Definitions of Generalized Multi-Performance Weighted Multi-State K^{-}-out-of-*n* System and its Reliability Evaluations

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Abstract — The *k*-out-of-*n* system model is widely applied for the reliability evaluation of many technical systems. Multi-state system modelling is also widely used for representing real systems, whose components can have different levels of performance. For these researches, recently multi-state *k*-out-of-*n* systems have been comprehensively studied. In these studies, it is usually assumed that the system has a single task function to complete in a given environment. Moreover, the system or component performance is characterised by one measure, for example "electric power" in generation systems or "flow-rate" in transmission systems. However, this can be a simplification for some real-life engineering systems. For example, an intertwined district heating and electricity system consists of combined heat and power generating units, which can produce both electricity and heat. In this paper, definitions of multi-performance weighted multi-state components are provided and two generalized multi-performance multi-state K^{-}-out-of-*n* system models are proposed. Universal generating function approach is developed for the evaluation of such systems, with two numerical examples.

Keywords: Multi-performance, Multi-state, reliability, weighted K^{-}-*out-of-n*, universal generating function

Acronym

MSS multi-state system

FMSS fuzzy multi-state system

MPMS multi-performance multi-state system

FUGF fuzzy universal generating function

GA genetic algorithms

CHP combined heat and power

UGF universal generating function

Nomenclature

M	the highest performance state of the component or system
V	the number of performances of the component and of the system
W_i^v	performance variable *v* of component *i*
$\bar{W}_i$	performance variable vector of component *i* represented as $\left(W_i^1,\ldots,W_i^v,\ldots,W_i^V\right)$
$w_{i,j}^v$	the weight of performance variable *v* of component *i* in state *j*
$\bar{w}_{i,j}$	the performance vector of component *i* in state *j* represented as $\left(w_{i,j}^1,\ldots,w_{i,j}^v,\ldots,w_{i,j}^V\right)$
$\overline{w_i}$	*V* by $M+1$ dimensional array, which represents the set of weights of component *i*
W_{sys}^v	total weight of all components to a certain performance *v*
$p_{i,j}$	probability that component *i* is in state *j*
n	number of components in MSS
k_j	minimum weight requirement to be in system state *j*
$\overline{k_j}$	set of minimum weight requirements to be in system state *j*
ϕ	system structure function
$u_i(z)$	universal generating function of component *i*
$U\left(z\right)$	universal generating function of system
Ω	composition operator of universal generating functions
X	largest number of possible combination of the components' states

1. Introduction

In recent years, multi-state system (MSS) models have been widely used for reliability modelling of real technical systems that can perform their tasks with various performance levels [1]. A MSS is typically made of multi-state components, each of which has a finite number of states, corresponding to the different levels of performance that can be achieved. Failures or performance degradation of individual component do not necessarily lead to system-wide failure, but to reduced system performance. The basic concepts of MSS reliability were introduced in [1] [2] and a comprehensive summary of this field can be found in [3]. For some MSS, besides that each component must satisfy its individual requirement, the surplus performance can be shared with other components [4]. Reference 4 proposed a MSS with two performance sharing groups and the developed method can also be applied to the cases of more groups, which can be applied to many real life engineering systems. In some recent researches, fuzzy set theory has been applied to MSS modelling and corresponding reliability evaluation for overcoming the "dimension damnation" and data inaccuracy problems. General definitions of fuzzy multi-state system (FMSS) were proposed in [5] and recent advancement of MSS was summarized in [6]. Corresponding fuzzy universal generating function technique (FUGF) was developed for evaluating reliability of FMSS [7]. In [7], fuzzy set theory has been introduced for representing uncertainties of performances and corresponding probabilities and the fuzzy universal generating function has been developed for evaluating reliabilities of such multi-state systems. The uncertainties of component state probabilities have been modelled by transferable belief based on the Dempster–Shafer theory and reliabilities of multi-state systems have been evaluated correspondingly [26]. Reference [27] has studied the propagation mechanism of estimation uncertainties of component probabilities, which also has evaluated reliability of weighted *k*-out-of-*n* systems with multi-state component. Reference 8 utilized genetic algorithms (GA) for solving general optimal redundancy allocation of MSS.

The *k*-out-of-*n* system structure is a very important and popular type of redundant system, which finds wide applications in the reliability evaluation of many technical systems[9].The *k*-out-of-*n* system can be made of multi-state components[2]. A generalized multi-state *k*-out-of-*n:* G system and the corresponding reliability evaluation algorithm were developed in[10]. In a multi-state *k*-out-of-*n* system, the system is in state *j* or above if at least *k* components are in state *j* or above. A multi-state *k*-out-of-*n* system model is proposed in [11] for satisfying practical engineering systems, which allows different requirements on the number of components for different state levels. In [12], multi-state weighted *k*-out-of-*n* system models were further developed, where each component contributes to system performance with a weight that represents the performance of the component. Modelling and corresponding reliability evaluation of multi-state *k*-out-of-*n* system was summarized in [13, 14]. The lifetimes of two different multi-state *k*-out-of-*n* system models were studied in [15]. In these previous researches, it is usually assumed that the MSS has a single task function to complete in a given environment. System performance or component performance is characterised by a single performance variable, e.g. transmission rate throughput in computer and communication systems [16], electric power in electricity distribution systems [17] and flow rate in transportation systems [18]. In reality, there are many systems whose performance cannot be adequately described by a single variable. For example, a range of performance indicators were introduced in [19] to describe oil and gas production and transportation systems, including "production availability", "production regularity", "quantity availability" and others, and eventually simplified to a single variable: "throughput availability". Likewise, an intertwined district heating and electricity system consisting of combined heat and power generating units (CHP) would need to consider two performance variables of electric power and heating power. Then, multi-performance MSS modelling must be introduced to describe options with modelling performance measures.

Though reference [4] considered a MSS with two performance sharing groups, the performance variable is still restricted as a single one, e.g. capacity of generating system. Multi-state vector-*k*-out-of-*n* models have recently been developed to combine component and performance-based models into a single framework [20]. This allows evaluating the reliability of a system with multiple consumers, each one possibly utilizing different resources (electricity, oil, etc) to satisfy requirements. The two models given in [20] consider the system state j as the number of consumers being satisfied from a multiple line flow transmission system or multiple resource consumption. This begins to address the need to consider a number of performance variables when assessing system reliabilities.

