Monte Carlo Simulation for Reliability and Availability analyses
Example: Offshore Installation

Production:
- Gas: 5 MSm³/d
- Oil: 26500 Sm³/d
- Water: 8000 Sm³/d

Separation:
- 23300 Sm³/d
- 7000 Sm³/d

Wells:
- Fuel Gas 25 b
- 2.2 MSm³/d
- 4.4 MSm³/d

Oil Trt°:
- 0.1 MSm³
- 0.1 MSm³
- 13 MW
- 50%

Gas Lift 100b:
- 60b

Gas Lift 60b:
- 60b

TEG:
- 4.4 MSm³/d
- 1 MSm³/d
- 6 MW

Gas Export:
- 3.0 MSm³/d, 60b

Flare:
- 6 MW

Oil Export:
- 6 MW

Water Inj:
- 7 MW

Wells:
- Sea
- 7 MW

Gam lift:
- 7 MW
Monte Carlo Simulation: Why?

- Computation of system reliability and availability for industrial systems characterized by:
  - many components
  - multistate components
  - complex architectures and production logics
  - complex maintenance policy

- The use of analytical methods is unfeasible

- Monte Carlo Simulation
Monte Carlo Simulation
The History of Monte Carlo Simulation

Random sampling of numbers

Buffon

Kelvin (kinetic theory of gas - multidimensional integrals)

Gosset (coefficient of the t-Student)

Fermi, von Neumann, Ulam (Manhattan project, interaction neutron-matter)

Neutron transport (design of shielding in nuclear devices - 6 dimension-integral)

System Transport (RAM,...)
Contents

- Sampling Random Numbers
- Simulation of system transport
- Simulation for reliability/availability analysis of a component
- Example
SAMPLING RANDOM NUMBERS
Sampling (pseudo) Random Numbers
Uniform Distribution

CDF: \( U_R(r) = \begin{cases} 0 & \text{if } r < 0 \\ r & \text{if } 0 \leq r \leq 1 \\ 1 & \text{if } r > 1 \end{cases} \)

PDF: \( u_R(r) = 1 \)

cdf: \( U_R(r) = P\{R \leq r\} = \int_0^r u_R(r) \, dr = r \)
Sampling Random Numbers from an Uniform Distribution

In the past: The Rand Book (1955)

1 million of random numbers
Sampling Random Numbers from an Uniform Distribution

Today:

\[ R \sim U[0, 1) \]
Sampling Random Numbers from a Generic Distribution

\[ F_X(x) \]
Sampling Random Numbers from a Generic Distribution

$$F_X(x)$$

- Sample $R$ from $U_R(r)$
- $X = F_X^{-1}(R)$
Sample R from $U_R(r)$ and find $X$:

$$X = F_X^{-1}(R)$$

**Question:** which distribution does $X$ obey?

$$P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}$$
Sample R from $U_R(r)$ and find $X$:

$$X = F_X^{-1}(R)$$

Question: which distribution does $X$ obey?

$$P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}$$
Sample \( R \) from \( U_R(r) \) and find \( X \):

\[
X = F_X^{-1}(R)
\]

**Question**: which distribution does \( X \) obey?

\[
P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\}
\]

\[
P\{R \leq F_X(x)\} = F_X(x)
\]
Sampling (pseudo) Random Numbers

Generic Distribution

Pr\{R \leq r\} = U_R(r) \quad F_X(x)

Sample R from U_R(r) and find X:

\[ X = F_X^{-1}(R) \]

Question: which distribution does X obey?

\[ P\{X \leq x\} = P\{F_X^{-1}(R) \leq x\} \]

Application of the operator \( F_X \) to the argument of \( P \) above yields

\[ P\{X \leq x\} = P\{R \leq F_X(x)\} = F_X(x) \]

Summary: From an \( R \sim U_R(r) \) we obtain an \( X \sim F_X(x) \)
Sampling from a Bernoulli distribution

- \( X = \begin{cases} 1 & \text{'healthy'} \\ 0 & \text{'failed'} \end{cases} \) with \( P(X = 0) = p_0 = 0.1 \)

\[
P_X(x) \quad F_X(x)
\]

\( p_0 = 0.1 \)
Sampling from a Bernoulli distribution

- $X = \begin{cases} 1 & \text{'healthy'} \\ 0 & \text{'failed'} \end{cases}$ with $P(X = 0) = p_0 = 0.1$

- Sample $R$ from $U_R(r)$
  - If $R \leq p_0 \rightarrow X = 0$
  - If $R > p_0 \rightarrow X = 1$

Does it work?

