• The efficiency of the method is:

\[
\varepsilon = \frac{1}{B_H} = \frac{\pi}{4} = 78.5\%
\]
Sampling
Evaluation of definite integrals
Simulation of system transport
Simulation for reliability/availability analysis
EVALUATION OF DEFINITE INTEGRALS
MC Evaluation of Definite Integrals (1D)

Analog Case

\[ G = \int_a^b g(x) f(x) \, dx \]

\[ f(x) \equiv \text{pdf} \quad \rightarrow \quad f(x) \geq 0 \quad ; \quad \int f(x) \, dx = 1 \]

MC analog dart game: sample \( x \) from \( f(x) \)

- the probability that a shot hits \( x \in dx \) is \( f(x) \, dx \)
- the award is \( g(x) \)

Consider \( N \) trials with result \( \{x_1, x_2, \ldots, x_n\} \): the average award is

\[ G_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i) \]
MC Evaluation of Definite Integrals (1D)

Biased Case

The expression for $G$ may be written

$$G = \int_D \left[ \frac{f(x)}{f_1(x)} g(x) \right] f_1(x) dx \equiv \int_D g_1(x) f_1(x) dx$$

MC biased dart game: sample $x$ from $f_1(x)$

- the probability that a shot hits $x \in dx$ is $f_1(x) dx$
- the award is

$$g_1(x) = \frac{f(x)}{f_1(x)} g(x) \quad \Rightarrow \quad G_{1N} = \frac{1}{N} \sum_{i=1}^{N} g_1(x_i)$$
SIMULATION OF SYSTEM TRANSPORT
PLANT = system of $N_c$ suitably connected components.

COMPONENT = a subsystem of the plant (pump, valve,...) which may stay in different exclusive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.

STATE of the PLANT at $t$ = the set of the states in which the $N_c$ components stay at $t$. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.

PLANT TRANSITION = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the $n$-th transition is called $t_n$ and the plant state thereby entered is called $k_n$.

PLANT LIFE = stochastic process.
\[ T(t \mid t'; k')dt = \text{conditional probability of a transition at } t \in dt, \text{ given that the preceding transition occurred at } t' \text{ and that the state thereby entered was } k'. \]

\[ C(k \mid k'; t) = \text{conditional probability that the plant enters state } k, \text{ given that a transition occurred at time } t \text{ when the system was in state } k'. \]

Both these probabilities form the "trasport kernel":

\[ K(t; k \mid t'; k')dt = T(t \mid t'; k')dt \cdot C(k \mid k'; t) \]

\[ \psi(t; k) = \text{ingoing transition density or probability density function (pdf) of a system transition at } t, \text{ resulting in the entrance in state } k \]
Random walk = realization of the system life generated by the underlying state-transition stochastic process.
The transition density \( \psi(t; k) \) is expanded in series of the partial transition densities:

\[ \psi^n(t; k) = \text{pdf that the system performs the } n\text{--th transition at } t, \text{ entering the state } k. \]

Then,

\[
\psi(t, k) = \sum_{n=0}^{\infty} \psi^n(t, k) = \\
= \psi^0(t, k) + \sum_{k'} \int_{t_0}^{t} dt' \psi(t', k') K(t, k \mid t', k')
\]

Transport equation for the plant states
Von Neumann approach:

- Initial Conditions: $t_0 = t^*$, $k_0 = k^*$, $P_0 = P^*$
- The subsequent transition densities in the random walk:
  \[
  \psi_1(t_1, k_1) = K(t_1, k_1 \mid t_0, k_0)
  \]
  \[
  \psi_2(t_2, k_2) = \sum_{t_1} \int_{t_1}^{t_2} \psi_1(t_1, k_1) dt_1 K(t_2, k_2 \mid t_1, k_1)
  \]
  \[
  \ldots \ldots
  \]
  \[
  \psi_n(t_n, k_n) = \sum_{t_{n-1}} \int_{t_{n-1}}^{t_n} \psi_{n-1}(t_{n-1}, k_{n-1}) dt_{n-1} K(t_n, k_n \mid t_{n-1}, k_{n-1})
  \]
- Changing notation:
  \[
  t_n \rightarrow t \quad \quad k_{n-1} \rightarrow k'
  \]
  \[
  t_{n-1} \rightarrow t' \quad \quad k_n \rightarrow k
  \]
Monte Carlo Solution to the Transport Equation (2)

