Markov Reliability and Availability Analysis

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General Framework
General Framework

SYSTEM

Component 1
- Failed
- Operating
- Hot standby...
- Degraded

Component 2
- Failed
- Operating
- Degraded...
- Partially failed

Component $N_c$
- Failed
- Operating
- Cold standby...
- Maintenance

Random transition at $t = t_1$
Random transition at $t = t_2$

Stochastic process of system evolution

Under specified conditions:

= MARKOV PROCESS
Markov Processes: Basic Elements
The system can occupy a **finite** or **countably infinite** number $N$ of states.

Set of possible states $U = \{0, 1, 2, \ldots, N\}$

**State-space** of the random process
Markov Processes: basic elements—the system states

• The states are:
  o **mutually exclusive** → the system must be *only* in one state at each time
  o **exhaustive** → the system must be in one state at all times

• Example:

Set of possible states $U = \{0, 1, 2, 3\}$

- Mutually exclusive: $P(\text{State} = i \cap \text{State} = j) = 0$, if $i \neq j$
- Exhaustive: $P(U) = P(\text{State} = 0 \cup \text{State} = 1 \cup \text{State} = 2 \cup \text{State} = 3)$
  $$= P(\text{State} = 0) + P(\text{State} = 1) + P(\text{State} = 2) + P(\text{State} = 3) = 1$$
• **Transitions** from one state to another occur **stochastically** (i.e., randomly in time)

![Diagram of Markov Processes](image)

- Random transition at time $t = t_1$
- Random transition at time $t = t_2 > t_1$
- Random transition at time $t = t_3 > t_2$
The random process of system transition in time can be described by an integer random variable $X(t)$:

$$X(t) = 5 \rightarrow \text{the system occupies state number 5 at time } t$$

The stochastic process may be observed at:

- **Discrete times** \(\Rightarrow\) **DISCRETE-TIME FINITE-STATE MARKOV CHAIN**

- **Continuously** \(\Rightarrow\) **CONTINUOUS-TIME FINITE-STATE MARKOV PROCESS**
Discrete-Time Finite-State Markov Chain (DTFSMC)
The conceptual model: discrete times

- The stochastic process is **observed** at **discrete** times

\[ \Delta t(2) = t_2 - t_1 \quad \Delta t(4) = t_4 - t_3 \]

0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad \ldots \quad t_n \quad \ldots \quad T_m \quad \rightarrow \quad t

\[ t_n = t_{n-1} + \Delta t(n) \]

- **Hypotheses:**
  - The time interval \( \Delta t(n) \) is **small** such that **only one** event (i.e., stochastic transition) can occur within it
  - For simplicity, \( \Delta t(n) = \Delta t = \text{constant} \)
The random process of system transition in time is described by an integer random variable $X(\cdot)$

- $X(n) := \text{system state at time } t_n = n\Delta t$
  - $X(3) = 5$: the system occupies state 5 at time $t_3$

**OBJECTIVE:**
Compute the probability that the system is in a given state at a given time, for all possible states and times

$$P[X(n) = j], n=1, 2, \ldots, N_{\text{time}}, j=0,1,\ldots,N$$
Objective:

\[ P[X(n) = j], n = 1, 2, ..., N_{\text{time}}, j = 0, 1, ..., N \]

What do we need?
The conceptual model: the Markov assumption

In general for stochastic processes:
• the probability of a future state of the system usually depends on its entire life history

\[ P[X(n+1) = j] = P[X(n+1) = j \mid X(0) = x_0, X(1) = x_1, X(2) = x_2, \ldots, X(n) = x_n] \]

In Markov Processes:
• the probability of a future state of the system only depends on its present state

\[ P[X(n+1) = j \mid X(0) = x_0, X(1) = x_1, X(2) = x_2, \ldots, X(n) = x_n] = P[X(n+1) = j \mid X(n) = x_n] \]

THE PROCESS HAS “NO MEMORY”
The conceptual model: the transition probabilities

- **Transition probability** that the system in state $i$ at time $t_m$ moves to state $j$ at time $t_n$

\[
p_{ij}(m, n) = P[X(n) = j \mid X(m) = i], \quad n > m \geq 0
\]

\[i = 0, 1, 2, \ldots, N, \quad j = 0, 1, 2, \ldots, N\]
The conceptual model: properties of the transition probabilities (1)

1. Transition probabilities $p_{ij}(m, n)$ are larger than or equal to 0

$$p_{ij}(m, n) \geq 0, \quad n > m \geq 0 \quad i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N$$

(definition of probability)

2. Transition probabilities must sum to 1

$$\sum_{all\ j} p_{ij}(m, n) = \sum_{j=0}^{N} p_{ij}(m, n) = 1, \ n > m \geq 0 \quad i = 0, 1, 2, \ldots, N$$

(the set of states is exhaustive)

Starting from $i = 1$, the system either remains in $i = 1$ or it goes somewhere else, i.e., to $j = 0$ or 2 or 3
The conceptual model: properties of the transition probabilities (2)

