Markov Reliability and Availability Analysis
Part II: Continuous Time Discrete State Markov Processes
The stochastic process may be observed at:

- Discrete times
- Continuously

DISCRETE-TIME DISCRETE-STATE MARKOV PROCESSES

CONTINUOUS-TIME DISCRETE-STATE MARKOV PROCESS
The stochastic process is **observed continuously** and **transitions** are assumed to **occur continuously in time**.
The random process of system transition between states in time is described by a **stochastic process** \( \{X(t); \, t \geq 0\} \)

- \( X(t) := \) system state at time \( t \)
  - \( X(3.6) = 5 \): the system is in state number 5 at time \( t = 3.6 \)

**OBJECTIVE:**
Computing the **probability** that the system is in a **given state** as a **function of time**, for all possible states

\[
P[X(t) = j], \, t \in [0, T_m], \, j = 0, 1, \ldots, N
\]
Objective:

\[ P[X(t) = j], t \in [0, T_m], j = 0, 1, \ldots, N \]

What do we need?

Transition Probabilities!
• **Transition probability** that the system moves to state \( j \) at time \( t + \nu \) given that it is in state \( i \) at current time \( t \) and given the previous system history

\[
P[X(t+\nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t]
\]

\((i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)\)
The conceptual model: Markov Assumption

- **IN GENERAL STOCHASTIC PROCESSES:**
  the *probability* of a *future* state of the system usually depends on its *entire life history*

  \[ P[X(t + \nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t] \]

  \( i = 0, 1, \ldots, N, j = 0, 1, \ldots, N \)

- **IN MARKOV PROCESSES:**
  the *probability* of a *future* state of the system *only* depends on its *present state*

  \[ P[X(t + \nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t] = P[X(t + \nu) = j \mid X(t) = i] \]

  \( i = 0, 1, \ldots, N, j = 0, 1, \ldots, N \)

  **THE PROCESS HAS “NO MEMORY”**
If the **transition probability** depends on the **interval** \( \nu \) and **not** on the **individual times** \( t \) and \( t + \nu \)

- the transition probabilities are **stationary**
- the Markov process is **homogeneous** in time

\[
p_{ij}(t, t+\nu) = P[X(t+\nu) = j \mid X(t) = i] = p_{ij}(\nu)
\]
HYPOTHESIS:

- The time interval \( \nu = dt \) is **small** such that **only one** event (i.e., one stochastic transition) can occur within it

\[
p_{ij}(dt) = P[X(t + dt) = j|X(t) = i] = 1 - e^{-\alpha_{ij} \cdot dt}
\]

\[
= (\text{Taylor 1}^{st} \text{ order expansion})
\]

\[
\alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0
\]

\[
\alpha_{ij} = \text{transition rate} \text{ from state } i \text{ to state } j
\]
The conceptual model: The Transition Probability Matrix (1)

\[ p_{ij}(dt) = \alpha_{ij} \cdot dt + \theta(dt), \quad \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0 \]

- In analogy with the discrete-time formulation:

**Discrete-time transition probability matrix**

\[
A = 1 = \begin{pmatrix}
0 & 1 & \ldots & N \\
0 & p_{00} & p_{01} & \ldots & p_{0N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N & p_{N0} & p_{N1} & \ldots & p_{NN}
\end{pmatrix}
\]

**Continuous-time transition probability matrix**

\[
A = \begin{pmatrix}
1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \ldots & \alpha_{0N} \cdot dt \\
\alpha_{10} \cdot dt & 1 - dt \cdot \sum_{j=0}^{N} \alpha_{1j} & \ldots & \alpha_{1N} \cdot dt \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{N0} \cdot dt & \alpha_{N1} \cdot dt & \ldots & 1 - dt \cdot \sum_{j=N}^{N} \alpha_{Nj}
\end{pmatrix}
\]

\[ p_{ii}(dt) = 1 - \sum_{j \neq i} p_{ij}(dt) = 1 - dt \cdot \sum_{j \neq i} \alpha_{ij} + \theta(dt) \]
• **In analogy with the discrete-time formulation:**

\[
P(t + dt) = P(t) \cdot A
\]

\[
\begin{bmatrix}
P_0(t + dt)P_1(t + dt)\ldots P_N(t + dt)
\end{bmatrix}
\]

\[
= \begin{pmatrix}
1 - dt \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt \ldots & \alpha_{0N} \cdot dt \\
\alpha_{10} \cdot dt & 1 - dt \sum_{j=0}^{N} \alpha_{1j} \ldots & \alpha_{1N} \cdot dt \\
\ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