The proposed reliability models in [20] take the perspective of "consumption" rather than of "production", the component of "production" only having a single task function providing a single resource (e.g. electricity). However there are several "production" components or systems with multiple "production" tasks. For example, combined heat and power systems known as "cogeneration" consisting of several CHP units can produce both electricity and useful heat to their customers in a single, integrated system. In recent years, combined heat and power systems have been widely used because of their high energy efficiency and environment friendly technologies. Combined heat and power systems are extremely flexible and are used in a spectrum of industries. Moreover the various "production" tasks of the component or system can be correlated, e.g. the produced electric power and thermal power of a CHP unit can be highly interdependent. The system operator needs a tool for evaluating reliabilities of such kind of systems for maintaining secure system operation. However the reliability characteristics of such components and systems, called as multi-performance multi-state systems (MPMS) have not been comprehensively studied yet.

In this paper, the basic definitions of MPMSs characterised by multiple performance variables, are introduced. Generalized multi-performance weighted multi-state $\bar{K}$-out-of-*n* system models and reliability evaluation algorithms are also introduced. The benefits are that more detailed information of components and systems can be considered in the evaluations of real-life engineering systems.

The paper is structured as follows. Section II introduces multi-performance components based on two definitions. Section III presents multi-performance systems, which are analysed from a multi-performance perspective and from a weighted-sum perspective that translates the multi-performance system to a single-performance one. The universal generating function (UGF) techniques for evaluating multi-performance systems are defined and presented with numerical examples in section IV. Concluding remarks are given in section V.

2. Definitions and Concepts of Multi-Performance Multi-State Components

In a multi-performance context, a component may have multiple functions to complete its different tasks in a given system environment, which are characterised by various performance variables.

The formal definition of Model I of the multi-performance multi-state component is given below.

Definition I: multi-performance multi-state strong-monotonic-increasing components

Component i has V performance variables represented by the vector $\bar{W}_i$, where $\bar{W}_i = \left(W_i^1, \ldots, W_i^v, \ldots, W_i^V\right)$ and $1 \le v \le V$, respectively. It also has *M+1* states, such that $0 \le j \le M$, and where $j = 0$ is the complete failure state and $j = M$ is the best performing state.

Each W_i^v in state j takes a value $w_{i,j}^v$, corresponding to a weight. The "weight" here means a component's contribution to a certain performance.

For a strong-monotonic-increasing component, $w_{i,j}^v$ corresponding to each performance variable W_i^v in state j must be not less than the value at a lower state , so that:

$$w_{i,j}^v \geq w_{i,j-1}^v, \forall i, j = 1 \ldots M \tag{1}$$

The performance of component i in state j can be represented by the vector $\bar{w}_{i,j}$, where $\bar{w}_{i,j} = \left(w_{i,j}^1, \ldots, w_{i,j}^v, \ldots, w_{i,j}^V \right)$.

All states of component i can be represented by the following matrix:

$$\hat{W}_i = \begin{pmatrix} w_{i,0}^1 & w_{i,0}^2 & \cdots & w_{i,0}^V \\ w_{i,1}^1 & w_{i,1}^2 & \cdots & w_{i,1}^V \\ \vdots & \vdots & & \vdots \\ w_{i,M}^1 & w_{i,M}^2 & \cdots & w_{i,M}^V \end{pmatrix} \tag{2}$$

Component i is in state j or above iff $W_i^v \geq w_{i,j}^v, \forall v$.

The following example illustrates this definition.

Example 1: A combined heat and power (CHP) generating unit e.g can produce both electric power and thermal power. A typical example is a GPC-180D gas unit with a nominal generating capacity of 17MW and 25 MW electric power and thermal power, respectively. Since thermal power is a by-product of electric power, the production of both is highly interdependent. The CHP unit i has three states, summarized in Table 1.

Table 1 - Performance parameters of component *i*

j	0	1	2
$w_{i.j}^1$	0	10	17
$w_{i.j}^2$	0	15	25

The performance matrix of the CHP generating unit can be as follows:

$$\hat{W}_i = \begin{pmatrix} 0 & 0 \\ 10 & 15 \\ 17 & 25 \end{pmatrix}$$

It is important to note that the units of each performance variable do not have to be the same. For example, power (W) and energy (Wh) in the case of an energy storage component, or volumetric flow rate (m^3/s) and temperature (°C) in the case of a hot water component. For the CHP generating unit, however, we consider both $w_{i.j}^1$ and $w_{i.j}^2$ to be MW. This means that, in the highest component state $j = 2$, the CHP produces 17MW electric power and 25MW thermal power. All weights increase with the increasing state number, which confirms that the unit is a multi-performance multi-state strongly-increasing monotonic component.

Definition I of the multi-performance multi-state component (MPMS) is strictly strongly-increasing: the weight corresponding to each performance variable is non-decreasing with the increase of state number. However, there may exist some realistic situations in which the component cannot be described by definition I. For example, a CHP unit may have a very high performance of electric power but a lower performance of thermal power in a specific state. It is also possible to evaluate the overall performance of a component by utilizing a weighted-sum conversion of various performance variables. The importance of the component performance variable can be scaled by a weighting-multiplier. Therefore in the Model II of the multi-performance multi-state component, the strongly-increasing monotonic characteristic for each performance variable is relaxed. The formal definition of Model II of the MPMS is given below.