- $P(X = 0) = P(R \leq p_0) = p_0$
- $P(X = 1) = P(R > p_0) = 1 - p_0$
Sampling from discrete distributions

\[ \Omega = \{x_0, x_1, \ldots, x_k\} \]

\[ P\{X = x_i\} = f_i \]

\[ F_i = P\{X \leq x_i\} = \sum_{j=0}^{i} f_j \]
**Sampling from discrete distributions**

\[ \Omega = \{x_0, x_1, \ldots, x_k, \ldots\} \]

\[ P\{X = x_k\} = f_k \]

\[ F_k = P\{X \leq x_k\} = \sum_{i=0}^{k} f_k \]

**Sampling procedure:**

- Sample an \( R \sim U[0,1) \)
- Find \( i \) such that \( F_{i-1} < R \leq F_i \)
- \( X = x_i \)

**Graphically:**

Why does it work?

\[ P\{X = x_i\} = P\{F_{i-1} < R \leq F_i\} = F_i - F_{i-1} = f_i \]
Exercise 1: Sampling from the Exponential Distribution

Probability density function:

\[ f_T(t) = \lambda e^{-\lambda t} \quad t \geq 0 \]

\[ = 0 \quad t < 0 \]

Expected value and variance:

\[ E[T] = \frac{1}{\lambda} \]

\[ Var[T] = \frac{1}{\lambda^2} \]

• Find the procedure to sample numbers from an exponential distribution. Assume to have available a generator of random numbers from an uniform distribution in the interval \([0,1)\)
Sampling Random Numbers from $F_T(t)$: Inverse Transform Method

Sample $R$ from $U_R(r)$ and find $T$:

$$T = F_T^{-1}(R)$$

Example: Exponential distribution

$$F_T(t) = 1 - e^{-\lambda t}$$

$$R = 1 - e^{-\lambda T}$$

$$T = F_T^{-1}(R) = -\frac{1}{\lambda} \ln(1 - R)$$
Monte Carlo Simulation for Failure Probability Estimation (non Repairable System)

Buffon:
- Repeat the experiment $M$ times
- $N_\cap \leftarrow$ Number of times the needle lies across a line
- $p = \frac{N_\cap}{M}$

Pseudocode

\[ N_f = 0 \]
For $i = 1 : M$
- Simulate the state of each component of the system
- Find the corresponding system state
- If system state=failed \( \rightarrow N_F = N_F + 1 \)
EndFor
\[ p_f = \frac{N_f}{M} \]
Monte Carlo estimate the failure probability of the network, i.e., the probability of no connection between nodes S and T.

<table>
<thead>
<tr>
<th>Arc number $i$</th>
<th>Failure probability $P_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.050</td>
</tr>
<tr>
<td>2</td>
<td>0.025</td>
</tr>
<tr>
<td>3</td>
<td>0.050</td>
</tr>
<tr>
<td>4</td>
<td>0.020</td>
</tr>
<tr>
<td>5</td>
<td>0.075</td>
</tr>
</tbody>
</table>
1) Minimal Failure Configuration (minimal cut set):

M₁={1,4}
M₂={2,5}
M₃={1,3,5}
M₄={2,3,4}

2) Network failure probability:

\[ P[M_1] = p_1 \cdot p_4 = 0.05 \cdot 0.02 = 1 \cdot 10^{-3} \]
\[ P[M_2] = p_2 \cdot p_5 = 0.025 \cdot 0.075 = 1.875 \cdot 10^{-3} \]
\[ P[M_3] = p_1 \cdot p_3 \cdot p_5 = 0.05 \cdot 0.05 \cdot 0.075 = 1.875 \cdot 10^{-4} \]
\[ P[M_4] = p_2 \cdot p_3 \cdot p_4 = 0.025 \cdot 0.05 \cdot 0.02 = 7.5 \cdot 10^{-5} \]

\[ P[X_T = 1] \approx \sum_{j=1}^{4} P[M_j] = 3.07 \cdot 10^{-3} \text{ rare event approximation} \]
The state of arc $i$ can be sampled by applying the inverse transform method to the set of discrete probabilities $\{p_i, 1 - p_i\}$ of the mutually exclusive and exhaustive states.

- We calculate the arrival state for all the arcs of the network.
- As a result of these transitions, if the system fails we add 1 to $N_f$.
- The trial simulation then proceeds until we collect $M$ trials.

\[
\hat{P}_{MC} = \frac{N_f}{M}
\]
Matlab Code

- clear all;
- p=[0.05,0.025,0.05,0.02,0.075]; %arc failure probabilities
- nf=0;
- M=1e6; %number of MC simulations
- for i=1:M
  - %sampling of arcs fault events
  - for j=1:5 % simulation of arc states
    - r=rand;
    - if (r<=p(j))
      - s(j)=1; % arc j is failed
    - else
      - s(j)=0; % arc j is working
    - end
  - end
  - %(is the network faulty?)
  - fault=s(1)*s(4)+s(2)*s(5)+s(1)*s(3)*s(5)+s(2)*s(3)*s(4);
  - if fault >= 1
    - nf=nf+1;
  - end
  - end
- pf=nf/M;%system failure probability
SIMULATION OF SYSTEM TRANSPORT
Monte Carlo simulation for system reliability

- **PLANT** = system of $N_c$ suitably connected components.
- **COMPONENT** = a subsystem of the plant (pump, valve,...) which may stay in different mutually exclusive and exhaustive (multi)states (nominal, failed, stand-by,...). Stochastic transitions from state-to-state occur at stochastic times.
- **STATE** of the PLANT at $t$ = the set of the states in which the $N_c$ components stay at $t$. The states of the plant are labeled by a scalar which enumerates all the possible combinations of all the component states.
- **PLANT TRANSITION** = when any one of the plant components performs a state transition we say that the plant has performed a transition. The time at which the plant performs the $n$-th transition is called $t_n$ and the plant state thereby entered is called $k_n$.
- **PLANT LIFE** = stochastic process.
Plant life: random walk

Random walk = realization of the system life generated by the underlying state-transition stochastic process.

