A New Analytical Approach for Interval Availability Analysis of Markov Repairable Systems

Shijia Du, Enrico Zio, and Rui Kang

Abstract—**Interval availability, defined as the fraction of time that a system is operational during a period of time $[0, T]$, is an important indicator of system performance, especially for industries with a Service Level Agreement, e.g., telecommunication industry, computer industry, etc. Most existing methods to compute interval availability are based on numerical simulations. In this paper, we present a new analytical solution for interval availability of Markov repairable systems. Three interval availability indexes, i.e., interval availability, interval availability in a general interval, and interval availability in multiple intervals, are considered. The interval availability indexes are derived based on aggregated stochastic processes and the results are obtained in closed form using Laplace transforms. A numerical example is presented and the results are compared with those of Monte Carlo simulation. The developed methods are applied to calculate the interval availability of a fault-tolerant database system from the literature.**

Index Terms—**Interval availability, Markov process, Markov repairable system.**

I. Introduction

Repairable systems have been intensively investigated in research and applications exist in practice (e.g., see [1]–[3]). Availability is a major concern for these systems [4]–[8].Various availability indexes have been defined in the literature to quantify different aspects of system availability [9]. For example, point-wise availability quantifies the availability at a given time instant, limiting availability focuses on the steady-state behavior of system availability, average availability quantifies the average behavior of system availability over a time interval [10], multi-point and multi-interval availabilities concentrate on behaviors of system under multiple given time points and intervals [11], etc.

In general, the existing availability indexes primarily focus on the instant, average, or limit state behavior of system availability, and they are all essentially probabilities. In practice, however, decision makers are often interested in the availability during a finite period, e.g., month, quarter, or year, where the system is required to be operative for at least a fraction α of the given period and penalty rules are incurred if the requirement is violated. For example, in telecommunication industry, with respect to the operation of optical mesh networks and backbone networks, customers have a contractual warranty, referred to as Service Level Agreement (SLA), which guarantees a certain minimum fraction of uptime over a finite contract period (see [12], [13]). SLAs are used to define obligations between network providers and customers in business relationships. Another example comes from supply chain management, where suppliers often provide warranties on the level of availability over a time interval and commit to pay a penalty if the guaranteed availability is not reached [14]. Also, in the computer industry, vendors of commercial computer systems warrant availability levels over finite observation periods for a long time [15]. To deal with such situations, interval availability has been proposed as an index of system availability performance.

Interval availability is defined as the fraction of time that a system is operational during a period of time $[0, T]$ [16]. For a Markov repairable system, since the operational period is a random variable, interval availability is also a random variable. It is no longer just a probability index as pointwise availability, limiting availability, and average availability are. The probability distribution of interval availability is, then, of great interest for both system designers and end-users. Goyal *et al*. [17] summarize commonly used numerical methods to compute interval availability in computer applications. Van der Heijden [18] presents two-moment approximations for the on and off periods, and then, approximates the interval availability distribution accordingly. Goyal and Tantawi [15] present another approximation method to evaluate numerically the availability distribution. De Souza eSilva and Gail [19] present a numerical method to calculate the distribution of interval availability based on a randomization (or uniformization) technique. Such work is extended by Sericola [20] to derive a closed-form expression for the distribution of interval availability. Rubino and Sericola [21] present an algorithm for the efficient evaluation of interval availability. Smith [22] uses a beta distribution to approximate the first two moments and, then, the availability distribution function is estimated based on the approximation. Carrasco [23] considers a special case of Markov repairable systems with absorbing states and proposes an efficient algorithm to calculate its interval availability distribution. Kirmani and Hood [24] develop exact expressions for the first two

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moments of the on and off periods, which can be subsequently used to estimate the interval availability distribution. Carrasco [25] develops a new randomization-based general-purpose method for the computation of the interval availability distribution. Al Hanbali and Van der Heijden [14] use an approximation method similar in [22] to numerically calculate the interval availability distribution.

Most of the existing work relies on simulation-based methods for the computation of interval availability distribution. These simulation-based methods are adequate for practical applications on complex systems, but closed-form solutions of the interval availability distribution are still sought, both for practical applications and for validation of the Monte Carlo simulation methods. To the best of the authors' knowledge, [20] is the only work that deals with the analytical analysis of the interval availability distribution for Markov repairable systems. In their work, an analytical expression is derived based on uniformization and the result is given in terms of an infinite sum of terms, where each term contains a matrix whose dimension increases as the index of the term. This causes great difficulty in the practical computation of the closed-form expression. In this paper, we use a different method, the theory of aggregated stochastic processes (see [26], [27]), to derive the closed-form expression for the interval availability distribution of a Markov repairable system.

Aggregated stochastic processes have been introduced by Burke and Rosenblatt [26] and applied successfully by Colquhoun and Hawkes [27] to develop probabilistic models for ion channels. Since then, they have received increasing attention from various areas. Recently, aggregated stochastic processes have been used in reliability and maintainability modeling. For instance, Zheng *et al.* [28] study a single-unit Markov repairable system with repair time omission. Cui *et al.* [29] build a Markov repairable model with history-dependent up and down states. Zheng *et al.* [30] and Wang and Cui [31] extend the work of [29] into semi-Markov processes. Bao and Cui [32] extend the work of [28] into series and two-item cold standby Markov repairable systems with neglected or delayed failures, respectively. Hawkes *et al.* [33] consider a Markov repairable system under alternative environments. Wang *et al.* [34] somewhat extend the work of [33]. Cui *et al.* [35], [36] study this kind of problems focusing on availabilities and time distributions in different situations. Liu *et al.* [37] consider interval reliability for aggregated Markov repairable systems with repair time omission. Du *et al.* [38] present closed-form solutions of some reliability indexes and time distributions for repairable degradation systems based on aggregated stochastic processes.

In this paper, we apply the theory of aggregated stochastic processes to derive the closed-form expression for the interval availability distribution of Markov repairable systems. Interval availability only consider about the system behavior from the initial time 0 to a given time t. In practice, however, people are also interested in interval availability in a general interval $[a, b]$ and multiple intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m]$. The closed-formed expressions for the latter two cases are also derived in this paper.

The rest of the paper is organized as follows. In Section II, we formally define a mathematical model for interval availability analysis. The analytical expressions for the three interval availability indexes are given in Section III. In Section IV, a numerical example is given to illustrate the developed methods. In Section V, the derived analytical expression is applied to calculate the interval availability of a fault-tolerant database system. Finally, the conclusion is presented in Section VI.

II. Problem Statement

In this section, we formulate a mathematical model for interval availability analysis. For simplicity, we consider a Markov repairable system denoted by a Markov process $\{X(t), t \geq 0\}$ with finite state space $\boldsymbol{S} = \{1, 2, \ldots, n\}$, as in the most existing literatures (e.g., [15], [19], [20], [28], [29], [39]–[41]). It should be noted that the developed models work only for Markovian systems. In practice, the assumption of Markovian systems might not always hold. The readers should carefully check the validity of this assumption before applying this model for practical uses. We suppose that the system has only two kinds of states: up states and down states, i.e., $\boldsymbol{S} = \boldsymbol{W} + \boldsymbol{F} = \{1, 2, \ldots, n_0\} + \{n_0 + 1, \ldots, n\}$. The states $1, 2, \ldots, n_0$ stand for the system working or operational; the states $n_0 + 1, \ldots, n$ represent the system failed. In fact, it is very common to divide the states of a system into two disjoint subsets (e.g., [15], [19]). Such systems are commonly encountered in practice when they have multistate performances and are subject to degradation mechanisms, such as crack growth, fatigue, wear, corrosion, etc. [42]. A typical example is given in [43], where a water piping system comprising of three units is considered. The state space S presents different values of water transmission capacity, and a threshold capacity is defined based on user requirements which separate working states from failure states.

To consider the interval availability for the system, we define a new stochastic process $\{Y(t), t \geq 0\}$ as follows:

$$Y(t) = \begin{cases} 1, & \text{if } X(t) \in \boldsymbol{W}, \\ 0, & \text{if } X(t) \in \boldsymbol{F}. \end{cases} \tag{1}$$

When $Y(t) = 1$, the system is in the working state; when $Y(t) = 0$, the system is in the failure state.

Let us define a random variable

$$T_{W,t} = \int_0^t Y(u) du$$

which represents the periods in $[0, t]$ where the system is working. Interval availability analysis, can, then, be conducted based on the following three indexes.

1) Interval availability in $[0, t]$:

$$A\left([0, t], \alpha\right) = P\left\{\frac{T_{W,t}}{t} \geq \alpha\right\} \tag{2}$$

where $\alpha \in [0, 1]$ is the required availability. The physical meaning of $A([0, t], \alpha)$ is the probability that the system works for at least αt period of time in $[0, t]$.