• **First-equation:**

\[
P_0(t + dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0j}\right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \ldots + \alpha_{N0} P_N(t) dt
\]
The conceptual model: the fundamental matrix equation (2)

\[ P_0(t + dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0,j} \right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \ldots + \alpha_{N0} P_N(t) \cdot dt \]

subtract \( P_0(t) \) on both sides

\[ P_0(t + dt) - P_0(t) = P_0(t) - P_0(t) - \sum_{j=1}^{N} \alpha_{0,j} P_0(t) dt + \alpha_{10} P_1(t) dt + \ldots + \alpha_{N0} P_N(t) dt \]

divide by \( dt \)

\[ \frac{P_0(t + dt) - P_0(t)}{dt} = -\sum_{j=1}^{N} \alpha_{0,j} P_0(t) + \alpha_{10} P_1(t) + \ldots + \alpha_{N0} P_N(t) \]

let \( dt \to 0 \)

\[ \lim_{dt \to 0} \frac{P_0(t + dt) - P_0(t)}{dt} = \frac{dP_0}{dt} = -\sum_{j=1}^{N} \alpha_{0,j} \cdot P_0(t) + \alpha_{10} \cdot P_1(t) + \ldots + \alpha_{N0} \cdot P_N(t) \]
The conceptual model: The Transition Probability Matrix

- Extending to the other equations:

\[
\frac{d \mathbf{P}}{dt} = \mathbf{P}(t) \cdot \mathbf{A}^*, \quad \mathbf{A}^* = \begin{pmatrix}
    \alpha_{00} & \alpha_{01} & \ldots & \alpha_{0N} \\
    \alpha_{10} & -\sum_{j=1}^{N} \alpha_{1j} & \ldots & \alpha_{1N} \\
    \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

TRANSITION RATE MATRIX

It will be indicated as \( \mathbf{A} \)

System of linear, first-order differential equations in the unknown state probabilities

\[
P_j(t), \quad j = 0, 1, 2, \ldots, N, \quad t \geq 0
\]
Exercise 1

Consider a system made by one component which can be in two states: working or failed. Assume constant failure rate $\lambda$ and constant repair rate $\mu$. You are required to:

• Draw the Markov diagram
• Find the transition rate matrix, $A$

\[
A = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}
\]
Exercise 1 (Solution)

Discrete states = 0 → component working
1 → component failed

Transition rates = \( \lambda \rightarrow \) rate of failure (i.e., from 0 to 1)
= \( \mu \rightarrow \) rate of repair (i.e., from 1 to 0)

\[
A = \begin{pmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{pmatrix}
\]
Consider a system made by $N$ identical components which can be in two states: working or failed. Assume constant failure rate $\lambda$, that $N$ repairman are available and that the single component repair rate is constant and equal to $\mu$. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix, $A$
Exercise 2 - Solution

Consider a system made by $N$ identical components which can be in two states: working or failed. Assume constant failure rate $\lambda$, that $N$ repairmen are available and that the component repair rate is constant and equal to $\mu$. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix

-System discrete states:
  - State 0: none failed, all components function
  - State 1: one component failed, $N$-1 function
  - State 2: two components failed, $N$-2 function
  - ...
  - State $N$: all components failed, none function
**HYPOTHESES:**
- one event (failure or repair of one component) can occur in the small $\Delta t$
- the events are mutually exclusive

**Explanation:**
 Probability of transition $0 \rightarrow 1 =$

= probability \{ anyone of the $N$ components fails in $\Delta t$ \} =

= probability \{ component 1 fails in $\Delta t$ or component 2 fails in $\Delta t$ or ... \} =

= probability \{ component 1 fails in $\Delta t$ \} + probability \{ component 2 fails in $\Delta t$ \} + ... =