Phase space
Phase Space

$k$

$k^{*} + 1$

$k^{*}$

$k^{*} - 1$

4

3

2

1

$t^{*}$

$t$

$T_M$
Example: System Reliability Estimation

- Divide the mission time, $T_M$, in bins and associate a counter (of the system failure) to each bin:
  
  $C^F(1)$, $C^F(2)$, ..., $C^F(\tau)$, ...

- Initialize each counter to 0:
  
  $C^F(1) = 0$, $C^F(2) = 0$, ..., $C^F(\tau) = 0$, ...

\[ T_M \]
Example: System Reliability Estimation

- Divide the mission time, $T_M$, in bins and associate a counter (of the system failure) to each bin:

  \[ C^F(1), C^F(2), \ldots, C^F(\tau), \ldots \]

- Initialize each counter to 0:

  \[ C^F(1) = 0, C^F(2) = 0, \ldots, C^F(\tau) = 0, \ldots \]

- MC simulation of the first system life

  - Events at component level, which do not entail system failure $\Rightarrow$ No system failure during first simulation

  \[ C^F(\tau) = C^F(\tau) \text{ for } \forall \tau \in [0, T_m] \]
Example: System Reliability Estimation

- MC simulation of the first system life
  \[ C^F(1) = 0 \]
  \[ C^F(2) = 0 \]

- MC simulation of the second system life
  \[ C^F(\tau) = 0 \]

Event at component level, which do not entail system failure

Event at component level, which causes the system failure

System failure at time \( \tau_2 \)

\[ C^F(\tau) = C^F(\tau) + 1 \text{ for } \forall \tau > \tau_2 \]
Example: System Reliability Estimation

- MC simulation of the first system life
  \[ C_F(1) = 0 \]
- MC simulation of the second system life
  \[ C_F(2) = 0 \]
- MC simulation of the \( M \)-th system life
  \[ C_F(\tau) = 0 \]

Event at component level, which do not entail system failure

Event at component level, which causes the system failure

System failure at time \( \tau_M \)

\[ C_F(\tau) = C_F(\tau) + 1 \quad \text{for } \forall \tau > \tau_M \]
Example: System Reliability Estimation

- MC simulation of the first system life
  
  \[ C_F(1) = 0 \]

- MC simulation of the second system life
  
  \[ C_F(\tau) = 0 \]

- MC simulation of the \( M \)-th system life
  
  Event at component level, which causes the system failure
  
  \[ F_T(\tau) \equiv \frac{C_F(\tau)}{M} \rightarrow R(\tau) = 1 - \frac{C_F(\tau)}{M} \]

- MC estimation of the system reliability

Event at component level, which do not entail system failure
Example: System Reliability Estimation

\[ F(t) = 1 - R(t) \]

\[ t \]

1. \[ T_M \]
2. \[ \tau \]
3. \[ 1/4 \]
4. \[ 1/2 \]
SIMULATION FOR COMPONENT AVAILABILITY ESTIMATION
One component with exponential distribution of the failure time

State X=0 (WORKING)
State X=1 (FAILED)

Limit unavailability:

\[ q = ? \]
One component with exponential distribution of the failure time

State $X=0$ (WORKING)
State $X=1$ (FAILED)

Limit unavailability:

\[
q = \frac{MTTR}{MTTR + MTTF} = \frac{1}{\mu} \times \frac{1}{\mu + \frac{1}{\lambda}} = 0.1071
\]

<table>
<thead>
<tr>
<th>values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
</tbody>
</table>
Monte Carlo simulation for estimating the system availability at time $t$

- Divide the mission time, $T_M$, in bins and associate a counter (of the system failure) to each bin:
  $$C^q(1), C^q(2), \ldots, C^q(\tau), \ldots$$
- Initialize each counter to 0:
  $$C^q(1) = 0, C^q(2) = 0, \ldots, C^q(\tau) = 0, \ldots$$
Monte Carlo simulation for estimating the system availability at time $t$: Simulation of the first MC trial

State $X=0$ (WORKING)
State $X=1$ (FAILED)
Monte Carlo simulation for estimating the system availability at time $t$: Simulation of the first MC trial

State $X=0$ (WORKING)
State $X=1$ (FAILED)

$t_1 = -\frac{1}{\lambda} \ln(1 - R_1)$
Monte Carlo simulation for estimating the system availability at time $t$: Simulation of the first MC trial