2) Interval availability in a general interval $[a, b]$:

$$A\left([a, b], \alpha\right) = P\left\{\frac{\int_a^b Y(u) du}{b - a} \geq \alpha\right\}. \tag{3}$$

The physical meaning of $A([a,b],\alpha)$ is the probability that the system works for at least $\alpha(b-a)$ period of time in a general interval $[a,b]$.

3) Interval availability in multiple intervals $[a_1,b_1]$, $[a_2,b_2],\ldots,[a_m,b_m]$,$a_1 \le b_1 < \cdots < a_m \le b_m$:

$$A\left([a_1,b_1],\alpha_1,[a_2,b_2],\alpha_2,\ldots,[a_m,b_m],\alpha_m\right) = P\{A_1A_2\ldots A_m\} \tag{4}$$

where the events

$$A_i = \left\{\int_{a_i}^{b_i} Y(u)du \Big/ (b_i - a_i) \ge \alpha_i\right\}, i = 1,2,\ldots,m.$$

The physical meaning of (4) is the probability that different interval availability requirements for multiple intervals are satisfied simultaneously.

The new stochastic process $\{Y(t), t \ge 0\}$ is essentially an aggregated stochastic process based on $\{X(t), t \ge 0\}$. Hence, in this paper, we apply the theory of aggregated stochastic process [27] to derive closed-form expressions for (2)–(4), and then, use them for interval availability analysis of a fault-tolerant computer system from the literature.

III. Analytical Expressions for the Interval Availability Indexes

A. Interval Availability in Interval $[0,t]$

Suppose the Markov repairable system has an initial condition vector $\mathbf{\Phi}$ and the transition rate matrix $\boldsymbol{Q}$ can be partitioned into the following two parts:

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q_{WW}} & \boldsymbol{Q_{WF}} \\ \boldsymbol{Q_{FW}} & \boldsymbol{Q_{FF}} \end{pmatrix}. \tag{5}$$

Accordingly, the initial condition is also partitioned into the following two parts:

$$\mathbf{\Phi} = (\mathbf{\Phi_W}, \mathbf{\Phi_F}). \tag{6}$$

From (2), by conditioning on the initial state $Y(0)$, we have the following:

$$\begin{aligned} A([0,t],\alpha) &= P\{T_{W,t} \ge \alpha t\} \\ &= P\{T_{W,t} \ge \alpha t, Y(0) = 1\} + P\{T_{W,t} \ge \alpha t, Y(0) = 0\} \\ &= \mathbf{\Phi}_W \mathbf{1}_W P\{T_{W,t} \ge \alpha t | Y(0) = 1\} + \mathbf{\Phi}_F \mathbf{1}_F P\{T_{W,t} \\ &\ge \alpha t | Y(0) = 0\} \end{aligned} \tag{7}$$

where $\mathbf{1_W}$ and $\mathbf{1_F}$ are column vectors of all ones (the numbers of ones are $|\boldsymbol{W}|$ and $|\boldsymbol{F}|$, where $|\boldsymbol{W}|$ and $|\boldsymbol{F}|$ are the numbers of elements in sets $\boldsymbol{W}$, $\boldsymbol{F}$, respectively).

By further conditioning on the value of $Y(t)$ and the number of complete failure-repair cycles before t, denoted by $N(t)$, we can distinguish between the following four situations.

1) Situation 1: $Y(t) = 1, N(t) = 0, 1, \ldots, Y(0) = 1$, as shown in Fig. 1(a).
2) Situation 2: $Y(t) = 0, N(t) = 0, 1, \ldots, Y(0) = 1$, as shown in Fig. 1(b).
3) Situation 3: $Y(t) = 1, N(t) = 0, 1, \ldots, Y(0) = 0$, as shown in Fig. 1(c).
4) Situation 4: $Y(t) = 0, N(t) = 0, 1, \ldots, Y(0) = 0$, as shown in Fig. 1(d).

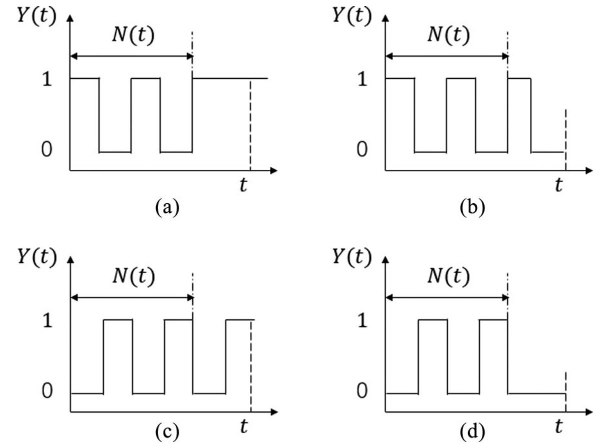

Fig. 1. Illustration of the four situations of interval availability. (a) System is working at time 0 and t, and undergoes $N(t)$ complete failure-repair cycles in $[0,t]$. (b) System works at time 0 and is failed at t. The system undergoes $N(t)$ complete failure-repair cycles in $[0,t]$. (c) System is failed at time 0 and works at t. The system undergoes $N(t)$ complete failure-repair cycles in $[0,t]$. (d) System is failed at time 0 and t. System undergoes $N(t)$ complete failure-repair cycles in $[0,t]$.

Hence, (7) becomes the following:

$$A([0,t],\alpha) = \mathbf{\Phi}_W \mathbf{1}_W \left(A_{S,1} + A_{S,2}\right) + \mathbf{\Phi}_F \mathbf{1}_F \left(A_{S,3} + A_{S,4}\right) \tag{8}$$

where $A_{S,1}$–$A_{S,4}$ corresponds to situations 1–4, respectively, and

$$A_{S,1} = \sum_{j=0}^{\infty} P\{T_{W,t} \ge \alpha t, N(t) = j, Y(t) = 1 | Y(0) = 1\}, \tag{9}$$

$$A_{S,2} = \sum_{j=0}^{\infty} P\{T_{W,t} \ge \alpha t, N(t) = j, Y(t) = 0 | Y(0) = 1\}, \tag{10}$$

$$A_{S,3} = \sum_{j=0}^{\infty} P\{T_{W,t} \ge \alpha t, N(t) = j, Y(t) = 1 | Y(0) = 0\}, \tag{11}$$

$$A_{S,4} = \sum_{j=0}^{\infty} P\{T_{W,t} \ge \alpha t, N(t) = j, Y(t) = 0 | Y(0) = 0\}. \tag{12}$$

Let us first consider situation 1. Let $T_j(t)$ and $D_j(t)$ be summations of the first jth working and repair durations within j complete failure-repair cycles by time t, respectively. For a given time t, let

$$\begin{aligned} f_{\mathbf{W}(i_1)\mathbf{W}(i_2)-(j,j)}(v,u) \equiv \lim_{\substack{dv\to 0\\ du \to 0}} & P\{T_j(t) \in (v, v+dv), D_j(t) \\ &\in (u, u+du), \\ X(T_j(t) + D_j(t)) &= i_2 | X(0) = i_1\}/dvdu, i_1, i_2 \in \boldsymbol{W}. \end{aligned} \tag{13}$$

Thus, $A_{S,1}$ becomes the following:

$$\begin{aligned} A_{S,1} &= \sum_{j=0}^{\infty} P\left\{\int_0^t Y(u)du \geq \alpha t, N(t)=j, Y(t)=1|Y(0)=1\right\} \\ &= \frac{\mathbf{\Phi}_W}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} e^{\boldsymbol{Q}_{\boldsymbol{WW}}t}\mathbf{1}_{\boldsymbol{W}} + \sum_{j=1}^{\infty} P\left\{\int_0^t Y(u)du \geq \alpha t, N(t)\right. \\ &\left. = j, Y(t)=1|Y(0)=1\right\} \\ &= \frac{\mathbf{\Phi}_{\boldsymbol{W}}}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} e^{\boldsymbol{Q}_{\boldsymbol{WW}}t}\mathbf{1}_{\boldsymbol{W}} + \sum_{j=1}^{\infty} P\{D_j(t) \geq (1-\alpha)t, T_j(t) \\ &\quad + D_j(t) < t \leq T_{j+1}(t) + D_j(t), Y(t)=1|Y(0)=1\} \\ &= \frac{\mathbf{\Phi}_{\boldsymbol{W}}}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} e^{\boldsymbol{Q}_{\boldsymbol{WW}}t}\mathbf{1}_{\boldsymbol{W}} + \frac{\mathbf{\Phi}_{\boldsymbol{W}}}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} \sum_{j=1}^{\infty} \int_0^{(1-\alpha)t} \left[\int_0^{t-v}\right. \\ &\quad \left. \times \boldsymbol{f}_{\boldsymbol{WW}-(j,j)}(u,v) e^{\mathbf{Q}_{\mathrm{WW}}(t-u-v)} du\right] dv \mathbf{1}_{\boldsymbol{W}}. \end{aligned} \tag{14}$$