3. \( p_{ij}(m,n) = \sum_k p_{ik}(m,r)p_{kj}(r,n) \quad i = 0,1,2,\ldots, N, j = 0,1,2,\ldots, N \)

\[
p[X(n) = j, X(m) = i] = \sum_k p[X(n) = j, X(r) = k, X(m) = i] \quad \text{(theorem of total probability)}
\]

\[
\downarrow \text{conditional probability}
\]

\[
= \sum_k p[X(n) = j \mid X(r) = k, X(m) = i]P[X(r) = k, X(m) = i]
\]

\[
\downarrow \text{Markov assumption}
\]

\[
= \sum_k p[X(n) = j \mid X(r) = k]P[X(r) = k, X(m) = i]
\]

\[
p_{ij}(m,n) = P[X(n) = j \mid X(m) = i] = \frac{P[X(n) = j, X(m) = i]}{P[X(m) = i]} \quad \text{(conditional probability)}
\]

\[
\downarrow \text{formula above}
\]

\[
= \sum_k p[X(n) = j \mid X(r) = k] \frac{P[X(r) = k, X(m) = i]}{P[X(m) = i]}
\]

\[
\downarrow \text{conditional probability}
\]

\[
= \sum_k P[X(n) = j \mid X(r) = k]P[X(r) = k \mid X(m) = i] = \sum_k p_{kj}(r,n)p_{ik}(m,r)
\]
The conceptual model: stationary transition probabilities

- If the **transition probability** $p_{ij}(m, n)$ depends on the **interval** $(t_n - t_m)$ and not on the **individual times** $t_m$ and $t_n$, then
  - the **transition probabilities** are **stationary**
  - the **Markov process** is **homogeneous in time**

  $k$ time steps

  $p_{ij}(m, n) = p_{ij}(m, m + (n - m)) = p_{ij}(m, m + k) = P[X(m + k) = j \mid X(m) = i]$

  $= P[X(k) = j \mid X(0) = i]$

  $= p_{ij}(k), \quad k \geq 0 \quad i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N$
The conceptual model: one-step stationary transition probabilities

We need to determine the **stationary** transition probabilities at the $k$-th time step

$$p_{ij}(k), k \geq 0$$

We need to know **only** the stationary **one-step** transition probabilities

$$p_{ij}(1) = p_{ij}$$

$(i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N)$

Markov assumption (see back-up slides…)

Francesco Cannarile & Enrico Zio
The conceptual model: the transition probability matrix

Properties:

- \( \dim(A) = (N + 1) \times (N + 1) \)
- \( 0 \leq p_{ij} \leq 1, \forall i, j \in \{0, 1, 2, \ldots, N\} \) (all elements are probabilities)
- Only \((N+1)xN\) elements need to be known
- \( \sum_{j=0}^{N} p_{ij} = 1, i = 0, 1, 2, \ldots, N \) (the set of states is exhaustive)

\[
A = \begin{pmatrix}
0 & 1 & \ldots & N \\
0 & p_{00} & p_{01} & \ldots & p_{0N} \\
1 & p_{10} & p_{11} & \ldots & p_{1N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
N & p_{N0} & p_{N1} & \ldots & p_{NN}
\end{pmatrix}
\]

is a Stochastic Matrix
• Given the **stationary one-step** transition probabilities $P_{ij}$ ($i = 0, 1, 2, \ldots, N, j = 0, 1, 2, \ldots, N$)

Go back to the **OBJECTIVE**: 

Compute $P[X(n) = j], n = 1, 2, \ldots, N_{time}, j = 0, 1, \ldots, N$
Unconditional state probabilities (1)

1. Compute the probability that the system is in a given state at a given time, for all possible states and times

$$P[X(n) = j] = P_j(n), n = 1, 2, ..., N_{time}, j = 0, 1, ..., N$$

2. Introduce the row vector:

$$\underline{P}(n) = [P_0(n) P_1(n) ... P_j(n) ... P_N(n)] = \text{probabilities of the system being in state } 0, 1, 2, ..., N \text{ at the } n\text{-th time step}$$

3. Initialize the vector $\underline{P}(n)$ at time step $n = 0$:

$$\underline{P}(0) = \underline{C} = [C_0 \ C_1 ... C_j ... C_N]$$
Unconditional state probabilities (2)

\[ P_j(1) = P\left[ X(1) = j \right] \]

\[ = \sum_{i=0}^{N} P\left[ X(1) = j \mid X(0) = i \right] \cdot P\left[ X(0) = i \right] \]

\[ = \sum_{i=0}^{N} p_{ij}C_i = p_{0j} \cdot C_0 + p_{1j} \cdot C_1 + p_{2j} \cdot C_2 + \ldots + p_{Nj} \cdot C_N, \]

with \( j = 0, 1, 2, \ldots, N \)

Using Matrix Notation:

\[ \underline{P}(1) = \underline{C} \cdot \underline{A} \]
Unconditional state probabilities (3)

