Modeling dependent competing failure processes with degradation-shock dependence

Mengfei Fan [1,2], Zhiguo Zeng [3], Enrico Zio [3,4] and Rui Kang [1,2]

[1] School of Reliability and Systems Engineering, Beihang University, Beijing, China

[2] Center for Resilience and Safety of Critical Infrastructure, Beihang University, Beijing, China

[3] EDF Foundation Chair on Systems Science and Energetic Challenge, CentraleSupelec, Universite Paris-Saclay, Grande Voie des Vignes, 92290 Chatenay-Malabry, France

[4] Energy Department, Politecnico di Milano, Italy

fanmengfei@buaa.edu.cn, zhiguo.zeng@centralesupelec.fr, enrico.zio@ecp.fr, kangrui@buaa.edu.cn

Abstract

In this paper, we develop a new reliability model for dependent competing failure processes (DCFPs), which accounts for degradation-shock dependence. This is a type of dependence where random shock processes are influenced by degradation processes. The degradation-shock dependence is modeled by assuming that the intensity function of the nonhomogeneous Poisson process describing the random shock processes is dependent on the degradation processes. The dependence effect is modeled with reference to a classification of the random shocks in three "zones" according to their magnitudes, damage zone, fatal zone, and safety zone, with different effects on the system's failure behavior. To the best of the authors' knowledge, this type of dependence has not yet been considered in reliability models. Monte Carlo simulation is used to calculate the system reliability. A realistic application is presented with regards to the dependent failure behavior of a sliding spool, which is subject to two dependent competing failure processes, wear and clamping stagnation. It is shown that the developed model is capable of describing the dependent competing failure behaviors and their dependence.

Keywords

Dependent competing failure processes; degradation; shock; degradation-shock dependence; Monte Carlo simulation; sliding spool

Highlights

- A new reliability model is developed for degradation-shock dependence.
- The influence of degradations on random shocks is considered.
- Zone effect of random shocks magnitudes is considered.
- A case study of a sliding spool is conducted.

Acronyms

DCFP	Dependent competing failure processes
DTS	Degradation-Threshold-Shock
NHPP	Nonhomogeneous Poisson process

Notation

$x(t)$	Degradation at time t
$\boldsymbol{\theta}$	Model parameters for the soft failure process
$f_{\boldsymbol{\theta}}(\mathbf{x})$	Probability density function of $\boldsymbol{\theta}$
$F_{\boldsymbol{\theta}}(\mathbf{x})$	Cumulative distribution function of $\boldsymbol{\theta}$
D	Soft failure threshold
Y_i	Magnitude of the i th shock
W_i	Damage caused by the i th shock
α	Shock damage proportional coefficient
$f_Y(y)$	Probability density function of Y_i
$F_Y(y)$	Cumulative distribution function of Y_i
$\lambda(t)$	Intensity of random shocks at time t
λ_0	Initial intensity of random shocks
$N(t)$	Number of random shocks that have arrived by time t
$N_1(t)$	Number of random shocks that have arrived in the damage zone by time t
$N_2(t)$	Number of random shocks that have arrived in the fatal zone by time t
$N_3(t)$	Number of random shocks that have arrived in the safety zone by time t
H	Hard failure threshold due to damage shocks
$\gamma_a, \gamma_b, \gamma_c, \gamma_d$	Dependence factors

1. Introduction

Industrial components, systems and products are subject to multiple degradation processes and failure modes, i.e. wear, corrosion, fracture, fatigue, and etc. [1, 2]. Such processes are often "competing" in leading to failure [3] and may be dependent on each other in a number of ways, in which case they are referred to as dependent competing failure processes (DCFPs) [4]. Various dependencies have been discussed in literature, e.g., the statistical dependency described by joint probability distributions [5, 6], copulas [7, 8], and the functional dependency described by failure propagation and isolations [9-11], etc. In this paper, we consider the dependency between two types of failure processes: degradation failure processes (or soft failures) and catastrophic failure processes (or hard failures) [12]. In practice, soft failures are modeled by an evolving degradation process and hard failures are modeled by a random shock process. Models that consider both degradation and random shocks are called Degradation-Threshold-Shock (DTS) models [13, 14]. In literature, most DCFPs are modeled by DTS models.

An early DTS model was developed by Lemoine and Wenocur [15], where degradation is modeled by a diffusion process and fatal shocks are modeled by a Poisson process. Soft failures are defined with respect to the degradation reaching pre-established critical values and hard failures arrive in correspondence of the occurrence of fatal shocks. Klutke and Yang [16] have developed an availability model for systems subject to degradations and random shocks, where degradations occur at a constant rate and shocks follow a Poisson process. Marseguerra et al. [17] and Barata et al. [18] investigate condition-based maintenance optimizations for deteriorating systems subject to degradations and random shocks. Li and Pham [12, 19] develop a reliability model and an maintenance model for multistate degraded systems considering two degradation processes and a shock process. In a recent paper by Lin et al. [20], a DTS model has been developed, where the degradation process is modeled by a continuous-time semi Markov process and the shock process is modeled using a homogeneous Poisson process.

To consider the dependence between degradations and shocks, Peng et al. [4] assume that the arriving shock would bring a sudden increase to the normal degradation process and develop a probabilistic model to calculate the system's reliability. Wang and Pham [21] use a similar approach to model DCFPs and determine the optimal imperfect preventive maintenance policy. Keedy and Feng [22] apply the model in [4] on a stent, where the degradation process is modeled by a PoF (Physics of failures) model. Song et al. [23] consider the reliability of a system whose components are subject to the DCFPs and distinct component shock sets [24].

Apart from the dependence considered in [4], Jiang et al. [25] consider a different kind of dependence, where the threshold of the degradation process is shifted by shocks. Rafiee et al. [26] develop a model in which the degradation rates are shifted by different shock patterns. Wang et al. [27] explore another type of dependence by assuming that a sudden change to the failure rate is caused by the arriving shocks. Jiang et al. [1] consider dependence where shocks are divided into different zones, where each different zone has a different effect on the degradation process.