State $X=0$ (WORKING)
State $X=1$ (FAILED)

$t_1 = -\frac{1}{\lambda} \ln(1 - R_1)$
$t_2 = t_1 - \frac{1}{\mu} \ln(1 - R_2)$
Monte Carlo simulation for estimating the system availability at time $t$: Simulation of the first MC trial

State $X=0$ (WORKING)
State $X=1$ (FAILED)

$t_1 = -\frac{1}{\lambda} \ln(1 - R_1)$

$t_2 = t_1 - \frac{1}{\mu} \ln(1 - R_2)$
Monte Carlo simulation for estimating the system availability at time $t$: counter updating

- Divide the mission time, $T_M$, in bins and associate a counter (of the system failure) to each bin:
  
  \[ C^q(1), C^q(2), \ldots, C^q(\tau), \ldots \]

- Initialize each counter to 0:
  
  \[ C^q(1) = 0, C^q(2) = 0, \ldots, C^q(\tau) = 0, \ldots \]

- If the component is failed in $(t_j, t_j + \Delta t)$, the corresponding counter, $c^q(t_j)$, increases
  
  \[ c^q(t_j) = c^q(t_j) + 1 \]

  otherwise, $c^q(t_j) = c^q(t_j)$
Monte Carlo simulation for estimating the system availability at time $t$: Simulation of the second MC trial

State $X=0$ (WORKING)
State $X=1$ (FAILED)
Monte Carlo simulation for estimating the system availability at time $t$

- Divide the mission time, $T_M$, in bins and associate a counter (of the system failure) to each bin:
  \[ C^q(1), C^q(2), \ldots, C^q(\tau), \ldots \]

- Initialize each counter to 0:
  \[ C^q(1) = 0, C^q(2) = 0, \ldots, C^q(\tau) = 0, \ldots \]

- If the component is failed in $(t_j, t_j + \Delta t)$, the corresponding counter, $c^q(t_j)$, increases \( c^q(t_j) = c^q(t_j) + 1 \); otherwise, \( c^q(t_j) = c^q(t_j) \)
One component with exponential distribution of the failure and repair time

State $X=0$ (WORKING)
State $X=1$ (FAILED)
Monte Carlo simulation for estimating the system availability at time \( t \)

- ... another trial
Monte Carlo simulation for estimating the system availability at time $t$

- ... another trial
Monte Carlo simulation for estimating the system availability at time $t$

The counter $c^q(t_j)$ adds 1 until $M$ trials have been sampled.

\[
q(t_j) = P\{X(t_j) = 1\} \approx \frac{C^q(t_j)}{M}
\]
Excercise 2

- Write the pseudo code for the estimation of the instantaneous unavailability of a continuously monitored component with constant failure rate ($\lambda$) and repair rate ($\mu$).

<table>
<thead>
<tr>
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</tr>
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<tbody>
<tr>
<td>$\lambda$</td>
</tr>
<tr>
<td>$\mu$</td>
</tr>
</tbody>
</table>

- Hints:
  - You can assume a mission time of $10^3$
  - You can compute the instantaneous availability at times: $0, 1, 2, 3, \ldots, 10^5$
% Initialize parameters
Tm=1000; % mission time;
M=10000; % number of trials;
lambda=3E-3;
mu=25E-3;
Dt=1; % bin length;
Time_axis=0:Dt:Tm;
counter_q=zeros(1,length(Time_axis));
for i=1:M %simulation of M MC trials
    t=0;
    state=0; %state=0 = working;
    state=1; %state=1 = failed;
while t<Tm
    if state == 0
        t=t-[\log(1-rand)]/\lambda; %failure time
        failure_time=t;
        state=1; % new state=failed
        lower_b=min(find(Time_axis>=failure_time)); % first unavailability counter to be increased
    else
        t=t-[\log(1-rand)]/\mu; %repair_time
        state=0;
        repair_time=t;
        if t<Tm
            upper_b=max(find(Time_axis < repair_time)); %last unavailability counter to be increased
        else
            %repair ends after mission time
            upper_b=length(Time_axis);
        end
        counter_q(lower_b:upper_b)= counter_q(lower_b:upper_b)+1; %increase all unavailability counter between lower_b and upper_b
    end
end
Unav_MC = counter_q(:)/M;
Unav_true = (lambda)/(lambda+mu) - (lambda/(lambda+mu))*exp(-(lambda+mu).*Time_axis);
plot(Time_axis, Unav_true, 'blue')
hold on
plot(Time_axis, Unav_MC, 'red')

M = 10000
Exercise 2 - Solution

\[ M = 100000 \]
SIMULATION FOR SYSTEM AVAILABILITY / RELIABILITY ESTIMATION
Stochastic Transitions: Governing Probabilities
Stochastic Transitions: Governing Probabilities

- \( T(t / t'; k') \, dt \) = conditional probability of a transition at \( T \in [t, t + dt) \), given that the preceding transition occurred at \( t' \) and that the state thereby entered was \( k' \).