• At the second time step \( n = 2 \):

\[
P_j(2) = P\left[ X(2) = j \right]
\]

\[
= \sum_{k=0}^{N} P\left[ X(2) = j \mid X(1) = k \right] \cdot P\left[ X(1) = k \right]
\]

\[
= \sum_{k=0}^{N} p_{kj} \cdot p_k(1)
\]

\[
= P_0(1) \cdot p_{0j} + P_1(1) \cdot p_{1j} + P_2(1) \cdot p_{2j} + \ldots + P_N(1) \cdot p_{Nj},
\]

with \( j = 0, 1, 2, \ldots, N \)

FUNDAMENTAL EQUATION
OF THE HOMOGENEOUS
DISCRETE-TIME DISCRETE-STATE
MARKOV PROCESS

\[
P(2) = P(1) \cdot A = (CA)A = CA^2
\]

Proceeding in the same recursive way…

\[
P(n) = P(0) \cdot A^n = C \cdot A^n
\]
Multi-step transition probabilities (1)

FUNDAMENTAL EQUATION

\[ P(n) = P(0) \cdot A^n = C \cdot A^n \]

Define:

\[ A^n = \begin{pmatrix}
    p_{00}(n) & p_{01}(n) & \cdots & p_{0N}(n) \\
    p_{10}(n) & p_{11}(n) & \cdots & p_{1N}(n) \\
    \vdots & \vdots & \ddots & \vdots \\
    p_{N0}(n) & p_{N1}(n) & \cdots & p_{NN}(n)
\end{pmatrix} \]

\[ n \text{-th step transition probability} \]

\[ p_{ij}(n) = P[X(n) = j \mid X(0) = i] \]

probability of arriving in state \( j \) after \( n \) steps given that the initial state was \( i \)
EXAMPLE WITH $N = 2$ STATES AND $n = 2$ time steps

$$A = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \quad (i = 0, 1, j = 0, 1)$$

$$A^2 = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} \cdot \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} p_{00} \cdot p_{00} + p_{01} \cdot p_{10} & p_{00} \cdot p_{01} + p_{01} \cdot p_{11} \\ p_{10} \cdot p_{00} + p_{11} \cdot p_{10} & p_{10} \cdot p_{01} + p_{11} \cdot p_{11} \end{pmatrix}$$

WHAT IS THE “PHYSICAL” MEANING?
Multi-step transition probabilities (3)

\[ p_{ij}(n) = P[X(n) = j | X(0) = i] \]

\[ p_{01}(2) = p_{00} \cdot p_{01} + p_{01} \cdot p_{11} \]

\[ p_{00}(2) = p_{00} \cdot p_{00} + p_{01} \cdot p_{10} \]

\[ p_{ij}(n) \] is the sum of the probabilities of all trajectories with length \( n \)
which originate in state \( i \) and end in state \( j \)
Example 1: wet and dry days in a town (1)

- Stochastic process of raining in a town (transitions between wet and dry days)

**DISCRETE STATES**
State 1: dry day
State 2: wet day

**DISCRETE TIME**
Time step = 1 day

**TRANSITION MATRIX**

\[
A = \begin{pmatrix}
0.8 & 0.2 \\
0.5 & 0.5 \\
\end{pmatrix}
\]

**MARKOV DIAGRAM**

*Question:* If today the weather is dry, what is the probability that it will be **dry two days from now**?
Example 1: wet and dry days in a town (2)

\[
A = \begin{pmatrix}
  \text{dry} & \text{wet} \\
  0.8 & 0.2 \\
  0.5 & 0.5
\end{pmatrix}
\]

Initial condition: today is dry
\[
C = [1 \ 0]
\]

At step \( n \):
\[
P(n) = P(0) \cdot A^n = C \cdot A^n
\]
\[
P(2) = P(0) \cdot A^2 = C \cdot A^2
\]

\[
P(2) = [1 \ 0] \cdot \begin{pmatrix}
  0.8 & 0.2 \\
  0.5 & 0.5
\end{pmatrix} \cdot \begin{pmatrix}
  0.8 & 0.2 \\
  0.5 & 0.5
\end{pmatrix} = [0.74 \ 0.26]
\]

Probability that it will be dry 2 days from now = \( P_1(2) = 0.74 \)
Solution to the fundamental equation
Solution to the fundamental equation (1)

\[
\begin{cases}
P(n) = P(0)A^n \\
P(0) = C
\end{cases}
\]

SOLVE THE ASSOCIATED EIGENVALUE PROBLEM

i) Set the eigenvalue problem \( V \cdot A = \omega \cdot V \)

ii) Write the homogeneous form \( V \cdot (A - \omega \cdot I) = 0 \)

iii) Find non-trivial solutions by setting \( \det(A - \omega \cdot I) = 0 \)

iv) From \( \det(A - \omega \cdot I) = 0 \) compute the eigenvalues \( \omega_j, j = 0, 1, \ldots, N \)

v) Set the \( N \) eigenvalue problems \( V_j \cdot A = \omega_j \cdot V_j \) \( j = 0, 1, \ldots, N \)

vi) From \( V_j \cdot A = \omega_j \cdot V_j \) compute the eigenvectors \( V_j, j = 0, 1, \ldots, N \)
The eigenvectors \( V_j \) span the \((N + 1)\)-dimensional space and can be used as a basis to write any \((N + 1)\)-dimensional vector as a linear combination of them.