As reviewed above, the majority of existing DCFP models focuses on modeling the influence of random shocks on degradations (shock-degradation dependence). In engineering practice, however, there are components, systems and products that are subject to DCFPs in which random shocks are influenced by degradations (degradation-shock dependence). A typical engineering example is the sliding spool, which is a widely applied component in hydraulic control systems [28]. A sliding spool is subject to two failure processes: wear, which is modeled as a degradation process and clamping stagnation, which is modeled by random shocks [29]. As shown in Sasak and Yamamot [30], the process of clamping stagnation is dependent on the wear process: as the wear process progresses, more wear debris is generated. The debris will contaminate the hydraulic oil and further, increase the likelihood of clamping stagnation [31, 32], which causes the degradation-shock dependence.

Compared to shock-degradation dependence, there are fewer models considering the degradation-shock dependence. In [33], Fan et al. consider the degradation-shock dependence by assuming that the magnitude of random shocks is increased by the degradation processes. Bagdonavicius et al. [34] have developed a model to consider the degradation-shock dependence among a linear degradation process and several extreme shocks. In their model, the intensities of the random shocks are assumed to depend only on the degradation level. Ye et al. [35] model the degradation-shock dependence by assuming that the destructive probability of a shock is determined by the remaining hazard of the degradation process. In [13, 36, 37], the intensity of the random shock is assumed to be a piecewise function of the degradation level.

The aforementioned models only consider a relatively simple type of random shock: the extreme shocks (also referred to as traumatic shocks), where the arrival of a shock in the critical region causes the immediate failure of the product [38]. In practice, however, more complex shock patterns, such as cumulative shocks [39], δ-shocks [38], mixed shocks [40], run shocks [41], are frequently encountered. Since the 1990s, works on imperfect fault coverage models [42-44] categorize the effects of a component fault into three types: single-point failure, which causes immediate system failure, permanent coverage, which causes cumulative damages to the system and transient restoration, which has no effect on the system. Based on a similar consideration, Jiang et al. discuss a shock pattern called zone shocks, where shocks with different magnitudes might have different impacts on the failure behavior of the product [1]. Zhou [45] reports an example of the zone shocks based on experimental results of a sliding spool: the sliding spools exhibit different failure behaviors when they are tested under different levels of shock magnitudes (measured by sizes of wear debris). To the best of our knowledge, the zone shocks process has only been considered in the shock-degradation dependence model [1]. However, after a careful literature review, we find that there are no existing models that consider both the degradation-shock dependence and the zone shocks.

In this paper, we develop a new DCFP model which considers degradation-shock dependence and zone shocks. The developed model contributes to the existing scientific literature and to engineering practice in two aspects:

- it allows for explicit modeling of the degradation-shock dependence, where random shocks are influenced by degradation processes and
- it allows for the consideration of zone shocks in the degradation-shock dependence model, where shocks with different magnitudes have different impacts on the components failure behaviors.

The remainder of the paper is organized as follows. Sect. 2 introduces the engineering motivation of the present research. In Sect. 3, the new reliability model is developed to describe failure behaviors of systems subject to competing soft failures and hard failures involving degradation-shock dependence. In Sect. 4, the developed model is applied to model the dependent failure behavior of a sliding spool subject to wear and clamping stagnation. Finally, the paper is concluded with a discussion on potential future works in Sect. 5.

2. Engineering problem setting

The model developed in this paper is motivated by the actual engineering problem of modeling the reliability of a sliding spool. Sliding spools are critical control components in hydraulic control systems [28]. As illustrated in Figure 1, a sliding spool is composed of a spool and a sleeve, where the spool slides in the sleeve to control hydraulic oil flows [46].

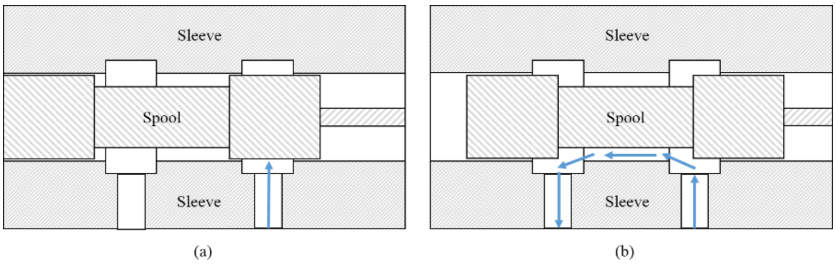
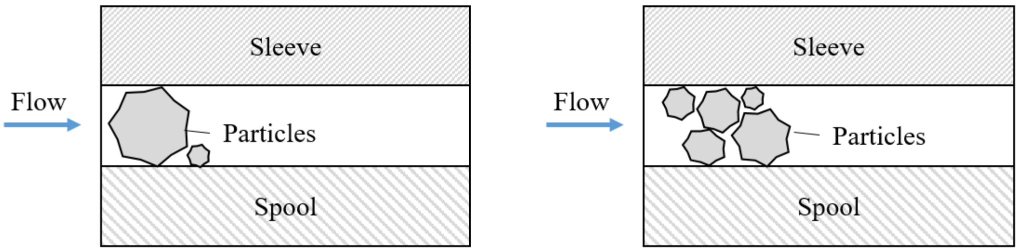


Figure 1 Illustration to a sliding spool: (a) Closed position; (b) Open position [46]

A sliding spool is subject to two failure mechanisms [29]. One is wear between the spool and the sleeve, and the other is clamping stagnation (also referred to as hydraulic locking or sticking), in which the spool is stuck in the sleeve. In practice, wear can be modeled by a physical deterioration model, for example, the linear Archard model suggested by Liao [29]. The modeling of clamping stagnation, however, is more complex because the factors contributing to clamping stagnation are much more varied. According to the survey by Sasak and Yamamot [30], one of the major causes of clamping stagnation is the sudden appearance of pollutant in the hydraulic oil, which can be modeled by a random shock model. The pollutant may come inside the hydraulic system from the outside environment or be generated by the hydraulic system itself. One significant source of internal pollutant is the wear debris generated due to the wear of the sliding spool. Therefore, the random shock process is dependent on the wear process: As the wear process progresses, more wear debris is generated. The debris will contaminate the hydraulic oil and further, increase the likelihood of clamping stagnation [31, 32]. This kind of dependence, caused by the influence of degradation on shocks, needs to be considered when developing the reliability model of the spool valve.