Definition II: multi-performance multi-state monotonic-increasing components with weighted-sum conversion

Component *i* has *V* performance variables represented by the vector $\left\{W_i^1, \ldots, W_i^v, \ldots, W_i^V\right\}$, where $1 \le v \le V$. It also has *M+1* states, such that $0 \le j \le M$, and where $j = 0$ is the complete failure state and $j = M$ is the best performing state. Each performance variable W_i^v in state *j* takes a value $w_{i,j}^v$, which corresponds to a weight. Each component performance variable $w_{i,j}^v$ has a weighting-multiplier c^v , which scales its importance. A weighted sum of all performance variables for the component in state *j* can be evaluated as:

$$W_{i,j} = c^1 w_{i,j}^1 + c^2 w_{i,j}^2 + \ldots + c^V w_{i,j}^V \;\; \forall i \tag{3}$$

For a monotonic-increasing component with weighted-sum conversion, the weighted sum W_i^v in state *j* must be not less than the value at a lower state, so that:

$$W_{i,j} \geq W_{i,j-1}, \forall i, j = 1 \ldots M \tag{4}$$

3. Definitions and Concepts of Multi-Performance Multi-State Weighted $\bar{K}$-*out-of-n* Systems

In this section, key definitions and concepts of multi-performance multi-state weighted $\bar{K}$-out-of-*n* systems (MPMS weighted *k*-out-of-*n* systems) are introduced. A MPMS weighted $\bar{K}$-out-of-*n* system can consist of several multi-performance multi-state components for completing its different tasks. The first definition of MPMS weighted $\bar{K}$-out-of-*n* systems is basic, which corresponds to the model of multi-performance multi-state strongly-increasing monotonic components. The formal definition of the basic MPMS weighted $\bar{K}$-out-of-*n* system (Model I) is provided below.

Definition III: basic MPMS weighted $\bar{K}$-*out-of-n* systems

Consider a system consisting of *n* multi-performance multi-state components. Each component and system may be in *M+1* states, such that $0 \leq j \leq M$ and where $j = 0$ is the complete failure state and $j = M$ is the best performing state. Component *i* ($1 \leq i \leq n$) in state *j* provides multiple performance contributions to the system, whose values are represented by the vector $\left(w_{i,j}^1, \ldots, w_{i,j}^v, \ldots, w_{i,j}^V\right)$. $w_{i,j}^v$ represents the weight of performance variable *v* of component *i* in state *j*. The "weight" here indicates a component's contribution in a state to a certain performance of the system.

Let ϕ be the system structure function representing the system state and $\left(W_{sys}^1, \ldots, W_{sys}^v, \ldots, W_{sys}^V\right)$ be the system weight vector, where W_{sys}^v represents the total weight of all components to a certain performance *v*. Therefore $\left(W_{sys}^{1},\ldots,W_{sys}^{v},\ldots,W_{sys}^{V}\right)=\left(\sum_{i=1}^{n}W_{i}^{1},\ldots,\sum_{i=1}^{n}W_{i}^{v},\ldots,\sum_{i=1}^{n}W_{i}^{V}\right)$. The system is in state *j* or above if the system weight vector is greater than or equal to a pre-defined value set $\bar{k}_j=\left(k_j^1,k_j^2,\cdots,k_j^V\right)$. When comparing two sets, each element is compared to the corresponding element with the same index, such that: if $W_{sys}^{v}\geq k_j^{v}$, $\forall v$. Then we have:

$$\begin{aligned}\Pr\{\phi\geq j\}&=\Pr\left\{\left(W_{sys}^{1}\geq k_j^1\right)\wedge\left(W_{sys}^{2}\geq k_j^2\right)\wedge\ldots\wedge\left(W_{sys}^{V}\geq k_j^V\right)\right\}\\&=\Pr\left\{\left(\sum_{i=1}^{n}W_i^1\geq k_j^1\right)\wedge\left(\sum_{i=1}^{n}W_i^2\geq k_j^2\right)\wedge\ldots\wedge\left(\sum_{i=1}^{n}W_i^V\geq k_j^V\right)\right\}\end{aligned}\quad(5)$$

where $\wedge$ represents AND logic.

Equation (5) indicates that the system is in state *j* or above if each element in the system weight vector representing the total weight of all components to a certain performance is greater than or equal to the corresponding element in the pre-defined set.

Example 2: An intertwined district electricity and heating system provides energy to a load for consuming both electric and thermal power. The system consists of three CHP generating units as shown in Fig.1, which are one GPC-180D gas unit and two GPC-70D gas CHP units. The GPC-180D gas unit has a nominal generating capacity of 17MW and 25 MW electric power and thermal power, respectively. The GPC-70D gas unit has a nominal generating capacity of 6.5 MW and 10 MW electric power and thermal power, respectively. Obviously one GPC-180D gas unit has higher contribution than a GPC-70D gas unit, which indicates it has a higher "weight". Every generating unit has three possible states: 0, 1, 2. The weight distribution of three units are shown in Table 2. As shown in Table 2, all weights of both performances increase with the increasing state number, which indicates that each unit is a multi-performance multi-state strongly-increasing monotonic component. The system may also be in three different states: 0, 1, and 2. In order to meet the electric power and heating consumption of the load, a range of requirements $\bar{k}_j$, should be met. When the total weights of both electric power and thermal power are greater than or equal to 30MW and 45MW, respectively, the system is considered to be in state 2; otherwise but greater than or equal to 16MW and 25MW, respectively, in state 1;

Otherwise, in state 0. Based on these descriptions, the system can be considered to be a basic MPMS weighted *k*-out-of-*n* system with the following parameters:

$$n=3,\ M=2,\ V=2,\ \bar{k}_j=(0,0),(16,25),(30,45)\ \text{for}\ j=0,1,2\ \text{respectively.}$$

Based on Definition III, we can describe this model as follows: The system weight vector can be represented as $\left(W_{sys}^1, W_{sys}^2\right)=\left(\sum_{i=1}^{3} W_i^1, \sum_{i=1}^{3} W_i^2\right)$. Suppose that the three CHP generating units are both in state 2. The system performance vector can be evaluated as $\left(W_{sys}^1, W_{sys}^2\right)=(17+6.5+6.5, 25+10+10)=(30,45)$ and the system is also in state 2. If units 2 and 3 are still in state 2 and the state of unit 1 decreases from 2 to 1, then the system performance vector can be evaluated as $\left(W_{sys}^1, W_{sys}^2\right)=(10+6.5+6.5, 15+10+10)=(23,35)$ and the state of the system decreases from 2 to 1.