= $\lambda \Delta t + \lambda \Delta t + ... = N\lambda \Delta t$
Consider a system made by $N$ identical components which can be in two states: working or failed. Assume constant failure rate $\lambda$, that 1 repairman is available and that the component repair rate is constant and equal to $\mu$. You are required to:

- Draw the Markov diagram
- Find the transition rate matrix
Exercise 3: Solution

SYSTEM CHARACTERISTICS:
- The system is made of $N$ identical components
- Each component can be in two states: working or failed
- The transition rates are constant $\lambda \rightarrow$ rate of failure $\mu \rightarrow$ rate of repair
- One repairman is available
Solution to the Fundamental Equation
Solution to the fundamental equation of the Markov process continuous in time

\[
\begin{aligned}
\frac{dP}{dt} &= P(t) \cdot A \\
0 \leq j &\leq N
\end{aligned}
\]

where

\[
A = \begin{pmatrix}
     -\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \ldots & \alpha_{0N} \\
     \alpha_{10} & -\sum_{j=1}^{N} \alpha_{1j} & \ldots & \alpha_{1N} \\
     \vdots & \vdots & \ddots & \vdots \\
     \alpha_{N0} & \alpha_{N1} & \ldots & -\sum_{j=1}^{N} \alpha_{Nj}
\end{pmatrix}
\]

System of linear, first-order differential equations in the unknown state probabilities

\[P_j(t), \ j = 0, 1, 2, \ldots, N, \ t \geq 0\]

USE LAPLACE TRANSFORM
Solution to the fundamental equation of the Markov process continuous in time: the Laplace Transform Method

- Laplace Transform: \( \mathcal{L}_j(\sigma) = L[P_j(t)] = \int_0^\infty e^{-\sigma t} P_j(t) \, dt, \quad j = 0, 1, \ldots, N \)

- First derivative: \( L\left( \frac{dP_j(t)}{dt} \right) = s \cdot \mathcal{L}_j(\sigma) - P_j(0), \quad j = 0, 1, \ldots, N \)

- Apply the Laplace operator to \( \frac{dP}{dt} = P(t) \cdot A \)

\[
L\left[ \frac{dP(t)}{dt} \right] = L[P(t) \cdot A]
\]

First derivative \( s \mathcal{L}_j(\sigma) - C = \mathcal{L}_j(\sigma) \cdot A \), Linearity

\[
\mathcal{L}_j(\sigma) = C \cdot \left[ s \cdot I - A \right]^{-1} \quad \Rightarrow \quad P(t) = \text{inverse transform of } \mathcal{L}_j(\sigma)
\]
Solution to the fundamental equation of the Markov process continuous in time: steady state probabilities

- At steady state \( \frac{dP(t)}{dt} = 0 \Rightarrow \frac{dP(t)}{dt} = P(t) \cdot A = \Pi \cdot A = 0 \)

- Solve the (linear) system:
  \[
  \begin{aligned}
  \Pi \cdot A &= 0 \\
  \sum_{j=0}^{N} \Pi_j &= 1 
  \end{aligned}
  \]

- It can be shown that
  \[
  \Pi_j = \frac{D_j}{\sum_{i=0}^{N} D_i} \quad j = 0, 1, 2, \ldots, N
  \]

\[D_j = \text{determinant of the square matrix obtained from } A \text{ by deleting the } j\text{-th row and column}\]
Consider a system made by one component which can be in two states: working (‘0’) or Failed (‘1’). Assume constant failure rate $\lambda$ and constant repair rate $\mu$ an that the component is working at $t = 0$: $C = [1 \ 0]$ 

- You are required to find the component steady state and instantaneous availability
Exercise 4: one component/one repairman – Solution to the fundamental equation (2)

- **Solve** \( \Phi(s) = C \cdot (sI - A)^{-1} \)

- **Compute** \( (sI - A)^{-1} \)

\[
(sI - A)^{-1} = \begin{pmatrix}
s + \lambda & -\lambda \\
-\mu & s + \mu
\end{pmatrix}^{-1} = \frac{1}{\det[(sI - A)]}\begin{pmatrix}s + \mu & \lambda \\s + \lambda & s + \mu\end{pmatrix}
\]

\[
= \frac{1}{s^2 + s\mu + s\lambda} \begin{pmatrix}s + \mu & \lambda \\
\mu & s + \lambda
\end{pmatrix}
\]
Exercise 4: one component/one repairman – Solution to the fundamental equation (3)