Furthermore, experimental results show that the most harmful effect of shocks is generated by wear debris, whose sizes are either close to or much smaller than the clearance of the sliding spool [30, 45, 47]. This is because the clamping stagnation is caused by two failure mechanisms, immediate stagnation and cumulative stagnation [45]. When a particle with a size close to the clearance is generated, it causes immediate stagnation of the sliding spool [45], as shown in Figure 2 (a). If particle sizes are smaller than the clearance, the particles can enter the clearance with the hydraulic oil and form filer cakes cumulatively [48]. When the filter cakes become large enough, cumulative stagnation occurs, as shown in Figure 2 (b). According to Zhou [45], particles whose sizes are greater than the clearance have little effect on clamping stagnation, because they are blocked outside the clearance. Hence, based on their magnitudes, the random shocks affecting clamping stagnation can be classified into three zones with different effects on the clamping stagnation. In other words, the sliding spool is subject to zone shocks.



(a) Immediate stagnation (b) Cumulative stagnation

Figure 2 Two failure mechanisms leading to clamping stagnation [45]

Hence, to accurately describe the failure behavior of components like the sliding spool, a new model for DCFPs is needed, which allows for: (1) the consideration of the degradation-shock dependence, where random shocks are influenced by degradation processes and (2) the modeling of zone shocks, where random shocks with different magnitudes have different effects on the products' failure behaviors. This is the motivation of the new DCFP model, which will be developed in Sect. 3.

3. DCFP model considering degradation-shock dependence

3.1 System descriptions

As shown in Figure 3, we consider a DCFP that comprises two failure processes: soft failures due to a degradation process (Figure 3 (a)) and hard failures due to random shocks. According to their magnitudes, random shocks are divided into three zones: damage zone, fatal zone and safety zone [1]. As shown in Figure 3 (b-1), a shock in the damage zone generates a cumulative damage to the system; as shown in Figure 3 (b-2), a shock in the fatal zone causes immediate failure of the system; a shock in the safety zone has no effect on the system's failure behavior. Note that the three shock zones in Figure 3 are shown for illustrative purposes only. In different applications, case-specific shock zones can be defined based on the magnitude of the shocks.

Degradation-shock dependence exists among the failure processes, that is, the arrival rate of the random shocks is dependent on the degradation levels, as illustrated in Figure 3. Failures occur whenever one of the three events happens:

- the degradation process reaches its threshold (denoted by T_1 in Figure 3);
- the cumulative damage resulting from the shocks in the damage zone exceeds its threshold (denoted by T_2 in Figure 3);
- a shock in the fatal zone occurs (denoted by T_3 in Figure 3).

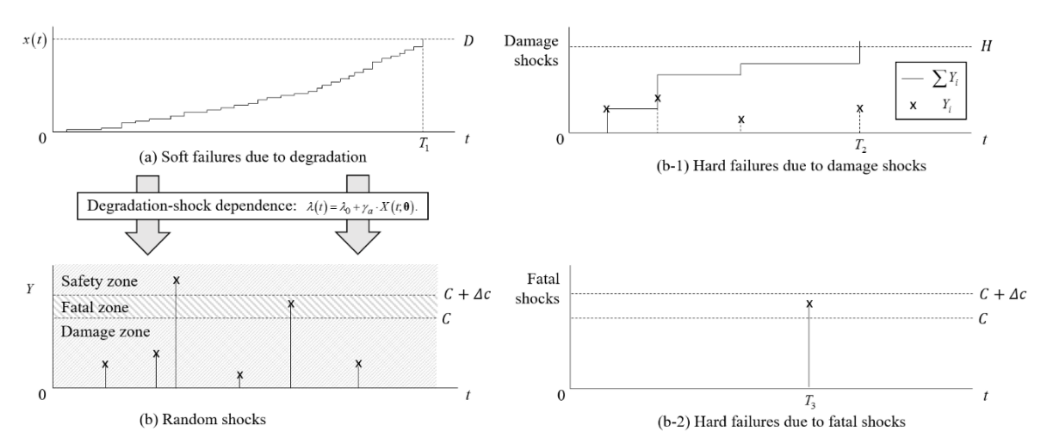


Figure 3 Soft failures due to degradation and hard failures due to random shocks

Additional assumptions of this model include:

(1) The degradation process can be modeled by the general path model proposed by Lu and Meeker [49]:

$$x(t) = X(t;\boldsymbol{\theta}), \tag{1}$$

where $x(t)$ is the degradation measure and $X(\cdot)$ is the deterministic degradation path, which can be obtained from knowledge of physics of failure [50]. Vector $\boldsymbol{\theta}$ in (1) contains a set of random parameters that represent the influence of uncertainties. It should be noted that the general path model in (1) is a general degradation model, for each specific case, the form of $X(\cdot)$ in (1) could be specified to describe the case-specific degradation mechanism.

(2) The random shocks can be modeled by a nonhomogeneous Poisson process (NHPP) with intensity function $\lambda(t)$, so that the arrival shocks prior to t, denoted by $N(t)$, follow a Poisson distribution with mean $a(t) = \int_0^t \lambda(u)du$ [51]:

$$P(N(t) = n) = \frac{e^{-a(t)}a(t)^n}{n!}. \tag{2}$$

(3) The magnitude of each shock, denoted by Y_i, is an independent and identically distributed (i.i.d.) random variable.