Based on the results of [27], the Laplace transforms of $\boldsymbol{f}_{\boldsymbol{WW}-(j,j)}(u,v)$ is as follows:

$$\begin{aligned} \boldsymbol{f}^*_{\boldsymbol{WW}-(j,j)}(s_1,s_2) &= [\boldsymbol{G}^*_{\boldsymbol{WF}}(s_1)\boldsymbol{G}^*_{\boldsymbol{FW}}(s_2)]^j \\ &= \left(f^*_{\mathrm{W}(i_1)\mathrm{W}(i_2)-(j,j)}(s_1,s_2)\right)_{|\boldsymbol{W}|\times|\boldsymbol{W}|} \end{aligned} \tag{15}$$

where $f^*_{\boldsymbol{W}(i_1)\boldsymbol{W}(i_2)-(j,j)}(s_1, s_2) = \int_0^{\infty}\int_0^{\infty} e^{-s_1 v} e^{-s_2 u} f_{\boldsymbol{W}(i_1)\boldsymbol{W}(i_2)-(j,j)}(v,u)dvdu$, $\boldsymbol{G}^*_{\boldsymbol{WF}}(s_1)$ and $\boldsymbol{G}^*_{\boldsymbol{FW}}(s_2)$ are defined in Appendix 1 and calculated by (41) and (42), respectively. Then, $\boldsymbol{f}_{\boldsymbol{WW}-(j,j)}(u,v)$ can be calculated by taking the inverse Laplace transform of (15) as follows:

$$f_{\boldsymbol{W}(i_1)\boldsymbol{W}(i_2)-(j,j)}(v,u) = \frac{\ell^{-1}[f^*_{\mathrm{W}(i_1)\mathrm{W}(i_2)-(j,j)}(s_1,s_2)](v,u)}{f^*_{\boldsymbol{W}(i_1)\mathrm{W}(i_2)-(j,j)}(0,0)} \tag{16}$$

where the symbol ℓ^{-1} denotes the inverse Laplace transform and can be calculated using analytical or numerical methods. For details in calculating inverse Laplace transforms, readers might refer to [44]–[46]. In this paper, we use Mathematica 2010 for the calculation of inverse Laplace transforms.

Similarly, we have the following:

$$\begin{aligned} A_{S,2} &= \int_{\alpha t}^{t} e^{\boldsymbol{Q}_{\boldsymbol{WW}}u}\boldsymbol{Q}_{\boldsymbol{WF}} e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u)} du \mathbf{1}_{\boldsymbol{F}} + \frac{\mathbf{\Phi}_{\boldsymbol{W}}}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} \sum_{j=1}^{\infty} \\ &\quad \times \int_{\alpha t}^{t}\left[\int_0^{t-v} \boldsymbol{f}_{\boldsymbol{WF}-(j+1,j)}(u,v) e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u-v)} du\right] dv \mathbf{1}_{\boldsymbol{F}}, \end{aligned} \tag{17}$$

$$\begin{aligned} A_{S,3} &= \frac{\mathbf{\Phi}_{\boldsymbol{F}}}{\mathbf{\Phi}_{\boldsymbol{F}}\mathbf{1}_{\boldsymbol{F}}} \int_0^{(1-\alpha)t} e^{\boldsymbol{Q}_{\boldsymbol{FF}}u}\boldsymbol{Q}_{\boldsymbol{FW}} e^{\boldsymbol{Q}_{\boldsymbol{WW}}(t-u)} du \mathbf{1}_{\boldsymbol{W}} \\ &\quad + \frac{\mathbf{\Phi}_{\boldsymbol{F}}}{\mathbf{\Phi}_{\boldsymbol{F}}\mathbf{1}_{\boldsymbol{F}}} \sum_{j=1}^{\infty} \int_0^{(1-\alpha)t}\left[\int_0^{t-u} \boldsymbol{f}_{\boldsymbol{FW}-(j+1,j)}(u,v)\right. \\ &\quad \left. \times e^{\boldsymbol{Q}_{\boldsymbol{WW}}(t-u-v)} dv\right] du \mathbf{1}_{\boldsymbol{W}}, \end{aligned} \tag{18}$$

$$\begin{aligned} A_{S,4} &= \frac{\mathbf{\Phi}_{\boldsymbol{F}}}{\mathbf{\Phi}_{\boldsymbol{F}}\mathbf{1}_{\boldsymbol{F}}} \sum_{j=1}^{\infty} \int_{\alpha t}^{t}\left[\int_0^{t-v} \boldsymbol{f}_{\boldsymbol{FF}-(j,j)}(u,v)\right. \\ &\quad \left. \times e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u-v)} du\right] dv \mathbf{1}_{\boldsymbol{W}}. \end{aligned} \tag{19}$$

Note that we specify $\frac{\mathbf{\Phi}_{\boldsymbol{W}}}{\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}} = \mathbf{0}$ and $\frac{\mathbf{\Phi}_{\boldsymbol{F}}}{\mathbf{\Phi}_{\boldsymbol{F}}\mathbf{1}_{\boldsymbol{F}}} = \mathbf{0}$, if $\mathbf{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}} = 0$ and $\mathbf{\Phi}_{\boldsymbol{F}}\mathbf{1}_{\boldsymbol{F}} = 0$, respectively. Details of (17)–(19) are given in Appendix 2. Hence, $A([0,t],\alpha)$ can be analytically calculated by substituting (14), (17)–(19) into (8).

Note that in [20], an analytical expression for the interval availability of Markov repairable systems is given, but using a different method:

$$\begin{aligned} A([0,t],\alpha) &= \frac{1}{t} P\left\{\int_0^t Y(u)du \leq s\right\} = \frac{1}{t}\left(1 - \sum_{j=0}^{\infty}\right. \\ &\quad \left. \times e^{-\lambda t}\frac{(\lambda t)^j}{j!} \sum_{i=0}^{j} \binom{n}{i}\left(\frac{s}{t}\right)^i \left(1-\frac{s}{t}\right)^{j-i} \boldsymbol{\beta}(j)\boldsymbol{H}(j)\mathbf{1}\right) \end{aligned} \tag{20}$$

where $s = \alpha t$; λ is a positive real, such that $\lambda \geq \max(\lambda_i, i \in S)$; $\boldsymbol{\beta}(j)$ is a j dimensional vector; $H(j)$ is a $j \times j$ matrix and $\mathbf{1} = (1,\ldots,1)$ is a vector with proper dimensions. From the computational point of view, one of the major disadvantages of (20) is that the dimension of $\boldsymbol{\beta}(j)$ and $H(j)$ increases with j, which makes the practical application of (20) difficult, since the evaluation requires infinite summation over j. This problem can be avoided in our method, since the dimensions of all the vectors and matrices in (8) are fixed. One drawback of our approach, however, is that it requires the calculation of inverse Laplace transforms, which adds computational efforts in comparison to Sericola's method.

The computational complexity of (8) heavily depends on the number of terms needed for the truncation. To evaluate the jth term, we need to calculate the jth matrix powers in each of the four scenarios. Another significant influencing factor is the dimension of $\boldsymbol{Q}_{\boldsymbol{WW}}$, $\boldsymbol{Q}_{\boldsymbol{WF}}$, $\boldsymbol{Q}_{\boldsymbol{FW}}$, and $\boldsymbol{Q}_{\boldsymbol{FF}}$, since we need to calculate the matrix exponential with respect to these matrices.

B. Interval Availability in General Interval $[a,b]$

From (3), we have the following:

$$A([a,b],\alpha) = P\left\{\int_a^b Y(u)du \geq \alpha(b-a)\right\}. \tag{21}$$

Since the system is Markovian, by transforming the interval $[a,b]$ into $[0,t]$, we have the following:

$$A([a,b],\alpha) = A([0,b-a],\alpha)|_{\mathbf{\Phi}_0^{(a)}=\boldsymbol{P}(a)} \tag{22}$$

where $\boldsymbol{P}(a) = (P_1(a), P_2(a), \ldots, P_n(a))$ is the probability vector that the repairable system stays in each state at time a, and $\mathbf{\Phi}_0^{(a)} = \boldsymbol{P}(a)$ describes the initial conditions. It is well known that $\boldsymbol{P}(a) = \mathbf{\Phi}_0 e^{\boldsymbol{Q}a}$. Equation (22) can be calculated using the results in Section III-A.