- Anti-Transform

\[ \tilde{P}(s) = \begin{bmatrix} \frac{s + \mu}{s(s + \lambda + \mu)} & \frac{\lambda}{s(s + \lambda + \mu)} \end{bmatrix} \]

- It is known that

\[ L^{-1}\left[ \frac{1}{s+a} \right] = e^{-ax} \text{ and } L^{-1}\left[ \frac{1}{s(s+a)} \right] = \frac{1}{a} \left(1 - e^{-ax}\right) \]

STATE PROBABILITY VECTOR

\[ P(t) = \begin{pmatrix} \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda+\mu)t} \\ \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda+\mu)t} \end{pmatrix} \]

\[ P_0(t) \]  

- system instantaneous **availability**  
  (probability of being in operational state \(0\) at time \(t\))

\[ P_1(t) \]  

- system instantaneous **unavailability**  
  (probability of being in failed state \(1\) at time \(t\))
Exercise 4: one component/one repairman – Solution to the fundamental equation (4)

• TIME-DEPENDENT STATE PROBABILITIES

\[ P_0(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \]  
(system instantaneous availability)

\[ P_1(t) = \frac{\lambda}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \]  
(system instantaneous unavailability)

• STEADY STATE PROBABILITIES

\[ \Pi_0 = \lim_{t \to \infty} P_0(t) = \frac{\mu}{\mu + \lambda} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{MTBF}{MTTR + MTBF} \]

= average fraction of time the system is functioning

\[ \Pi_1 = \lim_{t \to \infty} P_1(t) = \frac{\lambda}{\mu + \lambda} = \frac{1/\mu}{1/\mu + 1/\lambda} = \frac{MTTR}{MTTR + MTBF} \]

= average fraction of time the system is down (i.e., under repair)
Quantity of Interest
Frequency of departure from a state

- **Unconditional** probability of **arriving in state** \( j \) in the next \( dt \) **departing from state** \( i \) at time \( t \): \( P[X(t + dt) = j, X(t) = i] \)

\[
P[X(t + dt) = j, X(t) = i] = P[X(t + dt) = j | X(t) = i] \cdot P[X(t) = i] = p_{ij}(dt)P_i(t)
\]

- **Frequency of departure** from state \( i \) to state \( j \):

\[
\nu_{ij}^{dep}(t) = \lim_{dt \to 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t) = \nu_{ij}^{dep} = \alpha_{ij} \cdot \Pi_i
\]

(at steady state)

- **Total frequency of departure** from state \( i \) to any other state \( j \):

\[
\nu_i(t) = \sum_{\substack{j=0 \\ j \neq i}}^{N} \alpha_{ij} \cdot P_i(t) = -\alpha_{ii} \cdot P_i(t) = \nu_i^{dep} = -\alpha_{ii} \cdot \Pi_i
\]

(at steady state)
In analogy, the total frequency of arrivals to state \( i \) from any other state \( k \):

\[
\nu_{i}^{\text{arr}}(t) = \sum_{k=0 \atop k \neq i}^{N} \alpha_{k \cdot i} \cdot P_{k}(t)
\]

\[
\nu_{i}^{\text{arr}} = \sum_{k=0 \atop k \neq i}^{N} \alpha_{k \cdot i} \cdot \Pi_{k} \quad \text{(at steady state)}
\]

\[
\Pi \cdot A = 0 \quad \Rightarrow \quad \sum_{k=0}^{N} \alpha_{k \cdot i} \cdot \Pi_{k} = 0 \quad (i = 0, 1, 2, \ldots, N)
\]

\[
- \alpha_{i \cdot i} \cdot \Pi_{i} = \sum_{k=0 \atop k \neq i}^{N} \alpha_{k \cdot i} \cdot \Pi_{k} \quad (i = 0, 1, 2, \ldots, N)
\]

AT STEADY STATE:

frequency of departures from state \( i \) = frequency of arrivals to state \( i \)
System Failure Intensity

- **SYSTEM FAILURE INTENSITY** $W_f$:
  - **Rate** at which system failures occur
  - **Expected number** of system failures per unit of time
  - **Rate of exiting a success state** to go into one of fault

\[
W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F}
\]