(4) The damage caused by the *i*th shock in damage zone, denoted by W_i, is proportional to its magnitude Y_i with coefficient α:

$$W_i = \alpha Y_i. \tag{3}$$

(5) To consider the degradation-shock dependence, we introduce a dependence factor γ_a and assume:

$$\lambda(t) = \lambda_0 + \gamma_a \cdot X(t;\boldsymbol{\theta}). \tag{4}$$

Remark 1: the description of the shock damage in the form of equation (3) belongs to a type of commonly used damage model, which assumes that the shock damage is linearly dependent on the shock magnitude. Similar damage models can be found in [1, 52].

Remark 2: the assumption that the intensity of the NHPP is a linear function of the degradation level is adapted from the assumption proposed by Mercer [53]. In Mercer's model, the traumatic failure intensity is of the form $\lambda(t) + c \cdot Z(t)$, where $Z(t)$ is a degradation function describing the piecewise constant degradation process. Also, note that λ_0 in (4) denotes the initial intensity of the NHPP.

3.2 Modeling zone shocks

Once a random shock arrives, we should first classify it into one of the three zones according to its magnitude. Let P_1, P_2, P_3 denote the probabilities that the i th shock belongs to the damage, fatal and safety zone, respectively. It is clear that $P_1 + P_2 + P_3 = 1$. Because the magnitude of each shock is an i.i.d. random variable, the value of P_1, P_2 and P_3 can be easily determined from the distribution of Y_i.

Let us define $N(t)$ as the number of shocks which has arrived prior to time t and $N_1(t), N_2(t), N_3(t)$ be the number of shocks that have fallen into the damage, fatal and safety zone prior to time t, respectively. According to [54], if a NHPP with intensity $\lambda(t)$ is randomly split into two sub-processes with probabilities P_1 and P_2, where $P_1 + P_2 = 1$, then, the resulting sub-processes are independent NHPPs with intensities $P_1\lambda(t)$ and $P_2\lambda(t)$. The conclusion can be easily generalized to the three-sub-processes case, which leads to the following proposition:

Proposition 1: The number of shocks falling into damage and fatal zones, denoted by $N_1(t)$ and $N_2(t)$ respectively, follow NHPPs with intensity functions $P_1\lambda(t)$ and $P_2\lambda(t)$.

From Proposition 1, the probability mass function of $N_1(t)$ and $N_2(t)$ can be derived:

$$P\left(N_1(t) = n\right) = \frac{e^{-P_1 a(t)}\left(P_1 a(t)\right)^n}{n!}, \qquad P\left(N_2(t) = n\right) = \frac{e^{-P_2 a(t)}\left(P_2 a(t)\right)^n}{n!}, \tag{5}$$

where $a(t) = \int_0^t \lambda(u)du$. Proposition 1 and equation (5) will be used in the next section to develop the reliability model.

3.3 Reliability modeling

It can be seen from Figure 3 that a reliable system at time t should satisfy the following conditions: (1) the degradation process does not exceed its threshold D before t; (2) no shocks in the fatal zone occur before t and (3) the cumulative damage resulting from shocks in the damage zone does not exceed its threshold H before t. Therefore, we can express the reliability at time t as:

$$R(t) = \sum_{i=0}^{\infty} P\left(X(t;\boldsymbol{\theta}) < D, \sum_{j=1}^{N_1(t)} W_i < H, N_2(t) = 0 \middle| N_1(t) = i \right) \cdot P\left(N_1(t) = i\right). \tag{6}$$

Equation (6) comes from a direct application of the total probability formula.

For ease of illustration, we define three events E_1, E_2, E_3 and let them denote the events $X(t;\boldsymbol{\theta}) < D, \sum_{j=1}^{N_1(t)} W_i < H, N_2(t) = 0$, respectively. Because the degradation-shock dependence is considered (see (4)), E_1, E_2, E_3 are dependent. However, once the value of $\boldsymbol{\theta}$ is fixed, the three events become conditional-independent [50]. Therefore, we can apply the total probability formula on $\boldsymbol{\theta}$ in (6), which leads to:

$$R(t) = \sum_{i=0}^{\infty} \int_{\boldsymbol{\theta}} P\left(E_1, E_2, E_3 \middle| N_1(t) = i, \boldsymbol{\theta} = \mathbf{x}\right) \cdot P\left(N_1(t) = i \middle| \boldsymbol{\theta} = \mathbf{x}\right) \cdot f_{\boldsymbol{\theta}}(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{7}$$

where $f_{\boldsymbol{\theta}}(\mathbf{x})$ is the probability density function of $\boldsymbol{\theta}$. Considering the fact that E_1, E_2, E_3 are conditional-independent given $\boldsymbol{\theta}$, (7) becomes:

$$R(t) = \sum_{i=0}^{\infty} \int_{\boldsymbol{\theta}} P\left(E_1 \middle| A, B\right) \cdot P\left(E_2 \middle| A, B\right) \cdot P\left(E_3 \middle| A, B\right) \cdot P\left(A \middle| B\right) \cdot f_{\boldsymbol{\theta}}(\mathbf{x}) \mathrm{d}\mathbf{x}, \tag{8}$$

where event A denotes $N_1(t) = i$ and event B denotes $\boldsymbol{\theta} = \mathbf{x}$. To use (8) to calculate the reliability, expressions for $P\left(E_1 \middle| A, B\right)$, $P\left(E_2 \middle| A, B\right)$, $P\left(E_3 \middle| A, B\right)$ as well as $P\left(A \middle| B\right)$ should be determined.