C. Interval Availability in Multiple Intervals

From (4), we have the following:

$$A([a_1,b_1],\alpha_1;[a_2,b_2],\alpha_2;\ldots;[a_m,b_m],\alpha_m) = A\left(\bigcap_{i=1}^{m} \times \{[a_i,b_i],\alpha_i\}\right)$$
$$\equiv P\left\{\int_{a_1}^{b_1} Y(u)du \ge \alpha_1(b_1-a_1); \int_{a_2}^{b_2} Y(u)du \ge \alpha_2(b_2-a_2);\ldots;\int_{a_m}^{b_m} Y(u)du \ge \alpha_m(b_m-a_m)\right\}$$
$$= P\left\{\bigcap_{i=1}^{m}\int_{a_i}^{b_i} Y(u)du \ge \alpha_i(b_i-a_i)\right\}. \quad (23)$$

Let $A_i = \{\int_{a_i}^{b_i} Y(u)du \ge \alpha_i(b_i-a_i)\}, (i=1,2,\ldots,m)$ and $A_i^{(j)} = \{A_i \text{ and } Y(b_i)=j\}, (j=0,1)$. Then, we have $A_i = A_i^{(0)} + A_i^{(1)}$ and $A_i^{(0)} \cap A_i^{(1)} = \phi$, and furthermore, we can divide the events $\{Y(b_i)=1\}$ and $\{Y(b_i)=0\}$ as follows, respectively

$$\{Y(b_i)=1\} = \sum_{k=1}^{n_0}\{X(b_i)=k\} \equiv \sum_{k=1}^{n_0} B_i^{(k)}, \text{ and } \{Y(b_i)=0\} = \sum_{k=n_0+1}^{n}\{X(b_i)=k\} \equiv \sum_{k=n_0+1}^{n} B_i^{(k)}. \quad (24)$$

$$A\left(\bigcap_{i=1}^{m}\{[a_i,b_i],\alpha_i\}\right) = P\{A_1A_2\cdots A_m\} = P\left\{A_1A_2\cdots A_{m-2}A_{m-1}\left(\sum_{k=1}^{n}B_{m-1}^{(k)}\right)A_m\right\}$$
$$= \sum_{k=1}^{n} P\{A_1A_2\cdots A_{m-2}A_{m-1}B_{m-1}^{(k)}A_m\}$$
$$= \sum_{k=1}^{n} P\{A_m|A_1A_2\cdots A_{m-2}A_{m-1}B_{m-1}^{(k)}\}P\{A_1A_2\cdots A_{m-2}A_{m-1}B_{m-1}^{(k)}\}$$
$$= \sum_{k=1}^{n} P\{A_m|B_{m-1}^{(k)}\}P\{A_1A_2\cdots A_{m-2}A_{m-1}B_{m-1}^{(k)}\} \quad (25)$$

where the last equality results from the Markov property. Furthermore, we have the following:

$$P\{A_m|B_{m-1}^{(k)}\} = A([a_m,b_m],\alpha_m)|_{\mathbf{\Phi}_0^{(m)}(k)} \quad (26)$$

where the initial condition

$$\mathbf{\Phi}_0^{(m)}(k) = (\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0)\exp(\boldsymbol{Q}(a_m-b_{m-1}))$$

i.e., the vector $\mathbf{\Phi}_0^{(m)}(k)$ has all zero elements except for the kth column. We can continue the similar way for obtaining $P\{A_1A_2\ldots A_{m-2}A_{m-1}B_{m-1}^{(k)}\}$, i.e.

$$P\{A_1A_2\ldots A_{m-2}A_{m-1}B_{m-1}^{(k)}\}$$
$$= \sum_{j=1}^{n} P\{A_1A_2\ldots A_{m-2}B_{m-2}^{(j)}A_{m-1}B_{m-1}^{(k)}\}$$
$$= \sum_{j=1}^{n} P\{A_{m-1}B_{m-1}^{(k)}|A_1A_2\ldots A_{m-2}B_{m-2}^{(j)}\} \times P\{A_1A_2\ldots A_{m-2}B_{m-2}^{(j)}\}$$
$$= \sum_{j=1}^{n} P\{A_{m-1}B_{m-1}^{(k)}|B_{m-2}^{(j)}\}P\{A_1A_2\cdots A_{m-2}B_{m-2}^{(j)}\}. \quad (27)$$

On the other hand

$$P\{A_{m-1}B_{m-1}^{(k)}|B_{m-2}^{(j)}\}$$
$$= P\left\{\int_{a_{m-1}}^{b_{m-1}} Y(u)du \ge \alpha_{m-1}(b_{m-1}-a_{m-1}), X(b_{m-1}) = k|X(b_{m-2})=j\right\}$$
$$= P\left\{\int_{a_{m-1}-b_{m-2}}^{b_{m-1}-b_{m-2}} Y(u)du \ge \alpha_{m-1}(b_{m-1}-a_{m-1}), X(b_{m-1}-b_{m-2}) = k|X(0)=j\right\}$$
$$= A([a_{m-1}-b_{m-2},b_{m-1}-b_{m-2}],\alpha_{m-1})|_{\mathbf{\Phi}_0^{(m-1)}(j),\mathbf{\Phi}_1^{(m-1)}(k)} \quad (28)$$

where $\mathbf{\Phi}_0^{(m-1)}(j)$ and $\mathbf{\Phi}_1^{(m-1)}(k)$ denote the initial and ending conditions,

$$\mathbf{\Phi}_0^{(m-1)}(i) = (\underbrace{0,\ldots,0}_{i-1},1,0,\ldots,0),$$
$$\mathbf{\Phi}_1^{(m-1)}(k) = (\underbrace{0,\ldots,0}_{k-1},1,0,\ldots,0)^T,$$

and T denotes the transpose operator. Hence, (23) can be calculated in a recursive way.

IV. Numerical Example

In this section, a numerical example is presented. We consider a Markov repairable system, whose state space is $\mathbf{S} = \mathbf{W} + \mathbf{F} = \{1,2\} + \{3\}$. The transition rate matrix of the system is as follows:

$$\boldsymbol{Q} = \begin{pmatrix} \boldsymbol{Q_{WW}} & \boldsymbol{Q_{WF}} \\ \boldsymbol{Q_{FW}} & \boldsymbol{Q_{FF}} \end{pmatrix} = \left(\begin{array}{cc|c} -1 & 0 & 1 \\ 1 & -2 & 1 \\ \hline 0 & 1 & -1 \end{array}\right) \quad (29)$$

and the initial probability vector is $\mathbf{\Phi}_0 = (\mathbf{\Phi_W},\mathbf{\Phi_F}) = (\frac{1}{2},0,\frac{1}{2})$.

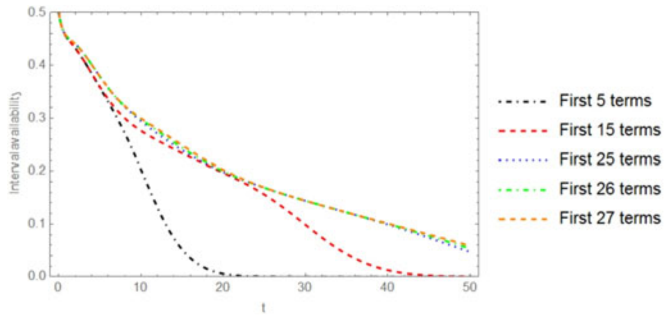

Fig. 2. Interval availability $A([0,50],0.6)$ for different terms.

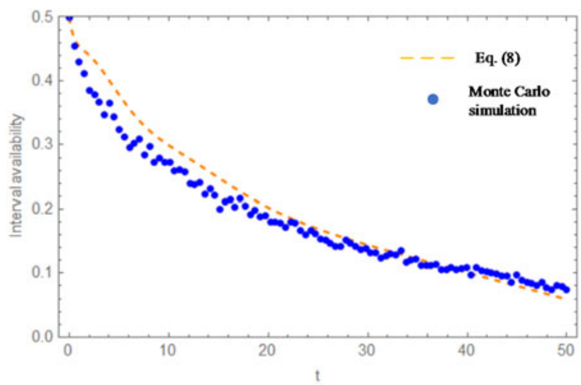

Fig. 3. Comparison between closed-form result and Monte Carlo simulation.