\[
P(n) = \sum_{j=0}^{N} \alpha_j \cdot V_j \quad \text{AND} \quad C = \sum_{j=0}^{N} c_j \cdot V_j
\]

We need to find the coefficients \( \alpha_j \) AND \( c_j, j = 0, 1, ..., N \).
Solution to the fundamental equation (3)

- **Find the coefficients** $c_j, j = 0, 1, \ldots, N$ for $C = \sum_{j=0}^{N} c_j \cdot V_j$

- Solve the eigenvalue problem for
  \[
  \begin{cases}
  P(n) = P(0)A^n \\
  P(0) = C
  \end{cases}
  \quad (\omega_j, V_j), j = 0, \ldots, N
  \]

a) Since eigenvectors are orthonormal $< V_i, V_j > = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

b) Multiply the left- and right-hand sides of $C = \sum_{j=0}^{N} c_i V_i$ by $V_j$

\[
< C, V_j > = \sum_{j=0}^{N} c_i < V_i, V_j > = c_j
\]

\[
c_j = < C, V_j >
\]
Solution to the fundamental equation (4)

- Find the coefficients $\alpha_j$, $j = 0, 1, ..., N$ for $P(n) = \sum_{j=0}^{N} \alpha_j \cdot V_j$

- Use $P(n) = \sum_{j=0}^{N} \alpha_j \cdot V_j$, $C = \sum_{j=0}^{N} c_j \cdot V_j$ and $P(n) = C A^n$

a) Substitute $C = \sum_{j=0}^{N} c_j \cdot V_j$ into $P(n) = C A^n$ to obtain $P(n) = \left( \sum_{j=0}^{N} c_j V_j \right) \cdot A^n$

b) Set $P(n) = \sum_{j=0}^{N} \alpha_j \cdot V_j = C \cdot A^n = \left( \sum_{j=0}^{N} c_j V_j \right) \cdot A^n$
Solution to the fundamental equation (5)

c) Multiply $V_j \cdot A = \omega_j \cdot V_j$ by $A$ to obtain $V_j \cdot A \cdot A = \omega_j \cdot V_j \cdot A$

Since $V_j \cdot A = \omega_j \cdot V_j$ then $V_j \cdot A^2 = \omega_j \cdot \omega_j \cdot V_j = \omega_j^2 \cdot V_j$

• • • (proceeding in the same recursive way)

$$V_j \cdot A^n = \omega_j^n \cdot V_j$$

d) Substitute $V_j \cdot A^n = \omega_j^n \cdot V_j$ into $P(n) = \sum_{j=0}^{N} \alpha_j \cdot V_j = C \cdot A^n = \sum_{j=0}^{N} c_j \cdot V_j \cdot A^n$

$$\sum_{j=0}^{N} \alpha_j \cdot V_j = \sum_{j=0}^{N} c_j \cdot \omega_j^n \cdot V_j$$

$$\alpha_j = c_j \cdot \omega_j^n$$
Quantity of Interest
Steady state probabilities

- **Steady state probabilities** $\pi_j$: probability of the system being in state $j$ asymptotically

- **TWO ALTERNATIVE APPROACHES:**

  1) Since $\omega_0 = 1$ and $|\omega_j| < 1, j = 1, 2, \ldots, N$

     AT STEADY STATE: $\lim_{n \to \infty} P(n) = \lim_{n \to \infty} \sum_{j=0}^{N} \alpha_j \cdot V_j = \lim_{n \to \infty} \sum_{j=0}^{N} c_j \cdot \omega_j^n \cdot V_j = c_0 V_0 = \Pi$

  2) Use the recursive equation $P(n) = P(n-1) \cdot A$

     AT STEADY STATE: $P(n) = P(n-1) = \Pi$

     SOLVE $\Pi = \Pi \cdot A$ subject to $\sum_{j=0}^{N} \Pi_j = 1$
Example 2: wet and dry days in a town

\[
\begin{pmatrix}
A & C \\
\begin{pmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \end{pmatrix} & [1 \\ 0]
\end{pmatrix}
\]

- **Question:** what is the probability that one year from now the day will be **dry**?

- By definition: \( P(365 \text{ days} = 1 \text{ year}) = CP^{365} \Rightarrow P^{365} \) ???