To determine $P\left(E_1 \middle| A, B\right)$, first note that it denotes the probability that $X(t;\boldsymbol{\theta}) < D$ given that $\boldsymbol{\theta} = \mathbf{x}$. Because $X(\cdot)$ is a deterministic degradation path model and $\boldsymbol{\theta}$ is the only random variable [48], we have

$$P\left(E_1 \middle| A, B\right) = \begin{cases} 1 & \text{if } X(t;\mathbf{x}) < D, \\ 0 & \text{if } X(t;\mathbf{x}) \geq D. \end{cases} \tag{9}$$

The probability $P\left(E_2 \middle| A, B\right)$ is expressed as

$$P\left(E_2 \middle| A, B\right) = P\left(\sum_{j=1}^{N_1(t)} W_i < H \middle| N_1(t) = i, \boldsymbol{\theta} = \mathbf{x}\right), \tag{10}$$

which can be further expanded once the distribution of Y_i is determined.

The probability $P\left(E_3 \middle| A, B\right)$ is the probability that no fatal shocks occur given $\boldsymbol{\theta} = \mathbf{x}$ and $N_1(t) = i$. From equation (4) and Proposition 1, we have

$$P\left(E_3 \middle| A, B\right) = \exp\left\{-P_2 \int_0^t \left(\lambda_0 + \gamma_a \cdot X(u;\mathbf{x})\right) \mathrm{d}u\right\}, \tag{11}$$

where P_2 is the probability that the arrival shock belongs to the fatal zone.

Finally, from (5), $P\left(A \middle| B\right)$ is given by:

$$P\left(A|B\right)=\frac{e^{-P_1\Lambda(t;\mathbf{x})}\left(P_1\Lambda\left(t;\mathbf{x}\right)\right)^i}{i!}, \tag{12}$$

where P_1 is the probability that the arrival shock belongs to the damage zone and $\Lambda\left(t;\mathbf{x}\right)$ is given by:

$$\Lambda\left(t;\mathbf{x}\right)=\int_0^t\left(\lambda_0+\gamma_a\cdot X\left(u;\mathbf{x}\right)\right)\mathrm{d}u. \tag{13}$$

The reliability model developed in (8) is an extension to the model in [1], obtained by considering the degradation-shock dependence described by (4). If we let $\gamma_a=0$, which indicates that there is no dependency between the degradation process and the shock process, the reliability model in (8) degenerates to

$$R\left(t\right)=\sum_{i=0}^{\infty}P\left(X\left(t\right)<D,\sum_{j=1}^{N_1(t)}W_i<H\right)\frac{e^{-\lambda(P_1+P_2)t}\left(\lambda P_1 t\right)^i}{i!}. \tag{14}$$

Equation (14) is in accordance with the model developed in [1], where $P\left(X\left(t\right)<D,\sum_{j=1}^{N_1(t)}W_i<H\right)$ corresponds to the $\int_0^H f_{X_s}\left(x_S\,|\,t,n\right)\mathrm{d}x_s$ in [1].

3.4 Reliability model quantification by Monte Carlo simulation

Due to the complexity of (8), the analytical solution of the reliability is hard to obtain. A numerical approach based on Monte Carlo simulation is, then, developed to compute the system reliability, by the following pseudo-code:

Algorithm 1 Monte Carlo simulation for DCFP model considering degradation-shock dependence

Set t, M, n.

For $j=1:\mathrm{n}$,

Sample a $\boldsymbol{\theta}_j$ from $F_{\boldsymbol{\theta}}$.

Calculate $x(t)$ by (1).

Calculate $P^j\left(E_1|A,B\right)$ and $P^j\left(E_3|A,B\right)$ by (9) and (11), respectively.

Sample a set of Y, $\left\{Y_1,Y_2,\ldots,Y_M\right\}$ from F_Y. Pick those belonging to the damage zone and reassemble them as a new set $\left\{Y_1,Y_2,\ldots,Y_m\right\}$.

For $i=1:m$,

Calculate $P_i^j\left(A|B\right)$ by (12) and (13).

Calculate $\sum_{Y_k\in\{Y_1,\ldots,Y_i\}}\alpha Y_k$ by adding the first i elements in $\left\{Y_1,Y_2,\ldots,Y_m\right\}$.

Calculate $P_i^j\left(E_2|A,B\right)$ by (15):

$$P\left(E_2 \middle| A,B\right)=\begin{cases}1 & \text{if} \sum_{Y_k\in\{Y_1,\dots,Y_i\}} \alpha Y_k < H,\\ 0 & \text{if} \sum_{Y_k\in\{Y_1,\dots,Y_i\}} \alpha Y_k \geq H.\end{cases} \tag{15}$$

End For

Calculate $R_j(t)$ by (16):

$$R_j(t)=\sum_{i=0}^{m} P^j\left(E_1 \middle| A,B\right) P_i^j\left(E_2 \middle| A,B\right) P^j\left(E_3 \middle| A,B\right) P_i^j\left(A \middle| B\right). \tag{16}$$

End For

Calculate $R(t)$ by (17):

$$R(t)=\frac{1}{n}\sum_{j=1}^{n} R_j(t). \tag{17}$$

The computational complexity of the simulation method is $O(M\times n\times n_t)\times O_{obj}$, where n_t is the evaluated number of t, O_{obj} is the computational complexity of each evaluation of the objective function, i.e., the $P^j\left(E_1 \middle| A,B\right) P_i^j\left(E_2 \middle| A,B\right) P^j\left(E_3 \middle| A,B\right) P_i^j\left(A \middle| B\right)$ in (16).

To implement the algorithm in practice, the values for M, n and n_t need to be set. The values of M and n determine the accuracy of the reliability estimation at each time point t. Therefore, they should be determined based on the requirements on the confidence bands. It is well known that the error of Monte Carlo simulation is proportional to $1/\sqrt{M\times n}$. Therefore, the appropriate sample sizes could be determined by trial-and-error: determine an initial sample size and check if the confidence band satisfies the requirement; then, adjust the sample sizes based on the proportional relation between sample sizes and simulation errors. The value of n_t, reflects the closeness of the point wise approximation values of the reliability function, output by the algorithm, to the true values: the larger n_t, the better the approximation. Proper balance is needed between the accuracy of the approximation and the computational complexity of the algorithm.