A. *Interval Availability in Interval* $[0,t]$

We first calculate the interval availability $A([0,t],\alpha)$ for $\alpha=0.6$ using (8). Although the evaluation of (8) involves the summation of infinite terms [see (14), (17)–(19), for details], the terms converge to 0 as j approaches infinity. Therefore, the summation can be truncated at a proper term based on the required accuracy. In Fig. 2, we compute $A([0,50],0.6)$ for the first 5, 15, 25, 26, and 27 terms, respectively. It can be seen from Fig. 2 that $A([0,50],0.6)$ approaches convergence as $j\geq 27$. Therefore, (8) is truncated at the 27th terms. Note that in this paper, we use Mathematica 2010 to calculate the inverse Laplace transforms in (8).

A Monte Carlo simulation is conducted to validate the result, as shown in Fig. 3, where $[0,t]$ is equally divided into 100 subintervals, and for each subinterval, 10^4 samples are used to calculate the interval availability.

B. *Interval Availability in General Interval* $[a,b]$

Next, we consider the interval availability in the general interval for $[a,b]$, using (22). If the values of a and b are fixed, the interval availability on such interval is a fixed value. Here, we just let $a=0.5$, whereas b can be changed. Then, the interval availability of the interval $[0.5,t]$ with $\alpha=0.6$, denoted as $A([0.5,t],0.6)$, can be obtained based on (22).

Fig. 4 shows the curve of $A([0.5,t],0.6)$ evaluated using different terms. From Fig. 4, we can see that when t is small, the interval availability can be truncated at smaller terms. For example, $A([0.5,6],0.6)$ can be approximated well using the first five terms. However, for larger t, more terms are needed. For example, $A([0.5,50],0.6)$ needs to be truncated using the first 18 terms. This can be explained by recalling the physical meaning of the terms: the number of the needed terms represents the complete failure-repair cycle in $[0.5,t]$. As t gets larger, more cycles are expected. Therefore, more terms are needed to truncate the analytical expression. Thus, our approach is effective when the time is not very large. When the time is large, we should use more terms for truncation, or resort to simulation-based methods.

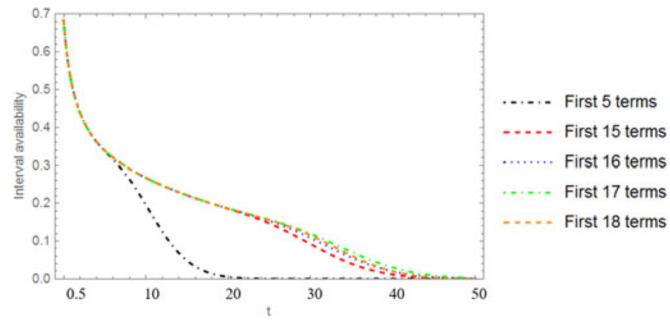

Fig. 4. Interval availability $A([0.5,t],0.6)$ curves for different terms.

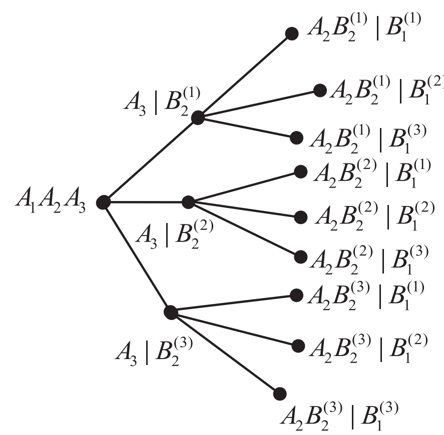

Fig. 5. Binary tree diagram for event $A_1A_2A_3$.

C. *Interval Availability in Multiple Intervals*

We use (25) to calculate interval availability in multiple intervals. As an example, we consider three intervals: $A([4,8],0.4;[10,15],0.6;[17,19],0.3)$. In fact, we can use a multiple tree diagram to get the $P\{A_1A_2\ldots A_m\}$ in (25), through breaking the event $A_j(j\geq 2)$ into n subevents $B_j^{(k)},(k=1,2,\ldots,n)$. The binary tree diagram for event $A_1A_2A_3$ is as depicted in Fig. 5. Hence, we have the following:

$$\begin{aligned}P\{A_1A_2A_3\} &= P\{A_1A_2B_2^{(1)}A_3\}+P\{A_1A_2B_2^{(2)}A_3\}\\&\quad+P\{A_1A_2B_2^{(3)}A_3\}\\&=P\{A_3|A_1A_2B_2^{(1)}\}P\{A_1A_2B_2^{(1)}\}+P\{A_3|A_1A_2B_2^{(2)}\}\\&\quad\times P\{A_1A_2B_2^{(2)}\}+P\{A_3|A_1A_2B_2^{(3)}\}P\{A_1A_2B_2^{(3)}\}\\&=P\{A_3|B_2^{(1)}\}P\{A_1A_2B_2^{(1)}\}+P\{A_3|B_2^{(2)}\}P\{A_1A_2B_2^{(2)}\}\\&\quad+P\{A_3|B_2^{(3)}\}P\{A_1A_2B_2^{(3)}\}.\end{aligned}\tag{30}$$

TABLE I
TERMS NEEDED FOR GETTING $P\{A_1 A_2 A_3\}$

Level	Terms needed
A_1-level	$P(A_1 B_1^{(1)}), P(A_1 B_1^{(2)}), P(A_1 B_1^{(3)})$
$A_1 - A_2$-level	$P\{A_2 B_2^{(1)} \vert B_1^{(1)}\}, P\{A_2 B_2^{(1)} \vert B_1^{(2)}\}, P\{A_2 B_2^{(1)} \vert B_1^{(3)}\},$ $P\{A_2 B_2^{(2)} \vert B_1^{(1)}\}, P\{A_2 B_2^{(2)} \vert B_1^{(2)}\}, P\{A_2 B_2^{(2)} \vert B_1^{(3)}\}$
$A_2 - A_3$-level	$P\{A_3 \vert B_2^{(1)}\}, P\{A_3 \vert B_2^{(2)}\}, P\{A_3 \vert B_2^{(3)}\}$

On the other hand

$$P\{A_1 A_2 B_2^{(1)}\} = P\{A_1 B_1^{(1)} A_2 B_2^{(1)}\} + P\{A_1 B_1^{(2)} A_2 B_2^{(1)}\}$$
$$+ P\{A_1 B_1^{(3)} A_2 B_2^{(1)}\}$$
$$= P\{A_2 B_2^{(1)} | B_1^{(1)}\} P\{A_1 B_1^{(1)}\} + P\{A_2 B_2^{(1)} | B_1^{(2)}\}$$
$$\times P\{A_1 B_1^{(2)}\} + P\{A_2 B_2^{(1)} | B_1^{(3)}\} P\{A_1 B_1^{(3)}\}. \quad (31)$$

Similarly, we have the following:

$$P\{A_1 A_2 B_2^{(2)}\} = P\{A_2 B_2^{(2)} | B_1^{(1)}\} P\{A_1 B_1^{(1)}\}$$
$$+ P\{A_2 B_2^{(2)} | B_1^{(2)}\} P\{A_1 B_1^{(2)}\}$$
$$+ P\{A_2 B_2^{(2)} | B_1^{(3)}\} P\{A_1 B_1^{(3)}\}. \quad (32)$$
$$P\{A_1 A_2 B_2^{(3)}\} = P\{A_2 B_2^{(3)} | B_1^{(1)}\} P\{A_1 B_1^{(1)}\}$$
$$+ P\{A_2 B_2^{(3)} | B_1^{(2)}\} P\{A_1 B_1^{(2)}\}$$
$$+ P\{A_2 B_2^{(3)} | B_1^{(3)}\} P\{A_1 B_1^{(3)}\}. \quad (33)$$

The decomposition process above can be depicted in Fig. 5.

We summarize the procedures and the terms needed for obtaining $P\{A_1 A_2 A_3\}$ in Table I.

The terms listed in Table I can be further expressed as follows:

$$P\{A_1 B_1^{(i)}\} = A([a_i, b_i], \alpha_i)|_{\mathbf{1}_W = (I_{\{i=1\}}, I_{\{i=2\}}, I_{\{i=3\}})},$$
$$\times i = 1, 2, 3. \quad (34)$$
$$P\{A_2 B_2^{(i)} | B_1^{(j)}\} = A([a_2, b_2], \alpha_2)$$
$$\times |_{\boldsymbol{\Phi}_0^{(a_2)} = (I_{\{j=1\}}, I_{\{j=2\}}, I_{\{i=3\}}) e^{\mathbf{Q}(a_2 - b_1)} \,\&\, \mathbf{1}_{\boldsymbol{W}} = (I_{\{i=1\}}, I_{\{i=2\}}, I_{\{i=3\}})},$$
$$\times i, j \in \{1, 2, 3\}. \quad (35)$$
$$P\{A_3 | B_2^{(i)}\} = A([a_3, b_3], \alpha_3)$$
$$\times |_{\boldsymbol{\Phi}_{(0)}^{(a_3)} = (I_{\{i=1\}}, I_{\{i=2\}}, I_{\{i=3\}}) e^{\mathbf{Q}(a_3 - b_2)}}, i = 1, 2, 3. \quad (36)$$

In (34)–(36), the interval availabilities can be calculated based on (8) and (22).