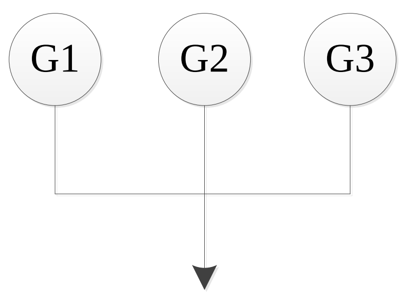

Figure 1. Single line diagram of a power system with three CHP units

Table 2 - Performance parameters of units and minimum system requirement

j	0	1	2
$\left(w_{1,j}^1, w_{1,j}^2\right)$	0,0	10,15	17,25
$\left(w_{2,j}^1, w_{2,j}^2\right)$	0,0	3,5	6.5,10
$\left(w_{3,j}^1, w_{3,j}^2\right)$	0,0	3,5	6.5,10
$\bar{k}_j$	0,0	16,25	30,45

The above definition of the basic MPMS *k*-out-of-*n* system (Model I) is strong: if just one element in the system performance vector is lower than the corresponding element in the pre-defined set for state *j*, the system state will be below *j*. In the Model II of the MPMS, this definition is relaxed by utilizing the weighted-sum conversion.

Definition IV: MPMS K^{-}-*out-of-n* systems with weighted-sum conversion

Consider a system consisting of *n* MPMS components. Each component and system may be in *M+1* states, such that $0 \le j \le M$ and where $j = 0$ is the complete failure state and $j = M$ is the best performing state. Component *i* ($1 \le i \le n$) in state *j* has multiple performance contributions to the system, whose values are represented by the vector $\left(w_{i,j}^{1},\ldots,w_{i,j}^{v},\ldots,w_{i,j}^{V}\right)$. $w_{i,j}^{v}$ represents the weight of performance variable *v* of component *i* in state *j*. In the proposed model, a pre-step of weighted-sum conversion is applied on each component as described in the previous section. A weighted sum of performance variables for the component can be evaluated by equation (3). Following this conversion, each multi-performance multi-state component resembles a traditional single-performance multi-state component. Let ϕ be the system structure function representing the system state and W_{sys} be the weighted system performance, where $W_{sys} = \sum_{i=1}^{n} W_i$. The system is in state *j* or above if the weighted system performance is greater than or equal to a pre-defined value k_j, which is therefore identical to previously documented models [12].

The proposed model can be useful for describing a system with a large number of performance variables.

Example 3: Consider the intertwined district electricity and heating system consisting of three CHP generating units as described in Example 2. The weighting-multipliers of electric power and heating power variables are $c^1 = 0.9$ and $c^2 = 1$, respectively. A weighted sum of the two performance variables for each unit can be evaluated as: $W_i = c^1 W_i^1 + c^2 W_i^2$. The minimum requirement for system states is $k = 0, 39, 72$ for $j = 0,1,2$ respectively.

Reliability parameters of units after weighted-sum conversion and minimum system requirement are described in Table 3.

Table 3 - Performance parameters of units after weighted-sum conversion and system state requirement

j	0	1	2
$w_{1,j}$	0	24	40.3
$w_{2,j}$	0	7.7	15.85
$w_{3,j}$	0	7.7	15.85
k_j	0	39	72

Suppose that three CHP generating units are both in state 2. The weighted system performance can be evaluated as $W_{sys} = (c^1W_1^1 + c^2W_1^2) + (c^1W_2^1 + c^2W_2^2) + (c^1W_3^1 + c^2W_3^2) = 40.3 + 15.85 + 15.85 = 72$

and the system is also in state 2.

If units 2 and 3 are still in state 2 and the state of unit 1 decreases from 2 to 1, then the system performance vector can be evaluated as $W_{sys} = (c^1W_1^1 + c^2W_1^2) + (c^1W_2^1 + c^2W_2^2) + (c^1W_3^1 + c^2W_3^2) = 24 + 15.85 + 15.85 = 55.7$ and the state of the system decreases from 2 to 1.

4. Reliability Evaluation of MPMS K^{-}-*out-of-n* systems

Since the number of MSS states increases rapidly with the increase in the number of its elements, a range of analytical methods have been created to reduce computational complexity [9].

The universal generating function (UGF) was firstly introduced in [28] for reliability evaluation of MSS. The performance distribution of MSS can be determined by using the UGF technique. The reliability of MSSs with series, parallel, series-parallel and bridge structures were evaluated in [24,29,30] by defining different composition operators. The UGF technique is also widely used for solving different MSS reliability optimization problems because the reliability index can be easily represented and evaluated.

The further developments and applications of UGF technique were presented in [31,32] and detailed description can be found in [22] that summarized achievements in the field.

By determining different composition operators, the output performance distributions of MSSs with series, parallel and series-parallel structures have been evaluated in [24] and [25]. The UGF method has shown to be a very flexible and intuitive tool, especially when deriving probability distributions of system performance algebraically.

4.1 Universal generating function (UGF) method for evaluating MPMS weighted $\bar{K}$-*out-of-n* systems

In this section, the UGF model for evaluating the reliability of MPMS *k*-out-of-*n* systems is presented.

The general form of the UGF for the multi-performance multi-state component *i* is:

$$u_i(z) = \sum_{j=0}^{M} p_{i,j} \cdot z^{\bar{w}_{i,j}} = \sum_{j=0}^{M} p_{i,j} \cdot z^{\left(w_{i,j}^1, w_{i,j}^2, \cdots, w_{i,j}^V\right)} \quad (6)$$

The function (6) is the *z*-transform of random variable vector $\bar{W}_i$ of component *i*, where $\bar{W}_i = \left(W_i^1, \ldots, W_i^v, \ldots, W_i^V\right)$ and W_i^v represents performance variable *v* of component *i*. Notice that W_i^v in state *j* takes a value $w_{i,j}^v$, corresponding to a weight, which means a component's contribution to a certain performance.