- \( C(k / k'; t) \) = conditional probability that the plant enters state \( k \), given that a transition occurred at time \( t \) when the system was in state \( k' \). Both these probabilities form the "trasport kernel":

\[
K(t; k / t'; k') \, dt = T(t / t'; k') \, dt \, C(k / k'; t)
\]
Phase Space
Components’ times of transition between states are exponentially distributed
\( \lambda_{j_i \rightarrow m_i}^i = \text{rate of transition of component } i \text{ going from its state } j_i \text{ to the state } m_i \)

<table>
<thead>
<tr>
<th>Initial</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>( \lambda_{1 \rightarrow 2}^{A(B)} )</td>
<td>( \lambda_{1 \rightarrow 3}^{A(B)} )</td>
</tr>
<tr>
<td>2</td>
<td>( \lambda_{2 \rightarrow 1}^{A(B)} )</td>
<td>-</td>
<td>( \lambda_{2 \rightarrow 3}^{A(B)} )</td>
</tr>
<tr>
<td>3</td>
<td>( \lambda_{3 \rightarrow 1}^{A(B)} )</td>
<td>( \lambda_{3 \rightarrow 2}^{A(B)} )</td>
<td>-</td>
</tr>
</tbody>
</table>
- The components are initially \((t=0)\) in their nominal states \((1,1,1)\).
- The failure configuration are \((C \text{ in state } 4:(\ast,\ast,4))\) and \((A \text{ and } B \text{ in } 3:\ (3,3,\ast))\).

<table>
<thead>
<tr>
<th>Initial</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-</td>
<td>(\lambda_{ \text{f} \rightarrow 2}^{C} )</td>
<td>(\lambda_{1 \rightarrow 3}^{C} )</td>
<td>(\lambda_{1 \rightarrow 4}^{C} )</td>
</tr>
<tr>
<td>2</td>
<td>(\lambda_{2 \rightarrow 1}^{C} )</td>
<td>-</td>
<td>(\lambda_{2 \rightarrow 3}^{C} )</td>
<td>(\lambda_{2 \rightarrow 4}^{C} )</td>
</tr>
<tr>
<td>3</td>
<td>(\lambda_{3 \rightarrow 1}^{C} )</td>
<td>(\lambda_{3 \rightarrow 2}^{C} )</td>
<td>-</td>
<td>(\lambda_{3 \rightarrow 4}^{C} )</td>
</tr>
<tr>
<td>4</td>
<td>(\lambda_{4 \rightarrow 1}^{C} )</td>
<td>(\lambda_{4 \rightarrow 2}^{C} )</td>
<td>(\lambda_{4 \rightarrow 3}^{C} )</td>
<td>-</td>
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</tbody>
</table>
Indirect Monte Carlo Trial: Sampling the time of transition

How to build $T(t|t' = 0, k' = (1,1,1))$?

The rate of transition of component $A(B)$ out of its nominal state 1 is:
\[
\lambda_1^{A(B)} = \lambda_{1\to 2}^{A(B)} + \lambda_{1\to 3}^{A(B)}
\]

- The rate of transition of component $C$ out of its nominal state 1 is:
\[
\lambda_1^C = \lambda_{1\to 2}^C + \lambda_{1\to 3}^C + \lambda_{1\to 4}^C
\]

- The rate of transition of the system out of its current configuration $(1, 1, 1)$ is:
\[
\lambda^{(1,1,1)} = \lambda_1^A + \lambda_1^B + \lambda_1^C
\]

\[
T(t|t' = 0, k' = (1,1,1) = \lambda^{(1,1,1)}e^{-\lambda^{(1,1,1)}t}
\]

- We are now in the position of sampling the first system transition time $t_1$, by applying the inverse transform method:

\[
t_1 = t_0 - \frac{1}{\lambda^{(1,1,1)}} \ln(1 - R_t)
\]

where $R_t \sim U[0,1)$
How to build $T(k|t = t_1, k' = (1,1,1))$?

- Assuming that $t_1 < T_M$ (otherwise we would proceed to the successive trial), we now need to determine which transition has occurred, i.e. which component has undergone the transition and to which arrival state.

- The probabilities of components A, B, C undergoing a transition out of their initial nominal states 1, given that a transition occurs at time $t_1$, are:

$$\frac{\lambda_1^A}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^B}{\lambda^{(1,1,1)}}, \quad \frac{\lambda_1^C}{\lambda^{(1,1,1)}}$$

- Thus, we can apply the inverse transform method to the discrete distribution.
Given that at $t_1$ component B undergoes a transition, its arrival state can be sampled by applying the inverse transform method to the set of discrete probabilities

$$\left\{ \frac{\lambda_{1 \rightarrow 2}^B}{\lambda_1^B}, \frac{\lambda_{1 \rightarrow 3}^B}{\lambda_1^B} \right\}$$

of the mutually exclusive and exhaustive arrival states

As a result of this first transition, at $t_1$ the system is operating in configuration (1,3,1).