In this example, by substituting (29) into (30), we have the following:

$$A([4, 8], 0.4; [10, 15], 0.6; [17, 19], 0.3) = 0.1520.$$

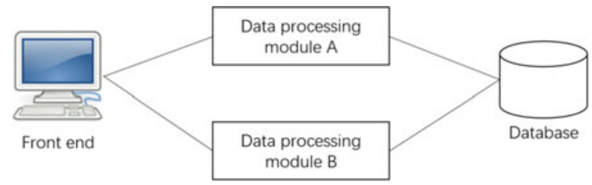

Fig. 6. Fault-tolerant database system.

TABLE II
FAILURE AND REPAIR RATES OF THE COMPONENTS OF THE FAULT-TOLERANT DATABASE SYSTEM

Components	Failure rate (h^{-1})	Repair rate (h^{-1})
Front end	$1/2400$	1
Data processing module A	$1/120$	1
Data processing module B	$1/120$	1
Database	$1/2400$	1

TABLE III
DEFINITIONS OF THE STATES

Index	Component states		
	Front end	DPMs	Database
1	Working	Both are working	Working
2	Working	Only one is working	Working
3	Working	Both failed	Working
4	Failed	Both are working	Working
5	Failed	Only one is working	Working
6	Failed	Both failed	Working
7	Working	Both are working	Failed
8	Working	Only one is working	Failed
9	Working	Both failed	Failed
10	Failed	Both are working	Failed
11	Failed	Only one is working	Failed
12	Failed	Both failed	Failed

V. APPLICATION

In this section, we apply the developed analytical expression to calculate the interval availability for a fault-tolerant database system from [15]. As shown in Fig. 6, the fault-tolerant database system comprises of a front end, two identical data processing modules, and a database. To remain operational, the front end, the database, and at least one of the data processing modules need to be in working state. From [15], the failure behaviors of all the components follow Markov processes, whose failure and repair rates are presented in Table II. Although assumptions of Markov property are idealized in practice, it has been widely applied in computer applications for describing the failure behaviors of the components. In this way, the issues needed to study are simplified.

By assuming that only one component can fail or be repaired in $(t, t + \Delta t)$, $\Delta t \to 0$, the system can be modeled as a Markov process of 12 states. The definitions of the states are given in Table III. The $\boldsymbol{Q}$-matrix of this Markov process is shown in Fig. 7.

From Table III, it can be seen that the database system is working only when it is in state 1 or state 2. Accordingly, we

$$
Q = \begin{array}{cc}
 & \begin{array}{cc} Q_{WW} & Q_{WF} \end{array} \\
 & \left[\begin{array}{cc|cccccccccc}
-\frac{7}{400} & \frac{1}{60} & 0 & \frac{1}{2400} & 0 & 0 & \frac{1}{2400} & 0 & 0 & 0 & 0 & 0 \\
1 & -\frac{1211}{1200} & \frac{1}{120} & 0 & \frac{1}{2400} & 0 & 0 & \frac{1}{2400} & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & -\frac{1201}{1200} & 0 & 0 & \frac{1}{2400} & 0 & 0 & \frac{1}{2400} & 0 & 0 & 0 \\
1 & 0 & 0 & -\frac{774}{761} & \frac{1}{60} & 0 & 0 & 0 & 0 & \frac{1}{2400} & 0 & 0 \\
0 & 1 & 0 & 1 & -\frac{1607}{800} & \frac{1}{120} & 0 & 0 & 0 & 0 & \frac{1}{2400} & 0 \\
0 & 0 & 1 & 0 & 1 & -\frac{4801}{2400} & 0 & 0 & 0 & 0 & 0 & \frac{1}{2400} \\
1 & 0 & 0 & 0 & 0 & 0 & -\frac{774}{761} & \frac{1}{60} & 0 & \frac{1}{2400} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -\frac{1607}{800} & \frac{1}{120} & 0 & \frac{1}{2400} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -\frac{4801}{2400} & 0 & 0 & \frac{1}{2400} \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -\frac{121}{60} & \frac{1}{60} & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -\frac{361}{120} & \frac{1}{120} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & -3
\end{array}\right] \\
 & \begin{array}{cc} Q_{FW} & Q_{FF} \end{array}
\end{array}
$$

Fig. 7. $\boldsymbol{Q}$-matrix of the Markov model for the fault-tolerant database system.

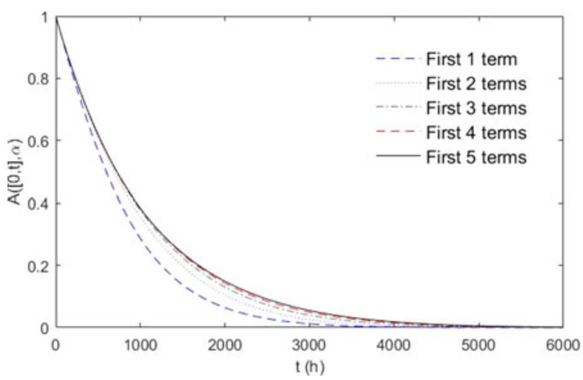

Fig. 8. Interval availability of the fault-tolerant database system.

can define the four partition matrices $\boldsymbol{Q_{WW}}$, $\boldsymbol{Q_{WF}}$, $\boldsymbol{Q_{FW}}$, and $\boldsymbol{Q_{FF}}$, as shown in Fig. 7.

Suppose that at $t = 0$, all the components are in working states, i.e., the system is in state 1. We use the numerical expression in (8) to calculate the interval availability of the system. Note that since the initial system state is state 1, we only need to consider the first two terms in (8):

$$A([0,t],\alpha) = A_{S,1} + A_{S,2} \tag{37}$$

where $A_{S,1}$ and $A_{S,2}$ are calculated by (9) and (10), respectively.

Fig. 8 shows the interval availability in $[0, 6000]$ (h). The required availability is $\alpha = 0.99999$. It can be seen from Fig. 8 that $A([0,t],\alpha)$ converges as we calculate the first five terms. This is because the physical meaning of the interval availability is the number of complete failure-repair cycles prior to t. Since the failure rates of the fault-tolerant database system are not very high, few complete failure-repair cycles are expected before t. Therefore, (37) is truncated at the first five terms to approximate the interval availability.

A Monte Carlo simulation is conducted to validate the result, as shown in Fig. 9, where $[0,t]$ is equally divided into 50 subintervals, and for each subinterval, 10^6 samples are used to

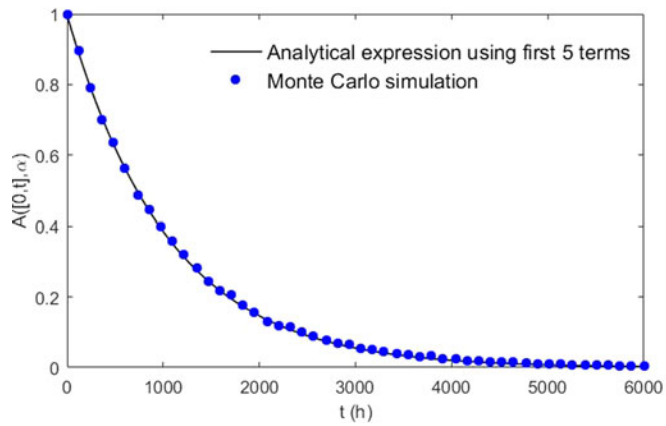

Fig. 9. Comparison to Monte Carlo simulation.

calculate the interval availability. It can be seen that in general the developed analytical expression compares well with the simulation results.

From Figs. 8 and 9, we can see that when t is close to zero, $A([0,t],\alpha)$ is close to one, which indicates that we have high confidence that the fault-tolerant database system can meet the availability requirement, i.e., $A([0,t]) \geq \alpha$. This is because the system is new at $t = 0$. Therefore, it can exhibit high availability performance. As t becomes larger, the interval availability decreases to zero. This is because, in this case, the availability requirement is higher than the steady-state availability, $A_{\mathrm{steady}} = 0.991$. Since in the long run, the availability will converge to A_{steady}, the probability of fulfilling a requirement higher than A_{steady} is, therefore, decreasing to zero.