$$\psi^n(t,k) = \sum_{k'} \int_{t^*}^{t} \psi^{n-1}(t',k') dt' K(t,k | t',k')$$

$$\Rightarrow \psi(t,k) = \sum_{n=0}^{\infty} \psi^n(t,k) = \psi^0(t,k) +$$

$$+ \sum_{k'} \int_{t^*}^{t} \sum_{n=1}^{\infty} \psi^{n-1}(t',k') dt' K(t,k | t',k')$$

$$\left( \sum_{n-1=0}^{\infty} \psi^{n-1}(t',k') = \psi(t',k') \right)$$
Initial Conditions: \((t^*, k^*)\)

Formally rewrite the partial transition densities:

\[
\psi^1(t_1, k_1) = \sum_{k_0} \int_{t^*_0}^{t_1} dt_0 \psi^0(t_0, k_0) K(t_1, k_1 | t_0, k_0) = K(t_1, k_1 | t^*, k^*)
\]

\[
\psi^2(t_2, k_2) = \sum_{k_1} \int_{t^*_1}^{t_2} dt_1 \psi^1(t_1, k_1) K(t_2, k_2 | t_1, k_1) = 
\]

\[
= \sum_{k_1} \int_{t^*_1}^{t_2} dt_1 K(t_1, k_1 | t^*, k^*) K(t_2, k_2 | t_1, k_1)
\]

\[
\ldots 
\]

\[
\psi^n(t, k) = \sum_{k_1, k_2, \ldots, k_{n-1}} \int_{t^*_n}^{t_n} dt_{n-1} \int_{t^*_n-1}^{t_{n-1}} dt_{n-2} \ldots 
\]

\[
\ldots \int_{t^*_1}^{t_2} dt_1 K(t_1, k_1 | t^*, k^*) K(t_2, k_2 | t_1, k_1) \cdots K(t, k | t_{n-1}, k_{n-1})
\]
MC Evaluation of Definite Integrals

\[ G = \int_{a}^{b} g(x)f(x)dx \]

\[ f(x) \equiv \text{pdf} \quad \rightarrow \quad f(x) \geq 0 \quad ; \quad \int f(x)dx = 1 \]

• MC analog dart game: sample \( x = (t_1, k_1; t_2, k_2; \ldots) \) from

\[ f(x) = K(t_1, k_1|t^*, k^*)K(t_2, k_2|t_1, k_1) \cdots K(t, k|t_{n-1}, k_{n-1}) \]

• the probability that a shot hits \( x \in dx \) is \( f(x)dx \)

• the award is \( g(x) = 1 \)

Consider \( N \) trials with result \( \{x_1, x_2, \ldots, x_n\} \): the average award is

\[ G_N = \frac{1}{N} \sum_{i=1}^{N} g(x_i) \]
CONTENTS

- Sampling
- Evaluation of definite integrals
- Simulation of system transport
- Simulation for reliability/availability analysis
SIMULATION FOR SYSTEM RELIABILITY ANALYSIS
Monte Carlo Simulation in RAMS

\[ G(t) = \sum_{k \in \Gamma} \int_{0}^{t} \psi(\tau, k) R_k(\tau, t) d\tau \]

- \(\Gamma = \) subset of all system failure states
- \(R_k(\tau,t) = 1 \Rightarrow G(t) = \) unreliability
- \(R_k(\tau,t) = \) prob. system not exiting before \(t\) from the state \(k\) entered at \(\tau < t\)
  \(\Rightarrow G(t) = \) unavailability

Monte Carlo solution of a definite integral:
expected value \(\approx\) sample mean
\[ C^R(t) = C_R(t) \quad t \in [0, T_M] \]
\[ C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M] \]
\[ C^R(t) = C^R(t) + 1 \quad t \in [\tau, T_M] \]
\[ C_R(t) = C^R(t) \quad t \in [0, T_M] \]
\[ \hat{F}_T(t) = \frac{C_R(t)}{M} \]
Each trial of a Monte Carlo simulation consists in generating a random walk which guides the system from one configuration to another, at different times.