4. Applications

4.1 Model development

In this section, the developed DCFP model is applied to describe the failure behavior of a sliding spool. The case study is artificially constructed, based on a realistic physical background, to illustrate the proposed modeling framework. As discussed in Sect. 2, the sliding spool is subject to two dependent failure processes, wear and clamping stagnation. According to a study of physics-of-failures [29], the wear of the sliding spool increases smoothly for most of its lifetime, and therefore the wear process can be modeled by a linear model:

$$x(t) = X(t;\varphi,\beta) = \varphi + \beta t, \tag{18}$$

where the initial value φ is a constant and the degradation rate β is assumed to be a normal random variable.

Based on the analysis in Sect. 2, clamping stagnation is modeled by the zone shocks discussed in Sect. 3.2. The arrival of the particles is assumed to follow a nonhomogeneous Poisson process with intensity function $\lambda(t)$. The size of the i th particle getting into the clearance of the sliding spool, denoted by Y_i, is assumed to be an i.i.d. random variable with a probability distribution $f_Y(y)$. Based on the failure mechanisms of clamping stagnation discussed in Figure 2, Y_i is classified into three zones according to its magnitude:

- if $Y_i < C$, the shock belongs to the damage zone;
- if $C \le Y_i < C + \Delta C$, the shock belongs to the fatal zone;
- if $Y_i \ge C + \Delta C$, the shock belongs to the safety zone,

where C is the clearance of the sliding spool and ΔC is the critical particle size. The probabilities that the i th shock belongs to the damage, fatal and safety zones, denoted by P_1, P_2, P_3, respectively, are calculated by:

$$\begin{aligned} P_1 &= \int_0^C f_Y(y)\,dy, \\ P_2 &= \int_C^{C+\Delta c} f_Y(y)\,dy, \\ P_3 &= \int_{C+\Delta c}^{\infty} f_Y(y)\,dy. \end{aligned} \tag{19}$$

Moreover, degradation-shock dependence exists between the wear and the clamping stagnation, because wear processes generate wear debris that accelerates the arrival rate of the particles which cause the clamping stagnation [30, 31]. In our developed model, the degradation-shock dependence is modeled by (4). Because the wear process is modeled by (18), (4) becomes:

$$\lambda(t) = \lambda_0 + \gamma_a \cdot (\varphi + \beta t). \tag{20}$$

Substituting (18)-(20) into (8), the reliability of the sliding spool under the two dependent failure processes can be calculated.

4.2 Reliability evaluation

The numerical algorithm presented in Sect. 3.4 is used to evaluate the reliability of the sliding spool based on the developed model. In this application, the values for the model parameters are estimated by domain experts. For illustrative purposes, we assume the expert estimations are accurate and given in Table 1. In practice, parameters related to degradation processes and random shocks, for example, the φ and β in Table 1, could be estimated from test data.

Table 1 Model parameters of the reliability model for the sliding spool

Parameter/distribution	Description	Values
φ	The initial wear magnitude	0 mm
β	The wear rate	$N(1 \times 10^{-4}, 1 \times 10^{-5})$ mm/s
D	The wear threshold	5 mm
$F_Y(y)$	The CDF for particle diameters	$N(1.2, 0.2)$
α	The shock damage proportional coefficient	1
H	The cumulative clamping stagnation threshold	7.5 mm
$[C, C+\Delta c]$	The range for fatal shocks	[1.5,1.6]mm
λ_0	The initial intensity of random shocks	2.5×10^{-5}/s
Parameters for simulation	$n = 1000, M = 200, n_t = 1000, t = \left[0, 6 \times 10^4\right] s$	
Dependence factors	$\gamma_a = 0, 10^{-5}, 10^{-4}, 10^{-3}$	

4.3 Results and discussions

We compute the reliability of the sliding spool under different levels of dependence ($\gamma_a = 0, 1\times10^{-5}, 1\times10^{-4}, 1\times10^{-3}$), to investigate the failure behavior of the sliding spool. Results are given in Figure 4. In general, the reliability of the sliding spool increases with the decrease of γ_a. As expected, when $\gamma_a = 0$, indicating that there is no dependence between the two failure processes, $R(t)$ takes the largest values. This is mainly due to the dependence mechanism between wear and clamping stagnation. As the wear process progresses, more wear debris is generated and, the arrival rate of particles causing clamping stagnation increases. In this way, clamping stagnation is aggravated by wear. Therefore, positive dependence exists between the two failure mechanisms, as described by (20). Proactively, this suggests that if we can mitigate the dependence between the two failure processes, the reliability of the sliding spool could be enhanced. In practice, filters can be installed to stop the debris generated by wear processes from entering the hydraulic oil. In this way, the dependence between wear and clamping stagnation can be controlled, so that the reliability of the sliding spool can be increased.

Furthermore, it can be seen from Figure 4 that when γ_a is relatively small, the reliability changes slowly at first, then declines sharply at about $4\times10^4 s$. As γ_a increases, the reliability curve decreases more smoothly. When γ_a becomes relatively large ($\gamma_a = 10^{-3}$), the declination of the reliability curve becomes more rapid. The main physical and mathematical reasons of the phenomenon are:

(1) A small γ_a indicates that the two failure mechanisms are almost independent. The sliding spool would experience two distinct failure behaviors. When t is small, because the wear process is far from its failure threshold, failures due to wear seldom occur. Hence, the failure behavior is primarily dominated by clamping stagnation. Because the rate of the clamping stagnation is relatively small, the sliding spool will have high reliability, which is indicated by the first part of the reliability curve.

As t increases, the wear level approaches its threshold. Therefore, failures are more likely to be caused by the wear processes. On the other hand, because γ_a is small, from (4), the rate of the clamping stagnation remains almost unchanged. Hence, when t is large, the failure behavior of the spool is primarily determined by wear, which leads to the second part of the reliability curve starting at around $t = 4\times10^4 s$.