- $S$ = set of success states of the system
- $F$ = set of failure states of the system
- $P_i(t) = \text{probability of the system being in the functioning state } i \text{ at time } t$
- $\lambda_{i \rightarrow F} = \text{conditional (transition) probability of leaving success state } i \text{ towards failure states}$
• **SYSTEM REPAIR INTENSITY** $W_r$:  
  - **Rate** at which system repairs occur  
  - **Expected number** of system repairs per unit of time  
  - **Rate of exiting a failed state** to go into one of success

\[ W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \rightarrow S} \]

$S =$ set of success states of the system  
$F =$ set of failure states of the system  
$P_j(t) =$ probability of the system being in the failure state $j$ at time $t$  
$\mu_{j \rightarrow S} =$ conditional (transition) probability of leaving failure state $j$ towards success states
Exercise 5: one component/one repairman

Consider a system made by one component which can be in two states: working (‘0’) or Failed (‘1’). Assume constant failure rate $\lambda$ and constant repair rate $\mu$ and that the component is working at $t = 0$: $C = [1 \ 0]$

- You are required to find the failure and repair intensities
Exercise 5: one component/one repairman – Failure and repair intensities

\[ W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F} \]

\[ W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \rightarrow S} \]

\[ S = \text{set of success states of the system} = \{0\} \]

\[ F = \text{set of failure states of the system} = \{1\} \]

\[ \lambda_{i \rightarrow F} = \lambda \]

\[ \mu_{j \rightarrow S} = \mu \]

\[ W_f(t) = \lambda P_0(t) = \frac{\lambda \mu}{\mu + \lambda} + \frac{\lambda^2}{\mu + \lambda} e^{-(\lambda + \mu)t} \]

\[ W_r(t) = \mu P_1(t) = \frac{\mu \lambda}{\mu + \lambda} + \frac{\mu \lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \]
Sojourn Time in a state (1)

- **Time of occupancy of state** \(i\) (sojourn time) = \(T_i\)

- Markov property and time homogeneity imply that if at time \(t\) the process is in state \(i\), the time remaining in state \(i\) is independent of time already spent in state \(i\)

\[
P(T_i > t + s | T_i > t) = P(X(t + u) = i, 0 \leq u \leq s | X(\tau) = i, 0 \leq \tau \leq t) =
\]

\[
= P(X(t + u) = i, 0 \leq u \leq s | X(t) = i) \text{ (by Markov property)}
\]

\[
= P(X(u) = i, 0 \leq u \leq s | X(0) = i) \text{ (by homogeneity)}
\]

\[
= \mathbb{P}(T_i > s) \text{ Memoryless Property}
\]

- The only distribution satisfying the memoryless property is the **Exponential distribution**

\[
T_i \sim \text{Exp}
\]
Sojourn Time in a state (2)

- System departure rate from state $i$ (at steady state): $-\alpha_{ii}$

\[ T_i \sim \text{Exp}(-\alpha_{ii}) \]

- Expected sojourn time $l_i$: average time of occupancy of state $i$

\[ l_i = \mathbb{E}\{T_i\} = \frac{1}{-\alpha_{ii}} \]
Sojourn Time in a state (3)

- Total frequency of departure at steady state: \( v_i^{dep} = -\alpha_{ii} \cdot \Pi_i \)
- Average time of occupancy of state: \( l_i = \frac{1}{-\alpha_{ii}} \)

\[
\begin{align*}
  v_i^{dep} &= -\alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i} \\
  v_i^{dep} &= v_i^{arr} \\
  \Pi_i &= v_i^{arr} \cdot l_i
\end{align*}
\]

The **mean** proportion of time \( \Pi_i \) that the system spends in state \( i \) is equal to the **total frequency of arrivals to state \( i \)** multiplied by the mean duration of one visit in state \( i \)
System Availability

- **System instantaneous availability** at time $t$
  
  \[ p(t) = \sum_{i \in S} P_i(t) = 1 - q(t) = 1 - \sum_{j \in F} P_j(t) \]

  \[ f(s) = \sum_{i \in S} f_i(s) = \frac{1}{s} - \sum_{j \in F} f_j(s) \]