During a trial, starting from a given system configuration $k'$ at $t'$, we need to determine when the next transition occurs and which is the new configuration reached by the system as a consequence of the transition.

This can be done in two ways which give rise to the so-called “indirect” and “direct” Monte Carlo approach.
The indirect approach consists in:

1. Sampling first the time \( t \) of a system transition from the corresponding conditional probability density of the system performing one of its possible transitions out of \( k' \) entered at time \( t' \).

\[
C(k|t,k')
\]

2. Sampling the transition to the new configuration \( k \) from the conditional probability that the system enters the new state \( k \) given that a transition has occurred at \( t \) starting from the system in state \( k' \).

3. Repeating the procedure from \( k' \) at time \( t \) to the next transition.
The **direct approach** differs from the previous one in that the system transitions are not sampled by considering the distributions for the whole system but rather by sampling directly the times of all possible transitions of all individual components of the system and then arranging the transitions along a timeline, in accordance to their times of occurrence. Obviously, this timeline is updated after each transition occurs, to include the new possible transitions that the transient component can perform from its new state. In other words, during a trial starting from a given system configuration $k'$ at $t'$:

1. We sample the times of transition $t^i_{j_i \rightarrow m_i}$, $m_i = 1, 2, \ldots, N_{s_i}$, of each component $i$, $i = 1, 2, \ldots, N_c$ leaving its current state $j'_i$ and arriving to the state $m_i$ from the corresponding transition time probability distributions $f^{i,j_i \rightarrow m_i}_T (t|t')$.

2. The time instants $t^i_{j_i \rightarrow m_i}$ thereby obtained are arranged in ascending order along a timeline from $t_{\text{min}}$ to $t_{\text{max}} \leq T_M$.
3. The clock time of the trial is moved to the first occurring transition time $t_{\text{min}} = t^*$ in correspondence of which the system configuration is changed, i.e. the component $i^*$ undergoing the transition is moved to its new state $m_{i^*}$.

4. At this point, the new times of transition $t_{m_i^* \to l_i^*}^*, l_i^* = 1, 2, \ldots, N_{S_i}^*$, of component $i^*$ out of its current state $m_{i^*}$ are sampled from the corresponding transition time probability distributions, $f_{T_{m_i^* \to l_i^*}}(t|t^*)$, and placed in the proper position of the timeline.

5. The clock time and the system are then moved to the next first occurring transition time and corresponding new configuration, respectively.

6. The procedure repeats until the next first occurring transition time falls beyond the mission time, i.e. $t_{\text{min}} > T_M$.

Compared to the previous indirect method, the direct approach is more suitable for systems whose components’ failure and repair behaviours are represented by different stochastic distribution laws.
E. Zio, Ecole Centrale Paris, Chateauray-Malibry, France
The Monte Carlo Simulation Method for System Reliability and Risk Analysis
Series: Springer Series in Reliability Engineering

> Illustrates the Monte Carlo simulation method and its application to reliability and system engineering to give the readers the sound fundamentals of Monte Carlo sampling and simulation
> Explains the merits of pursuing the application of Monte Carlo sampling and simulation methods when realistic modeling is required so that readers may exploit these in their own applications
> Includes a range of simple academic examples in support of the explanation of the theoretical foundations as well as case studies provide the practical value of the most advanced techniques so that the techniques are accessible

Monte Carlo simulation is one of the best tools for performing realistic analysis of complex systems as it allows most of the limiting assumptions on system behavior to be relaxed. The Monte Carlo Simulation Method for System Reliability and Risk Analysis comprehensively illustrates the Monte Carlo simulation method and its application to reliability and system engineering. Readers are given a sound understanding of the fundamentals of Monte Carlo sampling and simulation and its application for realistic system modeling.

Whilst many of the topics rely on a high-level understanding of calculus, probability and statistics, simple academic examples will be provided in support of the explanation of the theoretical foundations to facilitate comprehension of the subject matter. Case studies will be introduced to provide the practical value of the most advanced techniques.

This detailed approach makes the Monte Carlo Simulation Method for System Reliability and Risk Analysis a key reference for senior undergraduate and graduate students as well as researchers and practitioners. It provides a powerful tool for all those involved in system analysis for reliability, maintenance and risk evaluation.