(2) When γ_a is relatively large, the wear process has considerable influence on the shock process. Even when t is small, the failure probability of the sliding spool is relatively high due to the positive dependence between the two failure processes. Hence, the difference between the two failure behaviors in the independent case becomes small. Therefore, the reliability curve declines more smoothly comparing to the independent case.

(3) When γ_a becomes even larger, the reliability of the sliding spool decreases rapidly with time. This is because strong dependence exists between wear and clamping stagnation. Therefore, even at the early stages of the spool's life cycle, a lot of spools might fail due to the high risks of clamping stagnation, which results from the strong positive dependence on the wear process.

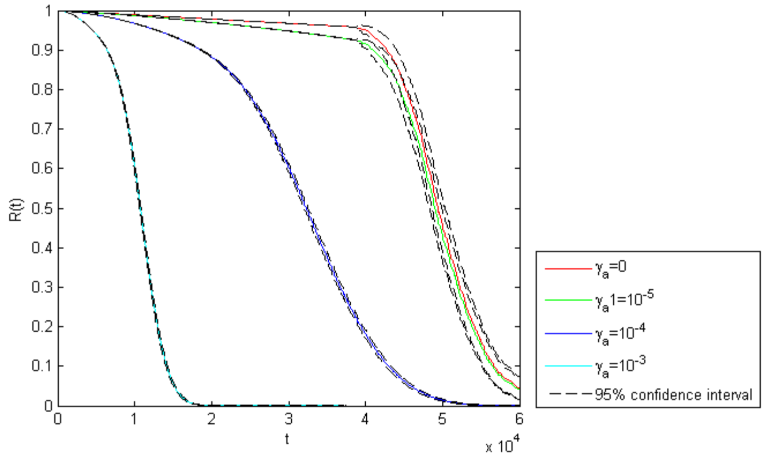


Figure 4 Reliability under different values of the dependence factor, γ_a

4.4 Some extensions

In real cases, the dependency behaviors might be more complicated. For example, as the debris accumulates, more wear and debris are expected. As a consequence, the intensity of the shock arrival process might not only depend on the degradation level, but also on the accumulated shock damage $\sum W$. Furthermore, the size of the wear debris might also be affected by the degradation levels. In addition, the degradation rate also depends on the accumulated shock damage, i.e., the shock-degradation dependence. In this section, we show how to extend the developed model to capture more complex dependency behaviors.

Except for the assumptions we made in Sect. 4.1, three additional assumptions are made:

- the intensity function of the arrival process of the wear debris, i.e., the $\lambda(t)$ in (20), depends on the degradation level $x(t)$ and the current cumulative shock damage $\sum W$:

$$\lambda(t)=\lambda_0+\gamma_a x(t)+\gamma_b \sum W, \tag{21}$$

  where γ_a, γ_b are dependence factors and λ_0 is the initial intensity of the arrival process.

- the mean of the debris size distribution $F_Y(y)$, denoted by $\mu_Y(t)$, is affected by the degradation level $x(t)$:

$$\mu_Y(t)=\mu_{Y_0}+\gamma_c \cdot x(t), \tag{22}$$

  where γ_c is the dependence factor, and μ_{Y_0} is the initial value of $\mu_Y(t)$.

- the degradation rate, i.e., the β in (18), depends on the current cumulative shock damage $\sum W$:

$$\beta = \beta_0 + \gamma_d \cdot \sum W \tag{23}$$

where γ_d is the dependence factor.

The first two assumptions consider degradation-shock dependence, where the intensity of the shock arrival process and the distribution of shock magnitudes are both influenced by the degradation process. The last assumption describes shock-degradation dependence, where the degradation rate varies with the accumulated damages of the random shocks.

Monte Carlo method (with a sample size of 10^4) is used to investigate the effect of γ_b, γ_c and γ_d on the spool valve's reliability. The detailed procedures are given in the Appendix. As shown in Figure 5, when γ_b varies from 0 to 10^{-3}, the predicted reliability decreases significantly. This can be explained by the fact that γ_b measures the effect of cumulative shock damage $\sum W$ on the intensity function of the arrival process of the wear debris, as shown in (21), and on the system's degradation rate, as shown in (23). As $\sum W$ increases, on one hand, more shocks occur, which increases the probability of a hard failure; on the other hand, the system degrades more rapidly, which accelerates the arrival of a soft failure. Therefore, γ_b might has a significant effect on the system reliability.

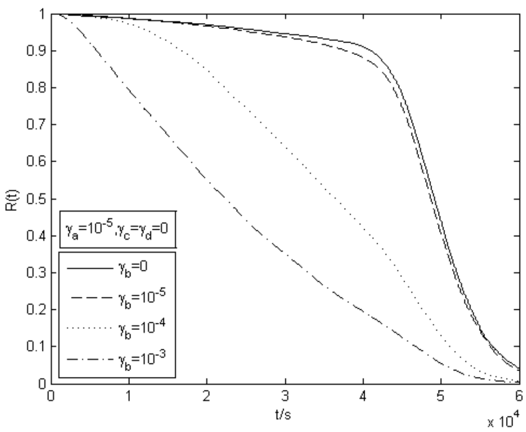


Figure 5 Effect and sensitivity of γ_b on the predicted reliability

As shown in Figure 6, when γ_c varies from 0 to 10^{-3}, the predicted reliability remains almost unchanged. This can be explained by the fact that γ_c measures the effect of the degradation magnitude $x(t)$ on the mean of the debris size distribution $\mu_Y(t)$, as shown in (22). In this case, the initial value of $\mu_Y(t)$ was in the damage shock zone; as $x(t)$ increases, $\mu_Y(t)$ moves first into the fatal shock zone, and then into the safety shock zone.

When $x(t)$ is relatively small, its effect on $\mu_Y(t)$ is negligible. However, when $x(t)$ is large enough, since $\mu_Y(t)$ has already entered the safety shock zone, most random shocks would also be in the safety shock zone. The failure process in this period is dominated by the degradation process. As a result, the system's reliability is not sensitive to γ_c and, therefore, the effect of γ_c can be ignored in modeling the sliding spool.