The function $u_i(z) = \sum_{j=0}^{M} p_{i,j} \cdot z^{\bar{w}_{i,j}}$ represents the probability mass function (p.m.f.) ($\bar{w}_{i,0}$,..., $\bar{w}_{i,j}$,…, $\bar{w}_{i,M}$),($p_{i,0}$,..., $p_{i,j}$,…, $p_{i,M}$) in the polynomial form, where the exponent $\bar{w}_{i,j} = \left(w_{i,j}^1, \ldots, w_{i,j}^v, \ldots, w_{i,j}^V\right)$ is the performance vector of component *i* in state *j* and $p_{i,j}$ is the probability that component *i* is in state *j* , respectively [22]. The p.m.f. is defined as the mapping $\bar{w}_{i,j} \to p_{i,j}$.

The universal generating function $u_i(z)$ is derived from the famous moment generating function $m_i(t)$ [22, 23]. The moment generating function $m_i(t)$ of the discrete random variable vector $\bar{W}_i$ with p.m.f. ($\bar{w}_{i,0}$,..., $\bar{w}_{i,j}$,…, $\bar{w}_{i,M}$), ($p_{i,0}$,..., $p_{i,j}$,…, $p_{i,M}$) is defined for all values of *t* by:

$$m^{'}(t) = \frac{d}{dt}(\sum_{j=0}^{M} e^{t\bar{w}_{i,j}} p_{i,j}) = \sum_{j=0}^{M} \bar{w}_{ij} e^{t\bar{w}_{i,j}} p_{i,j} \tag{7}$$

The function $m_i(t)$ is called as the moment-generating function because all of the moments of random variable vector $\bar{W}_i$ can be obtained by successively differentiating $m_i(t)$ [21]. For example:

$$m^{'}(t) = \frac{d}{dt}(\sum_{j=0}^{M} e^{t\bar{w}_{i,j}} p_{i,j}) = \sum_{j=0}^{M} \bar{w}_{ij} e^{t\bar{w}_{i,j}} p_{i,j} \tag{8}$$

Hence

$$m^{'}(0) = \sum_{j=1}^{M} \bar{w}_{ij} p_{i,j} = E(\bar{W}_i) \tag{9}$$

By replacing the function e^t by the variable z in Equation (2) we can obtain another function related to random variable vector $\bar{W}_i$ that uniquely determines its p.m.f.:

$$u_i(z) = E(z^{\bar{W}_i}) = \sum_{j=0}^{M} p_{i,j} z^{\bar{w}_{i,j}} \tag{10}$$

Equation (10) is usually called the *z*-transform of random variable vector $\bar{W}_i$, which represents performance distribution of component *i*.

The UGF of the system is found using the generalised vector composition operator, Ω [3, 23]:

$$\begin{aligned} U(z) &= \Omega\left(u_1(z), u_2(z), \ldots, u_n(z)\right) \\ &= \Omega\left(\sum_{j=0}^{M} p_{1,j} \cdot z^{\bar{w}_{1,j}}, \ldots, \sum_{j=0}^{M} p_{n,j} \cdot z^{\bar{w}_{n,j}} \right) \\ &= \Omega\left(\sum_{j=0}^{M} p_{1,j} \cdot z^{\left(w_{1,j}^{1}, w_{1,j}^{2}, \ldots, w_{1,j}^{V}\right)}, \ldots, \sum_{j=0}^{M} p_{n,j} \cdot z^{\left(w_{n,j}^{1}, w_{n,j}^{2}, \ldots, w_{n,j}^{V}\right)} \right) \end{aligned} \tag{11}$$

The composition operator performs a multiplication of the coefficients (probabilities) of components' *u*-functions, as if polynomial factors were being expanded, and, then, sums the exponents (performance variables) by index.

For a MPMS weighted *k*-out-of-*n* system comprised of *n* multi-performance multi-state components with M states and *V* performance variables, the parallel composition operation is applied as follows:

$$
\begin{aligned}
U(z) &= \Omega_{par}\left(u_1(z), u_2(z), \ldots, u_n(z)\right) \\
&= \Omega_{par}\left(\sum_{j_1=1}^{M} p_{1,j_1} \cdot z^{\bar{w}_{1,j_1}}, \ldots, \sum_{j_n=1}^{M} p_{n,j_n} \cdot z^{\bar{w}_{n,j_n}}\right) \\
&= \Omega_{par}\left(\sum_{j_1=1}^{M} p_{1,j_1} \cdot z^{\left(w^1_{1,j_1}, w^2_{1,j_1}, \ldots, w^V_{1,j_1}\right)}, \ldots, \sum_{j_n=1}^{M} p_{n,j_n} \cdot z^{\left(w^1_{n,j_n}, w^2_{n,j_n}, \ldots, w^V_{n,j_n}\right)}\right) \\
&= \sum_{j_1=0}^{M} \cdots \sum_{j_n=0}^{M} \left(\prod_{i=1}^{n} p_{i,j_i} \cdot z^{\phi_{par}\left(\bar{w}_{1,j_1}, \ldots, \bar{w}_{n,j_n}\right)}\right) \\
&= \sum_{j_1=0}^{M} \cdots \sum_{j_n=0}^{M} \left(\prod_{i=1}^{n} p_{i,j_i} \cdot z^{\left(\sum_{i=1}^{n} w^1_{i,j_i}, \ldots, \sum_{i=1}^{n} w^V_{i,j_i}\right)}\right) \\
&= \sum_{x=1}^{X} P_x z^{\left(w^1_{sys,x}, \ldots, w^V_{sys,x}\right)}
\end{aligned} \tag{12}
$$

where Ω_{par} is defined as the parallel composition operator, which represents total weight of all components equals to the sum of the weights of the individual components [22, 23], *X* represents the largest number of possible combination of the components' states, P_x is the probability of a certain combination *x* of the components' states, $\left(w^1_{sys,x}, \ldots, w^V_{sys,x}\right)$ represents the vector of performance variables in state *x* for the system.