The simulation now proceeds to sampling the next transition time $t_2$ with the updated transition rate

$$\lambda^{(1,3,1)} = \lambda_1^A + \lambda_3^B + \lambda_1^C$$
Sampling the Next Transition

• The next transition, then, occurs at

\[ t_2 = t_1 - \frac{1}{\lambda^{(1,3,1)}} \ln(1 - R_t) \]

where \( R_t \sim U[0,1] \).

• Assuming again that \( t_2 < T_M \), the component undergoing the transition and its final state are sampled as before by application of the inverse transform method to the appropriate discrete probabilities.

• The trial simulation then proceeds through the various transitions from one system configuration to another up to the mission time \( T_M \).
When the system enters a failed configuration (*,*,4) or (3,3,*), where the * denotes any state of the component, tallies are appropriately collected for the unreliability/unavailability estimates (at discrete times $t_j \in [0, T_M]$);
Example: System Reliability Estimation

- Consider a sequence of time instants:
  \[ 0, \Delta t, 2\Delta t, \ldots, \tau\Delta t, \ldots, T_M \]
- Associate to each time instant a counter (of the system failure):
  \[ C_F^1, C_F^2, \ldots, C_F^\tau, \ldots, C_F^{T_M/\Delta t} \]
- Initialize each counter to 0:
  \[ C_F^1 = 0, C_F^2 = 0, \ldots, C_F^\tau = 0, \ldots, C_F^{T_M/\Delta t} = 0 \]
• When the system enters a failed configuration \((*,*,4)\) or \((3,3,*)\), where the \(*\) denotes any state of the component, tallies are appropriately collected for the unreliability estimates.
Example: System Reliability Estimation

Events at components level, which do not entail system failure

Repair Time

\[ C^F(\tau) = C^F(\tau) \quad \forall \tau \in [0, T_M] \]

\[ C^F(\tau) = C^F(\tau) + 1 \quad \forall \tau \in [\tau_1, T_M] \]

\[ C^F(\tau) = C^F(\tau) + 1 \quad \forall \tau \in [\tau_2, T_M] \]

\[ F_T(\tau) \equiv \frac{C^F(\tau)}{M} \rightarrow R(\tau) = 1 - \frac{C^F(\tau)}{M} \]
• When the system enters a failed configuration (*,*,4) or (3,3,*), where the * denotes any state of the component, tallies are appropriately collected for the unreliability estimates.

• After performing a large number of trials $M$, we can obtain estimates of the system unreliability at any time $t_j$ by

$$F(t_j) = \frac{C^F(t_j)}{M}$$
Unavailability Estimation

- Consider a sequence of time instants: 
  \[0, \Delta t, 2\Delta t, \ldots, \tau \Delta t, \ldots, T_M\]

- Associate to each time instant a counter (of the system failure):
  \[C^q(1), C^q(2), \ldots, C^q(\tau), \ldots, C^q\left(\frac{T_M}{\Delta t}\right)\]

- Initialize each counter to 0:
  \[C^q(1) = 0, C^q(2) = 0, \ldots, C^q(\tau) = 0, \ldots\]
Example: System Availability Estimation

Events at components level, which do not entail system failure

\[ C^q(\tau) = C^q(\tau) \quad \forall \tau \in [0, T_M] \]

\[ C^q(\tau) = C^q(\tau) + 1 \quad \forall \tau \in [\tau_1, \tau_{r_1}] \]

\[ C^q(\tau) = C^q(\tau) + 1 \quad \forall \tau \in [\tau_1, \tau_{r_2}] \]

\[ q(\tau) \approx \frac{C^q(\tau)}{M} \rightarrow p(\tau) = 1 - \frac{C^q(\tau)}{M} \]
• When the system enters a failed configuration \((*,*,4)\) or \((3,3,*)\), where the * denotes any state of the component, tallies are appropriately collected for the instantaneous unavailability estimates (at discrete times \(t_j \in [0, T_M]\));

• System instantaneous unavailability at any time \(t_j\) by

\[
q(t_j) = \frac{C_q(t_j)}{M}
\]
A real example of Monte Carlo Simulation

PRODUCTION AVAILABILITY EVALUATION OF AN OFFSHORE INSTALLATION

RELIABILITY AND AVAILABILITY EVALUATION OF AN OFFSHORE INSTALLATION

System description: basic scheme

- Wells
- Separation
- Oil Trt°
- Wat. Trt°
- Sea
- Water Inj.
- TEG
- Flare
- Gas Export 3.0 mSm3/d, 60b
- Oil export
System description: gas-lift

First loop

<table>
<thead>
<tr>
<th>Gas-lift pressure</th>
<th>Production of the Well</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100%</td>
</tr>
<tr>
<td>60</td>
<td>80%</td>
</tr>
<tr>
<td>0</td>
<td>60%</td>
</tr>
</tbody>
</table>

Gas Export: 3.0 MSm³/d, 60b
System description:
fuel gas generation and distribution