VI. Conclusion

In this paper, aggregated stochastic processes are applied to derive a new analytical expression for interval availability. Related interval availability indexes, specially the interval availability in a general interval and the interval availability in multiple intervals, are also computed in closed form. A numerical example is conducted to illustrate the derived analytical expressions and the results are validated through Monte Carlo simulations. A comparison is made to the existing method in [20]. Unlike the work in [20], which is based on the technique of uniformization, our work is based on aggregated stochastic processes and Laplace transforms. From the perspective of computations, our method does not suffer from the dimension expansion problem and therefore, has better efficiency, especially for smaller time t or smaller interval length $b - a$ or smaller $\max_i\{b_i - a_i, i = 1, 2, \ldots, m\}$. The computational complexity of the developed analytical expression heavily depends on the number of terms needed for the truncation and the size of the Q-matrix. For large-scale problems, the efficiency of the developed methods might not be satisfactory and, therefore, simulation-based methods might be preferred. In the future, more efficient computational methods need to be further investigated, to ease the application of our method in practical problems.

APPENDIX 1

Preliminary Knowledge

In this section, some basic knowledge on aggregated stochastic processes is presented, limited to that used for the analysis of interval availability in this paper. Let us define two matrices $\boldsymbol{P}_{WW}(t)$ and $\boldsymbol{G}_{WF}(t)$ to be as follows:

$$\boldsymbol{P}_{\boldsymbol{WW}}(t) = \left({}^{\boldsymbol{W}}p_{ij}(t)\right)_{|\mathbf{W}|\times|\mathbf{W}|}, \boldsymbol{G}_{\boldsymbol{WF}}(t) = (g_{ij}(t))_{|\boldsymbol{W}|\times|\boldsymbol{F}|}$$

where $|\boldsymbol{W}|$ is the number of elements in set $\boldsymbol{W}$,$|\boldsymbol{F}|$ is the number of elements in set $\boldsymbol{F}$, and ${}^{\mathbf{W}}p_{ij}(t)$, $g_{ij}(t)$ are given by

$${}^{\boldsymbol{W}}p_{ij}(t) \equiv P\{\text{system remains within } \boldsymbol{W} \text{ from time 0 to time } t,$$
$$\text{and is in state } j \text{ at time } t| \text{ in state } i \text{ at time 0}\},$$
$$= P\{X(u) \in \boldsymbol{W},\ \text{for any } u \in [0,t], X(t) = j|X(0) = i\}, i, j \in \boldsymbol{W},$$
$$g_{ij}(t) \equiv \lim_{\Delta t\to 0} P\{\text{stay in } \boldsymbol{W} \text{ from time 0 to time } t, \text{and}$$
$$\text{leave } \boldsymbol{W}\ \mathrm{forstate}\ j \text{ between } t \text{ and } t+\Delta t|\text{in state } i \text{ at time 0 }\}$$
$$/\Delta t, i \in \boldsymbol{W}, j \in \boldsymbol{F}.$$

According to Colquhoun and Hawkes [27], the expressions of $\boldsymbol{P}_{\boldsymbol{WW}}(t)$, $\boldsymbol{G}_{\boldsymbol{WF}}(t)$ are given as follows:

$$\boldsymbol{P}_{\boldsymbol{WW}}(t) = \exp(\boldsymbol{Q}_{\boldsymbol{WW}}t), \tag{38}$$

$$\boldsymbol{G}_{\boldsymbol{WF}}(t) = \boldsymbol{P}_{\boldsymbol{WW}}(t)\boldsymbol{Q}_{\boldsymbol{WF}}. \tag{39}$$

For detailed derivation of (38) and (39), readers might refer to [27, Section 1(c)]. Note that in their original derivation, $\boldsymbol{A}$ and $\boldsymbol{B}$ are used instead of $\boldsymbol{W}$ and $\boldsymbol{F}$, respectively.

Laplace transforms of $\boldsymbol{P}_{\boldsymbol{WW}}(t)$ and $\boldsymbol{G}_{\boldsymbol{WF}}(t)$ can be calculated as [27] follows:

$$\boldsymbol{P}^*_{\boldsymbol{WW}}(s) = (s\boldsymbol{I} - \boldsymbol{Q}_{\boldsymbol{WW}})^{-1}, \tag{40}$$

$$\boldsymbol{G}^*_{\boldsymbol{WF}}(s) = (s\boldsymbol{I} - \boldsymbol{Q}_{\boldsymbol{WW}})^{-1}\boldsymbol{Q}_{\boldsymbol{WF}}. \tag{41}$$

Similarly, we have the following:

$$\boldsymbol{G}^*_{\boldsymbol{FW}}(s) = (s\boldsymbol{I} - \boldsymbol{Q}_{\boldsymbol{FF}})^{-1}\boldsymbol{Q}_{\boldsymbol{FW}} \tag{42}$$

where $\boldsymbol{I}$ is an identity matrix.

Taking an integral for $g_{ij}(t)$, we obtain the following:

$$\int_0^\infty g_{ij}(t)dt = P\{\text{exits to } j|\text{starts in } i\} = g^*_{ij}(0),\ i \in \boldsymbol{W},\ j \in \boldsymbol{F} \tag{43}$$

where $g^*_{ij}(0)$ are the elements of matrix $\boldsymbol{G}^*_{\boldsymbol{WF}}(0)$.

Note that $g_{ij}(t)$ is not, itself, a proper probability density function; the real probability density function is as follows:

$$\frac{g_{ij}(t)}{\int_0^\infty g_{ij}(t)dt} = \frac{g_{ij}(t)}{g^*_{ij}(0)}. \tag{44}$$

Throughout this paper, for brevity, we denote $\boldsymbol{G}^*_{\boldsymbol{WF}}(0)$ simply as $\boldsymbol{G}_{\boldsymbol{WF}}$, thus, it follows from (41)

$$\boldsymbol{G}_{\boldsymbol{WF}} \equiv \boldsymbol{G}^*_{\boldsymbol{WF}}(0) = -\boldsymbol{Q}^{-1}_{\boldsymbol{WW}}\boldsymbol{Q}_{\boldsymbol{WF}}. \tag{45}$$

APPENDIX 2
DERIVATIONS OF (17)–(19)

Derivation of (17)*:*
When $j=0$, we have the following:

$$P\left\{\int_0^t Y(u)du \geq \alpha t, N(t)=0, Y(t)=0|Y(0)=1\right\}$$
$$= \frac{\boldsymbol{\Phi}_{\boldsymbol{W}}}{\boldsymbol{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}}\int_{\alpha t}^t e^{\boldsymbol{Q}_{\boldsymbol{WW}}u}\boldsymbol{Q}_{\boldsymbol{WF}}e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u)}du\mathbf{1}_{\boldsymbol{F}}. \tag{46}$$

Similarly, when $j \geq 1$, we have the following:

$$\sum_{j=1}^\infty P\left\{\int_0^t Y(u)du \geq \alpha t, N(t)=j, Y(t)=0|Y(0)=1\right\}$$
$$= \sum_{j=1}^\infty P\{T_{j+1}(t) \geq \alpha t, T_{j+1}(t)+D_j(t) < t \leq T_{j+1}(t)$$
$$+ D_{j+1}(t), Y(t)=0|Y(0)=1\} = \frac{\boldsymbol{\Phi}_{\boldsymbol{W}}}{\boldsymbol{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}}\sum_{j=1}^\infty$$
$$\times \int_{\alpha t}^t \left[\int_0^{t-v} \boldsymbol{f}_{\boldsymbol{WF}-(j+1,j)}(u,v)e^{\mathbf{Q}_{\mathbf{FF}}(t-u-v)}du\right]dv\mathbf{1}_{\boldsymbol{F}} \tag{47}$$

where

$$\boldsymbol{f}^*_{\boldsymbol{WF}-(j+1,j)}(s_1,s_2) = [\boldsymbol{G}^*_{\boldsymbol{WF}}(s_1)\boldsymbol{G}^*_{\boldsymbol{FW}}(s_2)]^j\boldsymbol{G}^*_{\boldsymbol{WF}}(s_1)$$
$$= \left(f^*_{\boldsymbol{W}(i_1)\boldsymbol{F}(i_2)-(j,j)}(s_1,s_2)\right)_{|\boldsymbol{W}|\times|\boldsymbol{F}|},$$
$$[2pt] f^*_{\boldsymbol{W}(i_1)\boldsymbol{F}(i_2)-(j+1,j)}(v,u)$$
$$= \frac{\ell^{-1}\{f^*_{\boldsymbol{W}(i_1)\boldsymbol{F}(i_2)-(j+1,j)}(s_1,s_2)\}(v,u)}{f^*_{\boldsymbol{W}(i_1)\boldsymbol{F}(i_2)-(j+1,j)}(0,0)},$$
$$[2pt] \boldsymbol{f}_{\boldsymbol{WF}-(j+1,j)}(v,u) = \left(f_{\boldsymbol{W}(i_1)\boldsymbol{F}(i_2)-(j+1,j)}(v,u)\right)_{|\boldsymbol{W}|\times|\boldsymbol{F}|}.$$

$\boldsymbol{G}^*_{\boldsymbol{WF}}(s_1)$ and $\boldsymbol{G}^*_{\boldsymbol{FW}}(s_2)$ are defined in Appendix 1 and calculated by (41) and (42), respectively.