  $S = \text{set of success states of the system}$

  $F = \text{set of failure states of the system}$
• TWO CASES:

1) Non-Reparaible Systems
   → No repairs allowed

2) Reparaible Systems
   → Repairs allowed
System Reliability: Non-Reparaible Systems

- No repairs allowed ⇒ **Reliability = Availability**  \( R(t) \equiv p(t) = 1 - q(t) \)

- In the Laplace Domain:  \( \mathcal{L}\{R(s)\} = \sum_{i \in S} \mathcal{L}\{F_i\}(s) = \frac{1}{s} - \sum_{j \in F} \mathcal{L}\{P_j\}(s) \)

- **Mean Time to Failure (MTTF):**

\[
MTTF = \int_{0}^{\infty} R(t) \, dt = \left[ \int_{0}^{\infty} R(t) e^{-st} \, dt \right]_{s=0} = \tilde{R}(0) = \sum_{i \in S} \tilde{P}_i(0) = \left[ \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s) \right]_{s=0}
\]
• TWO CASES:

1) Non-reparaible systems
   → No repairs allowed

2) Reparaible systems
   → Repairs allowed
1. Transform the failed states $j \in F$ into absorbing states (the system cannot be repaired → it is not possible to escape from a failed state)

The new matrix $A'$ contains the transition rates for transitions only among the success states $i \in S$ (the “reduced” system is virtually functioning continuously with no interruptions)
2. Solve the **reduced problem** of $A'$ for the probabilities $P_i^*(t)$, $i \in S$ of being in these (transient) safe states

$$\frac{dP^*_i(t)}{dt} = P^*_i(t) \cdot A'$$

Reliability

$$R(t) = \sum_{i \in S} P_i^*(t)$$

Mean Time To Failure (MTTF)

$$MTTF = \int_{0}^{\infty} R(t) \, dt = \sum_{i \in S} \phi_i^0(0) = \mathcal{R}(0)$$

**NOTICE:** in the reduced problem we have only transient states $\Rightarrow \Pi_i^* = P_i^*(\infty) = 0$
Consider a system made by 2 identical components in parallel. Each component can be in two states: working or failed. Assume constant failure rate $\lambda$, that 2 repairman are available and that the single component repair rate is constant and equal to $\mu$. You are required to:

- find the system reliability
- find the system MTTF
Exercise 6: system with two identical components and two repairmen available (1)

- **TWO CASES:**
  
  a) Parallel logic (1 out of 2)
  
  b) Series logic (2 out of 2)
Example 8: system with two identical components and two repairmen available (2)

**PARALLEL LOGIC**

System discrete states:
- State 0: system is operating (both components functioning)
- State 1: system is operating (only one of the two components functioning)
- State 2: system is failed (both components failed)

\[
A = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\mu + \lambda) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix}
\]
Example 8: system with two identical components and two repairmen available (3)

1. Exclude all the failed states $j \in F$ from the transition rate matrix

- **System reliability** $R(t) :=$ probability of the system being in **safe states 0 or 1 continuously** from $t = 0$

\[
A = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\mu + \lambda) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix} \Rightarrow A' = \begin{pmatrix}
-2\lambda & 2\lambda \\
\mu & -(\mu + \lambda)
\end{pmatrix}
\]
Example 8: system with two identical components and two repairmen available (4)

2. Solve the **reduced problem** of $\mathcal{A}'$ for the probabilities $P_i^*(t), \ i \in S$ of being in these (transient) safe states

In the time domain:

\[
\begin{align*}
\frac{dP^*}{dt} &= P^*(t) \cdot A' \Rightarrow \\
\frac{dP^*}{dt} &= P^*(t) \cdot \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\lambda + \mu) \end{pmatrix} \\
P^*(0) &= (1 \ 0)
\end{align*}
\]