Once the UGF of the MPMS *k*-out-of-*n* system is obtained, the system reliability for any given vector $\bar{k} = \left(k^1, k^2, \cdots, k^V\right)$ can be evaluated by applying the δ_A operator:

$$
\begin{aligned}
R(\bar{k}) = \delta_A\left(U(z), \bar{k}\right) &= \delta_A\left(\sum_{x=1}^{X} P_x z^{\left(w^1_{sys,x}, \ldots, w^V_{sys,x}\right)}, \bar{k}\right) \\
&= \sum_{x=1}^{X} P_x \alpha\left(w^1_{sys,x} - k^1, \cdots, w^V_{sys,x} - k^V\right)
\end{aligned} \tag{13}
$$

The binary value $\alpha\left(w^1_{sys,x} - k^1, \cdots, w^V_{sys,x} - k^V\right)$ in the above equation is evaluated as:

$$\alpha\left\{w_{sys,x}^{1}-k^{1},\cdots,w_{sys,x}^{V}-k^{V}\right\}=\begin{cases}1,\left(k^{1}\le w_{sys,x}^{1}\right)\wedge\ldots\wedge\left(k^{V}\le w_{sys,x}^{V}\right)\\0,\left(k^{1}>w_{sys,x}^{1}\right)\vee\ldots\vee\left(k^{V}>w_{sys,x}^{V}\right)\end{cases} \tag{14}$$

where $\wedge$ represents AND logic and $\vee$ represents OR logic.

4.2 Illustrative example with the UGF method

Consider a combined heat and power system with one GPC-180D gas unit and two GPC-70D gas CHP units. Every generating unit has three possible states: 0, 1, 2. Reliability parameters and the weight distribution of three units are shown in Table 4. As shown in Table 4, the probabilities of the GPC-180D gas unit in state 0, 1, 2 are 0.1, 0.1, and 0.8, respectively. The probabilities of the GPC-70D gas unit in state 0, 1, 2 are 0.05, 0.05, and 0.9, respectively. There are two cases considered corresponding to the basic MPMS weighted $\overline{K}$-*out-of-n* system model (Definition III) and MPMS $\overline{K}$-*out-of-n* systems with weighted-sum conversion model (Definition IV), respectively.

Case 1: The system may also be in three different states: 0, 1, and 2. The units are modelled as multi-performance multi-state strong-monotonic-increasing components (Definition I). In order to meet the electric power and heating consumption of the load, a range of requirements $\overline{k}_j$, should be met. When the total weights of both electric power and thermal power are greater than or equal to 30MW and 45MW, respectively, the system is considered to be in state 2; otherwise but greater than or equal to 16MW and 25MW, respectively, in state 1; Otherwise, in state 0. Based on these descriptions, the system can be considered to be a basic MPMS weighted *k*-out-of-*n* system with the following parameters:

$$n=3,\ M=2,\ V=2,\ \overline{k}_j=(0,0),(16,25),(30,45)\ \text{for } j=0,1,2\ \text{respectively}$$

Table 4 - Performance parameters of units

j	0	1	2
$\left(w_{1,j}^{1},w_{1,j}^{2}\right)$	0,0	10,15	17,25
$p_{1,j}$	0.1	0.1	0.8

j	0	1	2
$\left(w_{2,j}^{1}, w_{2,j}^{2}\right)$	0,0	3,5	6.5,10
$p_{2,j}$	0.05	0.05	0.9

The UGF for each unit, representing its probability distribution, is defined as:

$$
\begin{aligned}
u_1(z) &= p_{1,2}\cdot z^{\left(w_{1,2}^{1},w_{1,2}^{2}\right)} + p_{1,1}\cdot z^{\left(w_{1,1}^{1},w_{1,1}^{2}\right)} + p_{1,0}\cdot z^{\left(w_{1,0}^{1},w_{1,0}^{2}\right)} = 0.8\cdot z^{(17,25)} + 0.1\cdot z^{(10,15)} + 0.1\cdot z^{(0,0)} \\
u_2(z) &= p_{2,2}\cdot z^{\left(w_{2,2}^{1},w_{2,2}^{2}\right)} + p_{2,1}\cdot z^{\left(w_{2,1}^{1},w_{2,1}^{2}\right)} + p_{2,0}\cdot z^{\left(w_{2,0}^{1},w_{2,0}^{2}\right)} = 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)} \\
u_3(z) &= p_{3,2}\cdot z^{\left(w_{3,2}^{1},w_{3,2}^{2}\right)} + p_{3,1}\cdot z^{\left(w_{3,1}^{1},w_{3,1}^{2}\right)} + p_{3,0}\cdot z^{\left(w_{3,0}^{1},w_{3,0}^{2}\right)} = 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)}
\end{aligned} \tag{15}
$$

Applying the operator Ω over the UGFs of the units, the probability distribution of the entire system can be obtained as:

$$
\begin{aligned}
U(z) &= \Omega\left(u_1(z), u_2(z), u_3(z)\right) \\
&= \Omega\begin{pmatrix} 0.8\cdot z^{(17,25)} + 0.1\cdot z^{(10,15)} + 0.1\cdot z^{(0,0)}, 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)}, \\ 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)} \end{pmatrix} \\
&= 0.648\cdot z^{(30,45)?} + \ldots + 0.00025\cdot z^{\;0,0)?}
\end{aligned} \tag{16}
$$

For the highest system state, where the minimum system requirement is $\bar{k}_2 = (30,45)$, the probability of the system attaining this system weight is:

$$
\begin{aligned}
\delta\left(U(z), \bar{k}_2\right) &= \delta\left(0.648\cdot z^{(30,45)?} + \ldots + 0.00025\cdot z^{\;0,0)?}, (30,45)\right) \\
&= \sum_{x=1}^{X} p_x \cdot \alpha_x = 0.6480
\end{aligned} \tag{17}
$$

For the second system state, where the minimum system requirement is $\bar{k}_1 = (16,25)$, the probability of the system attaining this system weight is:

$$\begin{aligned}\delta\left(U(z),\bar{k}_1\right) &= \delta\left(0.648\cdot z^{(30,45)?}+\ldots+0.00025\cdot z^{\ 0,0)?},(16,25)\right)\\ &= \sum_{x=1}^{X} p_x\cdot\alpha_x \\ &= 0.648?+\ \ .081+\ldots+0.002+0.002\\ &= 0.8992\end{aligned} \tag{18}$$

For the lowest system state, where the minimum system requirement is $\bar{k}_0=(0,0)$, the probability of the system attaining this system weight includes all possible system probabilities:

$$\begin{aligned}\delta\left(U(z),\bar{k}_0\right) &= \delta\left(0.648\cdot z^{(30,45)?}+\ldots+0.00025\cdot z^{\ 0,0)},(0,0)\right)\\ &= 0.648?+\ \ .081+\ldots 0.0003\\ &= \sum_{x=1}^{X} p_x\cdot\alpha_x \\ &= 1\end{aligned} \tag{19}$$

The lowest system state is, by definition, equal to 1. The final probabilities for the three system states are 0.648, 0.8992 and 1, respectively.