**ASSUMPTION:**

at \( n = 365 \) the steady state condition is established

\[
\begin{align*}
\Pi & = \Pi \cdot A \\
\sum_{j=0}^{N} \Pi_j & = 1
\end{align*}
\]

\[
\begin{align*}
\Pi_1 & = 0.8 \cdot \Pi_1 + 0.5 \cdot \Pi_2 \\
\Pi_1 + \Pi_2 & = 1
\end{align*}
\]

\[
\Rightarrow \Pi = \begin{pmatrix} 0.714 & 0.286 \end{pmatrix}
\]
First Passage Probabilities (1)

**FIRST PASSAGE RANDOM TIME:**

Random time that the system arrives *for the first time* in state $j$ given that it was in state $i$ at the initial time $0$

$$T_{ij} = \begin{cases} 
\min\{n \geq 1 \mid X(n) = j \text{ provided that } X(0) = i\} & \text{if } \exists n \geq 1 : X(n) = 1 \\
+\infty & \text{otherwise} 
\end{cases}$$

**NOTICE:**

\[ \{T_{ij} = n\} = \{X(n) = j, X(m) \neq j, 0 < m < n \mid X(0) = i\} \]

\[ \{T_{ij} < \infty\} = \{X(n) = j, n \geq 1 \mid X(0) = i\} \]
FIRST PASSAGE PROBABILITY AFTER \( n \) TIME STEPS:

Probability that the system arrives \textbf{for the first time} in state \( j \) \textbf{after} \( n \) \textbf{steps}, given that it was in state \( i \) at the initial time 0

\[
\begin{align*}
    f_{ij}(n) &= P[T_{ij} = n] \\
    f_{ij}(n) &= P[X(n) = j, X(m) \neq j, 0 < m < n | X(0) = i] \\
\end{align*}
\]

\textbf{NOTICE:}

\( f_{ij}(n) \neq p_{ij}(n) \)

\( p_{ij}(n) = \text{probability that the system reaches state } j \) \textbf{after} \( n \) \textbf{steps} starting from state \( i \), but \textbf{not necessarily for the first time}
First Passage Probabilities (3)

- Probability of going from state 1 to state 1 in 1 step for the first time
  \[ f_{11}(1) = p_{11} \]

- Probability that the system, starting from state 1, will return to the same state 1 for the first time after \( n \) steps: this is achieved by jumping in state 2 at the first step (\( p_{12} \)), remaining in state 2 during the successive \( n-2 \) steps (\( p_{22}^{n-2} \)) and moving back in the initial state 1 at the \( n \)-th step (\( p_{21} \)).
  \[ f_{11}(n) = p_{12} \cdot p_{22}^{n-2} \cdot p_{21} \]

- Probability that the system will arrive for the first time in state 2 after \( n \) steps; this is equal to the probability of remaining in state 1 for \( n-1 \) steps (\( p_{11}^{n-1} \)) and then jumping in state 2, at the final step (\( p_{12} \))
  \[ f_{12}(n) = p_{11}^{n-1} \cdot p_{12} \]
First Passage Probabilities (4)

- RELATIONSHIP WITH TRANSITION PROBABILITIES

\[ f_{ij}(1) = p_{ij}(1) = p_{ij} \]

\[ f_{ij}(2) = p_{ij}(2) - f_{ij}(1) \cdot p_{jj} \]

Probability that the system reaches state \( j \)
at step 2, given that it was in \( i \) at 0

\[ f_{ij}(3) = p_{ij}(3) - f_{ij}(1) \cdot p_{jj}(2) - f_{ij}(2) \cdot p_{jj} \]

\[ \cdots \]

\[ f_{ij}(k) = p_{ij}(k) - \sum_{l=1}^{k-1} f_{ij}(k-l)p_{jj}(l) \]

(Renewal Equation)

Probability that the system reaches state \( j \) for the first time at step 1 (starting from \( i \) at 0) and that it remains in \( j \) at the successive step
Recurrent, transient and absorbing states (1)

- **DEFINITIONS:**

  - First passage probability that the system goes to state $j$ **within $m$ steps** given that it was in $i$ at time 0:
    \[
    q_{ij}(m) = \sum_{n=1}^{m} f_{ij}(n) = \text{sum of the probabilities of the mutually exclusive events of reaching } j \text{ for the first time after } n = 1, 2, 3, \ldots, m \text{ steps}
    \]

  - Probability that the system **eventually** reaches state $j$ from state $i$:
    \[
    q_{ij}(\infty) = \lim_{m \to \infty} q_{ij}(m)
    \]

  - Probability that the system **eventually** returns to the initial state:
    \[
    f_{ii} = q_{ii}(\infty)
    \]
Recurrent, transient and absorbing states (2)

- State \( i \) is **recurrent** if the system starting at such state will **surely** return to it **sooner or later** (i.e., in finite time):
  \[
  f_{ii} = q_{ii} (\infty) = 1
  \]
  - For recurrent states \( \Pi_i \neq 0 \)

- State \( i \) is **transient** if the system starting at such state has a **finite probability** of **never** returning to it:
  \[
  f_{ii} = q_{ii} (\infty) < 1
  \]
  - For these states, at steady state \( \Pi_i = 0 \)

we cannot have a **finite Markov process** in which all states are **transients** because eventually it will leave them and **somewhere** it must go at steady state

- State \( i \) is **absorbing** if the system cannot leave it once it enters: \( p_{ii} = 1 \)
Sojourn Time in a state

- **Sojourn time** $S_i$: time spent in a state $i$

- Recalling that:
  
  
  
  $p_{ii} = \text{probability that the system \text{“}moves to\text{”} i \text{ in one step, given that it was in } i$