Second loop

Wells → Separation → TC → TC → TG → TG → TEG → Gas Export

0.1 MSm³/d

Fuel Gas 25 b
0.4 MSm³/d

3.0 MSm³/d, 60b
System description:
electricity power production and distribution
Wells
Production
Gas : 5 MSm³/d
Oil : 26500 Sm³/d
Water : 8000 Sm³/d

Separation
23300 Sm³/d

Wat. Trt°
7000 Sm³/d

Fuel Gas 25 b

TC
0.1 MSm³
2.2 MSm³/d
50%

TC
0.1 MSm³
2.2 MSm³/d
50%

Fuel Gas 25 b

TEG
4.4 MSm³/d

Gas Lift 60 b

EC
1 MSm³/d
6 MW

Gas Lift 100 b

Gas Export
3.0 MSm³/d, 60 b

Flare

Water Inj

Sea
13 MW

Oil Trt°

TG
0.1 MSm³
13 MW
50%

TG
0.1 MSm³
13 MW
50%

Oil export
6 MW

7 MW

7 MW
Component failures and repairs: TCs and TGs

State 0 = as good as new
State 1 = degraded (no function lost, greater failure rate value)
State 2 = critical (function is lost)

<table>
<thead>
<tr>
<th></th>
<th>TC</th>
<th>TG</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{01}$</td>
<td>$0.89 \cdot 10^{-3} \text{ h}^{-1}$</td>
<td>$0.67 \cdot 10^{-3} \text{ h}^{-1}$</td>
</tr>
<tr>
<td>$\lambda_{02}$</td>
<td>$0.77 \cdot 10^{-3} \text{ h}^{-1}$</td>
<td>$0.74 \cdot 10^{-3} \text{ h}^{-1}$</td>
</tr>
<tr>
<td>$\lambda_{12}$</td>
<td>$1.86 \cdot 10^{-3} \text{ h}^{-1}$</td>
<td>$2.12 \cdot 10^{-3} \text{ h}^{-1}$</td>
</tr>
<tr>
<td>$\mu_{10}$</td>
<td>$0.033 \text{ h}^{-1}$</td>
<td>$0.032 \text{ h}^{-1}$</td>
</tr>
<tr>
<td>$\mu_{20}$</td>
<td>$0.048 \text{ h}^{-1}$</td>
<td>$0.038 \text{ h}^{-1}$</td>
</tr>
</tbody>
</table>
Component failures and repairs: EC and TEG

State 0 = as good as new
State 2 = critical (function is lost)

<table>
<thead>
<tr>
<th></th>
<th>EC</th>
<th>TEG</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ</td>
<td>$0.17 \cdot 10^{-3} \text{ h}^{-1}$</td>
<td>$5.7 \cdot 10^{-5} \text{ h}^{-1}$</td>
</tr>
<tr>
<td>μ</td>
<td>0.032 $\text{ h}^{-1}$</td>
<td>0.333 $\text{ h}^{-1}$</td>
</tr>
</tbody>
</table>
Production priority: example

- **Wells**
  - Production:
    - Gas: 5 MSm³/d
    - Oil: 26500 Sm³/d
    - Water: 8000 Sm³/d

- **Separation**
  - 23300 Sm³/d
  - 7000 Sm³/d

- **Oil Trt°**
  - 13 MW
  - 6 MW
  - 7 MW

- **Wat. Trt°**
  - 0.1 MSm³
  - 0.1 MSm³

- **Fuel Gas 25 b**
  - 0.1 MSm³
  - 2.2 MSm³/d
  - 2.2 MSm³/d

- **TEG**
  - 4.4 MSm³/d
  - 6 MW

- **Gas Export**
  - 3.0 MSm³/d, 60b

- **Gas Lift 60b**
  - 1 MSm³/d

- **Gas Lift 100b**
  - 1 MSm³/d

- **Flare**

- **EC**
  - 6 MW

- **Oil export**

- **Water Inj**
  - 7 MW

- **Sea**
  - 13 MW
Production priority

When a failure occurs, the system is reconfigured to minimise (in order):

- the impact on the export oil production
- the impact on export gas production

The impact on water injection does not matter
Production priority: example

- Gas Production: 5 MSm³/d
- Oil Production: 26500 Sm³/d
- Water Production: 8000 Sm³/d

- Gas Lift 60b
- Gas Lift 100b

- Fuel Gas 25 b:
  - 2.2 MSm³/d
  - 0.1 MSm³

- TEG:
  - 4.4 MSm³/d

- Gas Export:
  - 3.0 MSm³/d, 60b
  - 6 MW
  - 1 MSm³/d

- Oil Export:
  - 7 MW

- Water Inj:
  - 13 MW

- Sea:
  - 13 MW
  - 7 MW

- Water Trt°:
  - 0.1 MSm³
  - 50%

- Oil Trt°:
  - 0.1 MSm³
  - 50%
  - 13 MW
Maintenance policy: reparation

Only 1 repair team

Two or more components are failed at the same time

Priority levels of failures:
1. Failures leading to total loss of export oil (both TG’s or both TC’s or TEG)
2. Failures leading to partial loss of export oil (single TG or EC)
3. Failures leading to no loss of export oil (single TC failure)
Maintenance policy: preventive maintenance