In the Laplace domain:

\[
\begin{align*}
\mathcal{P}^*(s) &= \mathcal{P}^*(0) \cdot (sI - A')^{-1} \Rightarrow \\
\mathcal{P}^{*0}(s) &= (1 \ 0) \cdot (sI - A')^{-1}
\end{align*}
\]
Example 8: system with two identical components and two repairmen available (5)

\[ f^c_0(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot (sI - A')^{-1} \]

with \( A' = \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\lambda + \mu) \end{pmatrix} \)

\[ sI - A' = \begin{pmatrix} s + 2\lambda & -2\lambda \\ -\mu & s + \mu + \lambda \end{pmatrix} \]

\[ (sI - A')^{-1} = \frac{1}{(s+2\lambda)(s+\mu+\lambda)-2\lambda\mu} \begin{pmatrix} s + \mu + \lambda & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix} \]

\[ \cdot \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} s + \lambda + \mu & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix} \]

where \( \omega_{0,1} = \frac{-3\lambda - \mu \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2} \)
Example 8: system with two identical components and two repairmen available (6)

\[
\tilde{P}^*(s) = C^* \cdot (sI - A')^{-1} = \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + \mu + \lambda & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix} = \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} s + \mu + \lambda & 2\lambda \end{pmatrix}
\]

SYSTEM RELIABILITY

\[
\tilde{R}(s) = \tilde{P}_0(s) + \tilde{P}_1(s)
\]

In the Laplace domain

\[
L^{-1} \left[ \frac{1}{s - a} \right] =
\]

• It is known that

SYSTEM RELIABILITY

\[
R(t) = \frac{\omega_0 \cdot e^{\omega_0 \cdot t} - \omega_1 \cdot e^{\omega_1 \cdot t}}{\omega_0 - \omega_1}
\]

In the time domain
Example 8: system with two identical components and two repairmen available (7)

- **MEAN TIME TO FAILURE**
  \[ MTTF = \overline{R}(0) = \sum_{i} \overline{f}_{i}(0) = \sum_{i=0}^{1} \tilde{P}_{i}^{*}(0) \]

- Starting from \( \overline{f}_{0}(s) = C^{*} \left( s \cdot I \right) A' \)\(^{-1} \)

\[
MTTF = C^{*} \cdot \left( -A' \right)^{-1} \cdot w^{T} \quad \text{with} \quad w = [1, 1, 1, \ldots, 1]^{T}
\]

\[
MTTF = (1, 0) \cdot \begin{pmatrix} 2\lambda & -2\lambda \\ -\mu & \mu + \lambda \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1, 0) \cdot \frac{1}{2\lambda(\lambda + \mu) - 2\lambda\mu} \cdot \begin{pmatrix} \mu + \lambda & 2\lambda \\ \mu & 2\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \]

\[
= \frac{1}{2\lambda^2(\mu + \lambda, 2\lambda)} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{3\lambda^2 + \mu}{2\lambda^2} = \]

\[
= \frac{3}{2\lambda} + \frac{\mu}{2\lambda^2}
\]
Consider a system made by 2 identical components in series. Each component can be in two states: working or failed. Assume constant failure rate $\lambda$, that 2 repairmen are available and that the single component repair rate is constant and equal to $\mu$. You are required to:

- find the system reliability
- find the system MTTF
Example 8: system with two identical components and two repairmen available (8)

- **TWO CASES:**
  
  a) Parallel logic (1 out of 2)
  
  b) Series logic (2 out of 2)
Example 8: system with two identical components and two repairmen available (9)

SERIES LOGIC

System discrete states:
State 0: system is operating (both components functioning)
State 1: system is failed (only one of the two components functioning)
State 2: system is failed (both components failed)
1. Exclude all the failed states $j \in F$ from the transition rate matrix

- **System reliability** $R(t) := \text{probability of the system being in safe states 0 or 1 continuously}$ from $t = 0$

**PARTITION OF THE TRANSITION MATRIX**

$$A = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda + \mu) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \quad \Rightarrow \quad A' = -2\lambda$$
Example 8: system with two identical components and two repairmen available (10)

2. Solve the **reduced problem** of $A'$ for the probabilities $P_i^*(t), \ i \in S$ of being in these (transient) **safe states**

- Easy to solve **in the time domain**:

\[
\begin{align*}
\frac{dP^*}{dt} &= P^* \cdot A' \\
P^*(0) &= C^*
\end{align*}
\]

which simplifies to

\[
\begin{align*}
\frac{dP_0^*}{dt} &= -2\lambda \cdot P_0^* \\
P_0^*(0) &= 1
\end{align*}
\]

\[
R(t) = P_0^*(t) = e^{-2\lambda t}
\]