  
  $1 - p_{ii} = \text{probability that the system exits i \text{ in one step, given that it was in } i$

  
  
  

  
  $\mathbb{P}(S_i = n) = p_{ii}^n (1 - p_{ii})$

  
  
  

  $S_i \sim \text{Geom}(1 - p_{ii})$

  
  
  

  $\mathbb{E}\{S_i\} = \text{average number of steps before the system exits state} = \frac{1}{1 - p_{ii}}$
• The **random** process of system transition in **time** can be described by an **integer random variable** $X(t)$

$$X(t) = 5 \rightarrow \text{the system occupies state number 5 at time } t$$

• The **stochastic process** may be **observed** at:

  - Discrete times $\rightarrow$ **DISCRETE-TIME FINITE-STATE MARKOV CHAIN**

  \[
  0 \quad t_1 \quad t_2 \quad t_3 \quad t_4 \quad \cdots \quad t_n \quad T_m \quad t
  \]

  - Continuously $\rightarrow$ **CONTINUOUS-TIME FINITE-STATE MARKOV PROCESS**

  \[
  0 \quad T_m \quad t
  \]
Continuous-Time Finite-State Markov Processes (CTFSMP)
• The stochastic process is **observed continuously** and **transitions** are assumed to **occur continuously** in time
The random process of system transition between states in time is described by a stochastic process \( \{X(t); t \geq 0\} \)

- \( X(t) \) := system state at time \( t \)
  - \( X(3.6) = 5 \): the system is in state number 5 at time \( t = 3.6 \)

**OBJECTIVE:**
Computing the probability that the system is in a given state as a function of time, for all possible states

\[
P[X(t) = j], t \in [0, T_m], j = 0, 1, \ldots, N
\]
The conceptual model: Markov Assumption

• **IN GENERAL STOCHASTIC PROCESSES:**
  the probability of a future state of the system usually depends on its entire life history

\[
P[X(t + \nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t] \\
(i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)
\]

• **IN MARKOV PROCESSES:**
  the probability of a future state of the system only depends on its present state

\[
P[X(t + \nu) = j \mid X(t) = i, X(u) = x(u), 0 \leq u < t] = P[X(t + \nu) = j \mid X(t) = i] \\
(i = 0, 1, \ldots, N, j = 0, 1, \ldots, N)
\]

THE PROCESS HAS “NO MEMORY”
The conceptual model: Transition Probabilities

- **Transition probability** that the system in state $i$ at time $t$ moves to state $j$ at time $t + \nu$

  $$p_{ij}(t, t + \nu) = P[X(t + \nu) = j \mid X(t) = i], t, \nu > 0$$

  \[i = 0, 1, \ldots, N, j = 0, 1, \ldots, N\]

- If the **transition probability** depends on the **interval $\nu$** and **not** on the **individual times** $t$ and $t + \nu$
  - the probabilities are **stationary**
  - the Markov process is **homogeneous** in time

  $$p_{ij}(t, t + \nu) = P[X(t + \nu) = j \mid X(t) = i] = p_{ij}(\nu)$$
HYPOTHESIS:

- The time interval $\nu = dt$ is small such that only one event (i.e., one stochastic transition) can occur within it

\[ p_{ij}(\nu) = p_{ij}(dt) = P[X(t + dt) = j | X(t) = i] \]

= (Taylor $1^{st}$ order expansion)

\[ \alpha_{ij} \cdot dt + \theta(dt), \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0 \]

$\alpha_{ij}$ = transition rate from state $i$ to state $j$
The conceptual model: The Transition Probability Matrix (1)

\[ p_{ij}(dt) = \alpha_{ij} \cdot dt + \theta(dt), \lim_{dt \to 0} \frac{\theta(dt)}{dt} = 0 \]

\[ p_{ii}(dt) = 1 - \sum_{j \neq i} p_{ij}(dt) = 1 - dt \cdot \sum_{j \neq i} \alpha_{ij} + \theta(dt) \]

- In analogy with the discrete-time formulation:

**Discrete-time transition probability matrix**

\[
\begin{pmatrix}
  p_{00} & p_{01} & \cdots & p_{0N} \\
  p_{10} & p_{11} & \cdots & p_{1N} \\
  \vdots & \vdots & \ddots & \vdots \\
  p_{N0} & p_{N1} & \cdots & p_{NN}
\end{pmatrix}
\]

**Continuous-time transition probability matrix**

\[
\begin{pmatrix}
  1 - dt \cdot \sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} \cdot dt & \cdots & \alpha_{0N} \cdot dt \\
  \alpha_{10} \cdot dt & 1 - dt \cdot \sum_{j=0}^{N} \alpha_{1j} & \cdots & \alpha_{1N} \cdot dt \\
  \vdots & \vdots & \ddots & \vdots \\
  \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
\]
In analogy with the discrete-time formulation:

\[ P(t + dt) = P(t) \cdot A^* \]

\[
\begin{bmatrix}
P_0(t + dt)P_1(t + dt)\ldots P_N(t + dt)
\end{bmatrix} = \\
\begin{bmatrix}
1 - dt \cdot \sum_{j=1}^{N} \alpha_{0,j} & \alpha_{01} \cdot dt & \ldots & \alpha_{0N} \cdot dt \\
\alpha_{10} \cdot dt & 1 - dt \cdot \sum_{j=0 \atop j \neq 1}^{N} \alpha_{1,j} & \ldots & \alpha_{1N} \cdot dt \\
\vdots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\]