Case 2: The weighting-multipliers of electric power and heating power variables are $c^1=0.9$ and $c^2=1$, respectively. The units are modelled as multi-performance multi-state monotonic-increasing components with weighted-sum conversion (Definition II).

A weighted sum of the two performance variables for each unit can be evaluated as: $W_i=c^1W_i^1+c^2W_i^2$. The system may be in three different states: 0, 1, and 2. The minimum requirement for system states is $k=0,38,72$ for $j=0,1,2$ respectively. When the weighted system performance is greater than or equal to 72MW, the system is considered to be in state 2; otherwise but greater than or equal to 40MW, in state 1; Otherwise, in state 0. Based on these descriptions, the system can be considered to be a MPMS $\bar{K}$-*out-of-n* system with weighted-sum conversion with the following parameters: $n=3,\ M=2,\ \ \bar{k}_j=0,40,72$ for $j=0,1,2$ respectively.

The UGF for each unit, representing its probability distribution, is defined as:

$$\begin{aligned}u_1(z) &= 0.8\cdot z^{40.3}+0.1\cdot z^{24}+0.1\cdot z^0\\ u_2(z)=u_3(z) &= 0.9\cdot z^{15.85}+0.05\cdot z^{7.7}+0.05\cdot z^0\end{aligned} \tag{20}$$

Applying the operator Ω over the UGFs of the units, the probability distribution of the entire system can be obtained as:

$$\begin{aligned} U(z) &= \Omega\left(u_1(z), u_2(z), u_3(z)\right) \\ &= \Omega\begin{pmatrix} 0.8\cdot z^{40.3} + 0.1\cdot z^{24} + 0.1\cdot z^{0}, 0.9\cdot z^{15.85} + 0.05\cdot z^{7.7} + 0.05\cdot z^{0}, \\ 0.9\cdot z^{15.85} + 0.05\cdot z^{7.7} + 0.05\cdot z^{0} \end{pmatrix} \\ &= 0.648\cdot z^{72} + \ldots + 0.0003\cdot z^{0?} \end{aligned} \tag{21}$$

For the highest system state, where the minimum system requirement is $\bar{k}_2 = 72$, the probability of the system attaining this system weight is:

$$\begin{aligned} \delta\left(U(z), \bar{k}_2\right) &= \delta\left(0.648\cdot z^{72} + \ldots + 0.0003\cdot z^{0?}, 72\right) \\ &= \sum_{x=1}^{X} p_x \cdot \alpha_x = 0.648 \end{aligned} \tag{22}$$

For the second system state, where the minimum system requirement is $\bar{k}_1 = 40$, the probability of the system attaining this system weight is:

$$\begin{aligned} \delta\left(U(z), \bar{k}_1\right) &= \delta(0.648\cdot z^{72} + \ldots + 0.0003\cdot z^{0?}, 30) \\ &= 0.9807 \end{aligned} \tag{23}$$

For the lowest system state, where the minimum system requirement is $\bar{k}_0 = 0$, the probability of the system attaining this system weight includes all possible system probabilities:

$$\begin{aligned} \delta\left(U(z), \bar{k}_0\right) &= \delta(0.648\cdot z^{72} + \ldots + 0.0003\cdot z^{0?}, 40) \\ &= 0.648 + \ldots + 0.0003 \\ &= 1 \end{aligned} \tag{24}$$

4.3 Large illustrative example with the UGF method

A large heat and power system consists of six GPC-180D gas units and four GPC-70D gas CHP units. Every generating unit has three possible states: 0, 1, 2. Reliability parameters and the weight distribution of these two kinds of units are shown in Tables 3, respectively. There are two cases considered corresponding to the basic weighted K^{-}*-out-of-n* system model (Definition III) and MPMS K^{-}*-out-of-n* systems with weighted-sum conversion model (Definition IV), respectively.

Case 1: The system may be in three different states: 0, 1, and 2. The units are modelled as multi-performance multi-state strong-monotonic-increasing components (Definition I). In order to meet the electric power and heating consumption of the load, a range of requirements $\bar{k}_j$, should be met. When the total weights of both electric power and thermal power are greater than or equal to 128MW and 190MW, respectively, the system is considered to be in state 2; otherwise but greater than or equal to 80MW and 120MW, respectively, in state 1; Otherwise, in state 0. Based on these descriptions, the system can be considered to be a basic MPMS weighted *k*-out-of-*n* system with the following parameters: $n=10$, $M=2$, V=2, $\bar{k}_j=(0,0),(80,120),(128,190)$ for $j=0,1,2$ respectively.