Thus, we have the following:

$$\sum_{j=0}^\infty P\left\{\int_0^t Y(u)du \geq \alpha t, N(t)=j, Y(t)=0|Y(0)=1\right\}$$
$$= \frac{\boldsymbol{\Phi}_{\boldsymbol{W}}}{\boldsymbol{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}}\int_{\alpha t}^t e^{\boldsymbol{Q}_{\boldsymbol{WW}}u}\boldsymbol{Q}_{\boldsymbol{WF}}e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u)}du\mathbf{1}_{\boldsymbol{F}} + \frac{\boldsymbol{\Phi}_{\boldsymbol{W}}}{\boldsymbol{\Phi}_{\boldsymbol{W}}\mathbf{1}_{\boldsymbol{W}}}$$
$$\times \sum_{j=1}^\infty \int_{\alpha t}^t \left[\int_0^{t-v} \boldsymbol{f}_{\boldsymbol{WF}-(j+1,j)}(u,v)e^{\boldsymbol{Q}_{\boldsymbol{FF}}(t-u-v)}du\right]dv\mathbf{1}_{\boldsymbol{F}}. \tag{48}$$

Derivation of (18):
Similarly, when $j = 0$, we have the following:

$$P\left\{ \int_0^t Y(u)du \geq \alpha t, N(t) = 0, Y(t) = 1 | Y(0) = 0 \right\} = \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \int_0^{(1-\alpha)t} e^{\boldsymbol{Q_{FF}}u} \boldsymbol{Q_{FW}} e^{\boldsymbol{Q_{WW}}(t-u)} du \boldsymbol{1_W}. \quad (49)$$

When $j > 0$, we have the following:

$$\sum_{j=1}^{\infty} P\left\{ \int_0^t Y(u)du \geq \alpha t, N(t) = j, Y(t) = 1 | Y(0) = 0 \right\}$$
$$= \sum_{j=1}^{\infty} P\Big\{ D_{j+1}(t) \leq (1-\alpha)t, D_{j+1}(t) + T_j(t) < t \leq D_{j+1}(t) + T_{j+1}(t), \, Y(t) = 1 | Y(0) = 0 \Big\}$$
$$= \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \sum_{j=1}^{\infty} \int_0^{(1-\alpha)t} \left[\int_0^{t-u} \boldsymbol{f_{FW-(j+1,j)}}(u,v) \times e^{\boldsymbol{Q_{WW}}(t-u-v)} dv \right] du \boldsymbol{1_W}. \quad (50)$$

Thus, we have the following:

$$\sum_{j=0}^{\infty} P\left\{ \int_0^t Y(u)du \geq \alpha t, M(t) = j, Y(t) = 1 | Y(0) = 0 \right\}$$
$$= \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \int_0^{(1-\alpha)t} e^{\boldsymbol{Q_{FF}}u} \boldsymbol{Q_{FW}} e^{\boldsymbol{Q_{WW}}(t-u)} du \boldsymbol{1_W} + \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \times \sum_{j=1}^{\infty} \int_0^{(1-\alpha)t} \left[\int_0^{t-u} \boldsymbol{f_{FW-(j+1,j)}}(u,v) e^{\boldsymbol{Q_{WW}}(t-u-v)} dv \right] du \boldsymbol{1_W} \quad (51)$$

where

$$\boldsymbol{f^*_{FW-(j+1,j)}}(s_1, s_2) = [\boldsymbol{G^*_{FW}}(s_1) \boldsymbol{G^*_{WF}}(s_2)]^j \boldsymbol{G^*_{FW}}(s_1) = \left(f^*_{\mathbf{F}(i_1)\mathbf{W}(i_2)-(j+1,j)}(s_1, s_2) \right)_{|\boldsymbol{F}| \times |\boldsymbol{W}|},$$

$$f_{\boldsymbol{F}(i_1)\boldsymbol{W}(i_2)-(j+1,j)}(u,v) = \frac{\ell^{-1}\{f^*_{\boldsymbol{F}(i_1)\boldsymbol{W}(i_2)-(j+1,j)}(s_1, s_2)\}(u,v)}{f^*_{\boldsymbol{F}(i_1)\boldsymbol{W}(i_2)-(j+1,j)}(0,0)},$$

$$\boldsymbol{f_{FW-(j+1,j)}}(u,v) = \left(f_{\boldsymbol{F}(i_1)\boldsymbol{W}(i_2)-(j+1,j)}(u,v) \right)_{|\boldsymbol{F}| \times |\boldsymbol{W}|}.$$

$\boldsymbol{G^*_{WF}}(s_1)$ and $\boldsymbol{G^*_{FW}}(s_2)$ are defined in Appendix 1 and calculated by (41) and (42), respectively.

Derivation of (19):
Similarly, when $j = 0$, we have the following:

$$P\left\{ \int_0^t Y(u)du \geq \alpha t, M(t) = 0, Y(t) = 0 | Y(0) = 0 \right\} = 0. \quad (52)$$

When $j > 0$, we have the following:

$$\sum_{j=1}^{\infty} P\left\{ \int_0^t Y(u)du \geq \alpha t, N(t) = j, Y(t) = 0 | Y(0) = 0 \right\}$$
$$= \sum_{j=1}^{\infty} P\{ T_j(t) \geq \alpha t, D_j(t) + T_j(t) < t \leq D_{j+1}(t) + T_j(t), Y(t) = 0 | Y(0) = 0 \}$$
$$= \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \sum_{j=1}^{\infty} \int_{\alpha t}^{t} \left[\int_0^{t-v} \boldsymbol{f_{FF-(j,j)}}(u,v) e^{\boldsymbol{Q_{WW}}(t-u-v)} du \right] dv \boldsymbol{1_W}. \quad (53)$$

Thus, we have the following:

$$\sum_{j=0}^{\infty} P\left\{ \int_0^t Y(u)du \geq \alpha t, N(t) = j, Y(t) = 0 | Y(0) = 0 \right\}$$
$$= \frac{\boldsymbol{\Phi_F}}{\boldsymbol{\Phi_F 1_F}} \sum_{j=1}^{\infty} \int_{\alpha t}^{t} \left[\int_0^{t-v} \boldsymbol{f_{FF-(j,j)}}(u,v) e^{\boldsymbol{Q}_{FF}(t-u-v)} du \right] dv \boldsymbol{1_W} \quad (54)$$

where

$$\boldsymbol{f^*_{FF-(j,j)}}(s_1, s_2) = [\boldsymbol{G^*_{FW}}(s_1) \boldsymbol{G^*_{WF}}(s_2)]^j = \left(f^*_{\boldsymbol{F}(i_1)\boldsymbol{F}(i_2)-(j,j)}(s_1, s_2) \right)_{|\boldsymbol{F}| \times |\boldsymbol{F}|},$$

$$f_{\boldsymbol{F}(i_1)\boldsymbol{F}(i_2)-(j,j)}(u,v) = \frac{\ell^{-1}\{f^*_{\boldsymbol{F}(i_1)\boldsymbol{F}(i_2)-(j+1,j)}(s_1, s_2)\}(u,v)}{f^*_{\boldsymbol{F}(i_1)\boldsymbol{F}(i_2)-(j,j)}(0,0)},$$

$$\boldsymbol{f_{FF-(j,j)}}(u,v) = \left(f_{\boldsymbol{F}(i_1)\boldsymbol{F}(i_2)-(j,j)}(u,v) \right)_{|\boldsymbol{F}| \times |\boldsymbol{F}|}.$$

$\boldsymbol{G^*_{WF}}(s_1)$ and $\boldsymbol{G^*_{FW}}(s_2)$ are defined in Appendix 1 and calculated by (41) and (42), respectively.

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