First-equation:

\[
P_0(t + dt) = \left[ 1 - dt \sum_{j=1}^{N} \alpha_{0,j} \right] P_0(t) + \alpha_{10}P_1(t) \cdot dt + \ldots + \alpha_{N0}P_N(t) \cdot dt
\]
The conceptual model: The Transition Probability Matrix (3)

\[
P_0(t + dt) = \left[1 - dt \sum_{j=1}^{N} \alpha_{0j}\right] P_0(t) + \alpha_{10} P_1(t) \cdot dt + \ldots + \alpha_{N0} P_N(t) \cdot dt
\]

- Subtract \( P_0(t) \) on both sides

\[
P_0(t + dt) - P_0(t) = P_0(t) - P_0(t) - \sum_{j=1}^{N} \alpha_{0j} P_0(t) \cdot dt + \alpha_{10} P_1(t) \cdot dt + \ldots + \alpha_{N0} P_N(t) \cdot dt
\]

- Divide by \( dt \)

\[
\frac{P_0(t + dt) - P_0(t)}{dt} = -\sum_{j=1}^{N} \alpha_{0j} P_0(t) + \alpha_{10} P_1(t) + \ldots + \alpha_{N0} P_N(t)
\]

- Let \( dt \to 0 \)

\[
\lim_{dt \to 0} \frac{P_0(t + dt) - P_0(t)}{dt} = \frac{dP_0}{dt} = -\sum_{j=1}^{N} \alpha_{0j} \cdot P_0(t) + \alpha_{10} \cdot P_1(t) + \ldots + \alpha_{N0} \cdot P_N(t)
\]
The conceptual model: The Transition Probability Matrix (4)

- Extending to the other equations:

\[
\frac{dP}{dt} = P(t) \cdot A, \quad A = \begin{pmatrix}
-\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \cdots & \alpha_{0N} \\
\alpha_{00} & -\sum_{j=0, j \neq 1}^{N} \alpha_{1j} & \cdots & \alpha_{1N} \\
\alpha_{10} & \alpha_{11} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\alpha_{N0} & \alpha_{N1} & \cdots & \cdots \\
\end{pmatrix}
\]

TRANSITION RATE MATRIX

System of linear, first-order differential equations in the unknown state probabilities

\[P_j(t), \quad j = 0, 1, 2, \ldots, N, \quad t \geq 0\]
Example 3: one component/one repairman—Markov Diagram and transition rate matrix

**ASSUMPTION**: exponential failure/repair times distributions

Discrete states = 0 $\rightarrow$ component working
   1 $\rightarrow$ component failed

Transition rates = $\lambda$ $\rightarrow$ rate of failure (i.e., from 0 to 1)
   = $\mu$ $\rightarrow$ rate of repair (i.e., from 1 to 0)

\[
\begin{bmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{bmatrix}
\]

**MARKOV DIAGRAM**

**TRANSITION RATE MATRIX**
Example 4: system with $N$ identical components and $N$ repairmen available

**SYSTEM CHARACTERISTICS:**

- The system is made of $N$ identical components
- Each component can be in two states: **working** or **failed**
- The transition rates are constant $= \lambda \rightarrow$ rate of failure $= \mu \rightarrow$ rate of repair
- $N$ repairmen are available

- **System discrete states:**
  - State 0: none failed, all components function
  - State 1: one component failed, $N-1$ function
  - State 2: two components failed, $N-2$ function
  - …
  - State $N$: all components failed, none function
Example 4: system with \( N \) identical components and \( N \) repairman available

**HYPOTHESES:**
- one event (failure or repair of one component) can occur in the small \( \Delta t \)
- the events are mutually exclusive

**Explanation:**
Probability of transition \( 0 \rightarrow 1 \) =
= probability \{ anyone of the \( N \) components fails in \( \Delta t \) \} =
= probability \{ component 1 fails or component 2 fails or component 3 fails … \} =
= probability \{ component 1 fails \} + probability \{ component 2 fails \} + … =
= \( \lambda \Delta t + \lambda \Delta t + \ldots = N\lambda \Delta t \)
Example 5: system with $N$ identical components and 1 repairman available

**SYSTEM CHARACTERISTICS:**

- The system is made of $N$ **identical** components.
- Each component can be in **two** states: working or failed.
- The **transition rates** are **constant**: $\lambda \rightarrow$ rate of failure, $\mu \rightarrow$ rate of repair.
- One repairman is available.