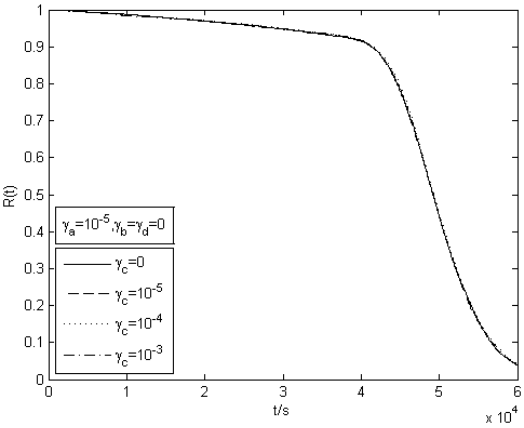


Figure 6 Effect and sensitivity of γ_c on the predicted reliability

As shown in Figure 7, when γ_d varies from 0 to 10^{-3}, the predicted reliability decreases significantly. This can be explained by the fact that γ_d measures the effect of cumulative shock damage $\sum W$ on the degradation rate β, as shown in (23). As γ_d increases, the degradation process becomes more rapid, according to (23), which, then, creates more random shocks due to the degradation-shock dependence (Eq. (20)). The system's reliability is significantly reduced by the mutually enhanced dependences.

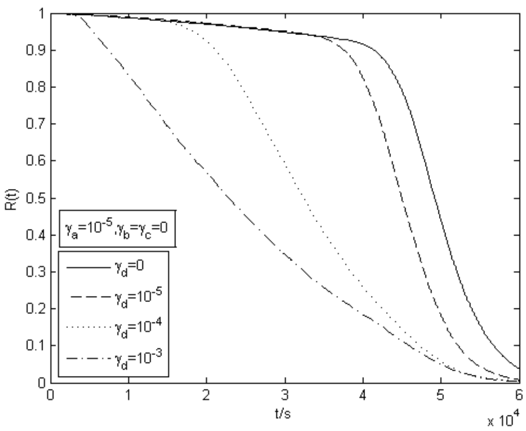


Figure 7 Effect and sensitivity of γ_d on the predicted reliability

5. Conclusions

In this paper, a new reliability model is developed for dependent competing failure processes. In the developed model, failure of a component could be caused by either “soft” failure (described by a degradation model) or “hard” failure (described by a shock model), and the degradation-shock dependence exists among the failure processes (where the intensity of the shock process is dependent on the degradation process and the effect of dependence is classified into different zones according to the magnitude of the random shocks). Since degradation-shock dependence is considered, the shock process is influenced by the degradation process but the degradation process is not influenced by the shock process. The developed model contributes to existing ones in two aspects. First, it considers a new type of dependence, the degradation-shock dependence, where random shocks are influenced by degradation processes. Second, it combines zone shocks into the degradation-shock dependence, where random shocks are classified into different zones according to their magnitudes and different zones have different effects on the failure behavior. The developed model is applied to describe the dependent failure behavior of an actual product, a sliding spool, which is subject to wear and clamping stagnation.

In the future, additional work will be done to further improve the developed model. First of all, in this paper we only consider a simple dependence scenario, where linear dependence exists between the intensity function and the degradation processes. In the future, the influence of other dependence relations, possibly nonlinear, will also be considered. Second, in this paper we only consider the degradation-shock dependence. In the future, the combination of two types of dependencies will be considered. Third, in this paper we assume that the parameters of the model are perfectly known. In practice, however, many times precise values of the parameters are not available due to limited data and knowledge. Hence, the model is subject to epistemic uncertainty, whose effect will also be studied.

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Appendix

Algorithm 2 Monte Carlo simulation for DCFP model considering multiple dependences in Sect. 4.4

Step 1: Set samplesize, $i=0, x=0, \sum W=0, \quad \tau=0$;

Step 2: If $i>\text{samplesize}$, stop;

else generate a β_s from F_β;

Step 3: Update the intensity function $\lambda(t)=\lambda_0+\gamma_a\left(x+\left(\beta_s+\gamma_d\cdot\sum W\right)(t-\tau)\right)+\gamma_b\sum W$;

Generate an arrival time $\Delta\tau$ from a NHPP with intensity function $\lambda(t)$ using thinning method [51]:

Set $T_{\max}$; Calculate $\lambda_M=\lambda(T_{max})$; $tt=\tau$;

While 1

$\Delta\tau \leftarrow$ Generate a random number from $\exp(\lambda_M)$; $tt=tt+\Delta\tau$;

If $tt<T_{\max}$, $p_{acc}=\lambda(tt)/\lambda_M$; Accept $\Delta\tau$ with probability p_{acc}; Return $\Delta\tau = tt-\tau$;

Else $tt=T_{\max}$; Increase the value of $T_{\max}$; Calculate $\lambda_M=\lambda(T_{max})$; Continue;

Endwhile

$\tau=\tau+\Delta\tau$;

Step 4: $x=x+\left(\beta_s+\gamma_d\cdot\sum W\right)\Delta\tau$;

If $x>D$, $i=i+1, t_f(i)=(D-x)/\left(\beta_s+\gamma_d\cdot\sum W\right)+\tau$, go to Step 2;

Else go to Step 5;

Step 5: $\mu_Y=\mu_{Y_0}+\gamma_c\cdot x$;

Generate $Y\sim Normal(\mu_Y,\sigma_Y^2)$;

If $Y<C$, go to Step 6;

Else if $Y<C+\Delta C$, $i=i+1, t_f(i)=\tau$, go to Step 2;

Step 6: $\sum W=\sum W+\alpha\cdot Y$;

If $\sum W\geq H$, $i=i+1, t_f(i)=\tau$, go to Step 2;

Else go to Step 3;