- **Only 1** preventive maintenance team
- The preventive maintenance takes place only if the system is in perfect state of operation

<table>
<thead>
<tr>
<th>Type of maintenance</th>
<th>Frequency [hours]</th>
<th>Duration [hours]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turbo-Generator and Turbo-Compressors</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type 1</td>
<td>2160 (90 days)</td>
<td>4</td>
</tr>
<tr>
<td>Type 2</td>
<td>8760 (1 year)</td>
<td>120 (5 days)</td>
</tr>
<tr>
<td>Type 3</td>
<td>43800 (5 years)</td>
<td>672 (4 weeks)</td>
</tr>
<tr>
<td>Electro Compressor</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Type 4</td>
<td>2666</td>
<td>113</td>
</tr>
</tbody>
</table>
Number of System States

Plant state

453 plant states
Monte Carlo Simulation: Why?

- Computation of system reliability and availability for industrial systems characterized by:
  - many components
  - multistate components
  - complex architectures and production logics
  - complex maintenance policy

- The use of analytical methods is unfeasible

- Monte Carlo Simulation
A systematic procedure

7 different production levels

<table>
<thead>
<tr>
<th>Plant state (Production Level)</th>
<th>Gas [kSm³/d]</th>
<th>Oil [k m³/d]</th>
<th>Water [m³/d]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0=(100%)</td>
<td>3000</td>
<td>23.3</td>
<td>7000</td>
</tr>
<tr>
<td>1</td>
<td>900</td>
<td>23.3</td>
<td>7000</td>
</tr>
<tr>
<td>2</td>
<td>2700</td>
<td>21.2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1000</td>
<td>21.2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>2600</td>
<td>21.2</td>
<td>6400</td>
</tr>
<tr>
<td>5</td>
<td>900</td>
<td>21.2</td>
<td>6400</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Failure of only 1 TG

Failure of the EC

Failure of TEG, Failure of both TCs, Failure of both TGs
MONTE CARLO APPROACH

Associate a production level to each of the 453 plant states

Production levels

- oil
- gas
- water

Plant state

TG₁ TG₂ TC₁ TC₂ TEG EC

A systematic procedure
Real system with preventive maintenance

<table>
<thead>
<tr>
<th>Production level</th>
<th>Average availability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>8.13E-1</td>
</tr>
<tr>
<td>1</td>
<td>5.68E-2</td>
</tr>
<tr>
<td>2</td>
<td>6.58E-2</td>
</tr>
<tr>
<td>3</td>
<td>1.19E-2</td>
</tr>
<tr>
<td>4</td>
<td>3.55E-2</td>
</tr>
<tr>
<td>5</td>
<td>2.34E-3</td>
</tr>
<tr>
<td>6</td>
<td>1.50E-2</td>
</tr>
</tbody>
</table>

$N_{trial} = 10^5 \quad T_{miss} = 5 \cdot 10^6$ hours - Real system with periodic preventive maintenance
### Real system with preventive maintenance

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil [k m³/d]</td>
<td>22.60</td>
<td>0.42</td>
</tr>
<tr>
<td>Gas [k Sm³/d]</td>
<td>2687</td>
<td>194.3</td>
</tr>
<tr>
<td>Water [k m³/d]</td>
<td>6.04</td>
<td>0.76</td>
</tr>
</tbody>
</table>

\[ N_{\text{trial}} = 10^5 \quad T_{\text{miss}} = 5 \cdot 10^5 \text{ hours} \]
## Case A: perfect system and preventive maintenances

<table>
<thead>
<tr>
<th>Type of maintenance</th>
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</table>
Case A: perfect system and preventive maintenances

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>Oil [k m³/d]</td>
<td>23.230</td>
<td>0.263</td>
</tr>
<tr>
<td>Gas [k Sm³/d]</td>
<td>2929</td>
<td>194.0</td>
</tr>
<tr>
<td>Water [k m³/d]</td>
<td>6.811</td>
<td>0.883</td>
</tr>
</tbody>
</table>

- P.Maintenance Type 1 (TC,TG)
- P.Maintenance Type 2 (EC)
- P.Maintenance Type 3 (TC,TG)

\[ N_{\text{trials}} = 10^5 \quad T_{\text{miss}} = 10^4 \text{ hours} \]

Perfect system with preventive maintenance

![Graph showing OIL, GAS, and WATER over time](image)
Case B: Component Failure and no preventive maintenances
Conclusions

- Complex multi-state system with maintenance and operational loops

- Systematic procedure to assign a production level to each configuration

- Investigation of effects maintenance on production
Where to study?

- Slides
- Book:

- Chapter 1
- Sections 3.1, 3.3.1, 3.3.2 (no examples of the multivariate normal distribution)
- Sections 4.1, 4.2, 4.3, 4.4, 4.4.1
- Section 5.1