The UGF for each unit, representing its probability distribution, is defined as:

$$\begin{aligned} u_1(z) &= u_2(z) = \ldots = u_6(z) \\ &= 0.8\cdot z^{(17,25)} + 0.1\cdot z^{(10,15)} + 0.1\cdot z^{(0,0)} \\ u_7(z) &= u_8(z) = u_9(z) = u_{10}(z) \\ &= 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)} \end{aligned} \tag{25}$$

Applying the operator Ω over the UGFs of the units, the probability distribution of the entire system can be obtained as:

$$\begin{aligned} &U(z) = \Omega\left(u_1(z), u_2(z), \ldots, u_{10}(z)\right) \\ &= \Omega\begin{pmatrix} 0.8\cdot z^{(17,25)} + 0.1\cdot z^{(10,15)} + 0.1\cdot z^{(0,0)}, \ldots, \\ 0.9\cdot z^{(6.5,10)} + 0.05\cdot z^{(3,5)} + 0.05\cdot z^{(0,0)} \end{pmatrix} \\ &= 0.172\cdot z^{(128,190)?} + \ldots + 6.25\cdot 10^{-2}\cdot z^{(0,0)?} \end{aligned} \tag{26}$$

For the highest system state, where the minimum system requirement is $\bar{k}_2=(128,190)$, the probability of the system attaining this system weight is:

$$
\begin{aligned}
\delta\left(U\left(z\right),\bar{k}_2\right) &= \delta\left(0.172 \cdot z^{(128,190)?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{(0,0)?}, \left(128,190\right)\right) \\
&= \sum_{x=1}^{X} p_x \cdot \alpha_x = 0.172
\end{aligned} \tag{27}
$$

For the second system state, where the minimum system requirement is $\bar{k}_1 = \left(80,120\right)$, the probability of the system attaining this system weight is:

$$
\begin{aligned}
\delta\left(U\left(z\right),\bar{k}_1\right) &= \delta\left(0.172 \cdot z^{(128,190)?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{(0,0)?}, \left(80,120\right)\right) \\
&= 0.9802
\end{aligned} \tag{28}
$$

For the lowest system state, where the minimum system requirement is $\bar{k}_0 = (0,0)$, the probability of the system attaining this system weight includes all possible system probabilities:

$$
\begin{aligned}
\delta\left(U\left(z\right),\bar{k}_0\right) &= \delta\left(0.172 \cdot z^{(128,190)?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{(0,0)?}, \left(0,0\right)\right) \\
&= 0.172 + \ldots + 6.25 \cdot 10^{-12} \\
&= 1
\end{aligned} \tag{29}
$$

Case 2: The weighting-multipliers of electric power and heating power variables are $c^1 = 0.9$ and $c^2 = 1$, respectively. The units are modelled as multi-performance multi-state monotonic-increasing components with weighted-sum conversion (Definition II). A weighted sum of the two performance variables for each unit can be evaluated as: $W_i = c^1 W_i^1 + c^2 W_i^2$. The system may be in three different states: 0, 1, and 2. The minimum requirement for system states is $k = 0, 150, 305$ for $j = 0,1,2$ respectively. When the weighted system performance is greater than or equal to 305 MW, the system is considered to be in state 2; otherwise but greater than or equal to 150MW, in state 1; Otherwise, in state 0. Based on these descriptions, the system can be considered to be a MPMS $\bar{K}$-*out-of-n* system with weighted-sum conversion with the following parameters: $n = 10$, $M = 2$, $\bar{k}_j = 0, 150, 305$ for $j = 0,1,2$ respectively.

The UGF for each unit, representing its probability distribution, is defined as:

$$u_1(z) = u_2(z) = \ldots = u_6(z)$$
$$= 0.8 \cdot z^{40.3} + 0.1 \cdot z^{24} + 0.1 \cdot z^{0}$$
$$u_7(z) = u_8(z) = u_9(z) = u_{10}(z) \tag{30}$$
$$= 0.9 \cdot z^{15.85} + 0.05 \cdot z^{7.7} + 0.05 \cdot z^{(0,0)}$$

Applying the operator Ω over the UGFs of the units, the probability distribution of the entire system can be obtained as:

$$U(z) = \Omega\left(u_1(z), u_2(z), \ldots, u_{10}(z)\right)$$
$$= \Omega\begin{pmatrix} 0.8 \cdot z^{40.3} + 0.1 \cdot z^{24} + 0.1 \cdot z^{0}, \ldots, \\ 0.9 \cdot z^{15.85} + 0.05 \cdot z^{7.7} + 0.05 \cdot z^{0} \end{pmatrix} \tag{31}$$
$$= 0.172 \cdot z^{305.2?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{0?}$$

For the highest system state, where the minimum system requirement is $\bar{k}_2 = 305$, the probability of the system attaining this system weight is:

$$\delta\left(U(z), \bar{k}_2\right) = \delta\left(0.172 \cdot z^{305.2?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{0?}, 305\right)$$
$$= \sum_{x=1}^{X} p_x \cdot \alpha_x = 0.172 \tag{32}$$

For the second system state, where the minimum system requirement is $\bar{k}_1 = 150$, the probability of the system attaining this system weight is:

$$\delta\left(U(z), \bar{k}_1\right) = \delta(0.172 \cdot z^{305.2?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{0?}, 150)$$
$$= 0.9984 \tag{33}$$

For the lowest system state, where the minimum system requirement is $\bar{k}_0 = 0$, the probability of the system attaining this system weight includes all possible system probabilities:

$$\delta\left(U(z), \bar{k}_0\right) = \delta\left(0.172 \cdot z^{305.2?} + \ldots + 6.25 \cdot 10^{-\ 2} \cdot z^{0?}, 0\right)$$
$$= 0.172 + \ldots + 6.25 \cdot 10^{-12} \tag{34}$$
$$= 1$$

5. Conclusion

This paper has presented an extension of the $\bar{K}$-*out-of-n* model, which is relevant to several real-world systems. By introducing a structure with multiple performance variables, systems can be analysed with more details and the variable that is the cause of system failure can be explicitly identified. Processing restrictions aside, the model can consider any number of performance measures at once, and there exist potentially many more composition operators that can be applied to multi-variable weighted $\bar{K}$-*out-of-n* systems to account for other structures. Future work includes optimization of redundancy structure of multi-performance weighted multi-state $\bar{K}$-*out-of-n* system satisfying multiple performance requirements of practical engineering systems. Some practical engineering systems, e.g. energy systems can be modelled as multi-performance sharing MSS. Reliability evaluation of MSS considering multi-performance sharing can be another direction of future research. Consideration of uncertainty in the reliability evaluation of the proposed multi-performance weighted multi-state $\bar{K}$-*out-of-n* can also be the promising extension of future research work. The uncertainties of multi-performances and probabilities can be modelled by fuzzy set theory or Dempster–Shafer theory. The propagation mechanism of estimation uncertainties from the component level to the system level and its corresponding impact can also be further studied.

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