---

**Diagram:**

- **State 0:** all functioning, with transitions $1-N\lambda \Delta t$ and $N\lambda \Delta t$.
- **State 1:** $N\lambda \Delta t$ transitions to state 0 and $(N-1)\lambda \Delta t$ to state 2.
- **State 2:** $\mu\Delta t$ transitions to state 1 and $1-\mu\Delta t-(N-1)\lambda\Delta t$ to state 3.
- **State $N-1$:** $\lambda\Delta t$ transitions to state $N$.
- **State $N$:** all failed, with $\mu\Delta t$ transitions to state $N-1$.

---

Francesco Cannarile & Enrico Zio
Solution to the Fundamental Equation
Solution to the fundamental equation of CTFSMP

\[
\left\{ \begin{align*}
\frac{dP}{dt} &= P(t) \cdot A \\
P(0) &= C
\end{align*} \right.
\]

where

\[
A = \begin{pmatrix}
-\sum_{j=1}^{N} \alpha_{0j} & \alpha_{01} & \cdots & \alpha_{0N} \\
\alpha_{10} & -\sum_{j=0}^{N} \alpha_{1j} & \cdots & \alpha_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{N0} & \alpha_{N1} & \cdots & -\sum_{j=N}^{N} \alpha_{NN}
\end{pmatrix}
\]

System of linear, first-order differential equations in the unknown state probabilities

\[P_j(t), j = 0, 1, 2, \ldots, N, \ t \geq 0\]

USE LAPLACE TRANSFORM
Solution to the fundamental equation of CTFSMP: the Laplace Transform Method

- Laplace Transform: $\mathcal{L}\{P_j(t)\} = \mathcal{L}\left[\int_0^\infty e^{-st} P_j(t)\,dt\right] = \frac{P_j(s)}{s}$, $j = 0,1,...,N$

- First derivative: $L\left(\frac{dP_j(t)}{dt}\right) = s \cdot \mathcal{L}\{P_j(t)\} - P_j(0)$, $j = 0,1,...,N$

- Apply the Laplace operator to

$$
\frac{dP}{dt} = P(t) \cdot A
$$

$$
L\left[\frac{dP(t)}{dt}\right] = L[P(t) \cdot A]
$$

First derivative

$$
\mathcal{L}\{\frac{dP(t)}{dt}\} = \mathcal{L}\{P(t) \cdot A\}
$$

Linearity

$$
\mathcal{L}\{P(t)\} = C \cdot \left[\frac{s \cdot I - A}{A}\right]^{-1}
$$

$P(t)$ = inverse transform of $\tilde{P}(s)$
CTFSMP: steady state probabilities

- At steady state \( \frac{d P(t)}{dt} = 0 \) \( \Rightarrow \) \( \frac{d P(t)}{dt} = P(t) \cdot A = \Pi \cdot A = 0 \)

- Solve the (linear) system:

\[
\begin{align*}
\Pi \cdot A &= 0 \\
\sum_{j=0}^{N} \Pi_j &= 1
\end{align*}
\]

- It can be shown that \( \Pi_j = \frac{D_j}{\sum_{i=0}^{N} D_i} \quad j = 0, 1, 2, \ldots, N \)

\( D_j \) = determinant of the square matrix obtained from \( A \) by deleting the \( j \)-th row and column
Example 6: one component/one repairman – Solution to the fundamental equation (1)

- Component discrete states = 0 → component working
  = 1 → component failed

- **Constant** transition rates = λ → rate of failure (i.e., from 0 to 1)
  = μ → rate of repair (i.e., from 1 to 0)

- Component is working at \( t = 0 \): \( C = [1 \; 0] \)

\[
A = \begin{pmatrix}
-\lambda & \lambda \\
\mu & -\mu
\end{pmatrix}
\]
Example 6: one component/one repairman – Solution to the fundamental equation (2)

• Solve \( \bar{P}(s) = C \cdot (sI - A)^{-1} \)

• Compute \( (sI - A)^{-1} \)

\[
(sI - A)^{-1} = \begin{pmatrix} s + \lambda & -\lambda \\ -\mu & s + \mu \end{pmatrix}^{-1} = \frac{1}{\text{det}[(sI - A)]} \begin{pmatrix} s + \mu & \lambda \\ \mu & s + \lambda \end{pmatrix}
\]

\[
= \frac{1}{s^2 + s\lambda + s\mu} \begin{pmatrix} s + \mu & \lambda \\ \mu & s + \lambda \end{pmatrix}
\]

\[
\bar{P}(s) = C \cdot (sI - A)^{-1} = \frac{1}{s^2 + s\lambda + s\mu} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} s + \mu & \lambda \\ \mu & s + \lambda \end{pmatrix} = \begin{bmatrix} \frac{s + \mu}{s(s + \lambda + \mu)} & \frac{\lambda}{s(s + \lambda + \mu)} \end{bmatrix}
\]
Example 6: one component/one repairman – Solution to the fundamental equation (3)

- **Anti-Transform**
  \[ \tilde{P}(s) = \begin{bmatrix} \frac{s + \mu}{s(s + \lambda + \mu)} & \frac{\lambda}{s(s + \lambda + \mu)} \end{bmatrix} \]

- It is known that
  \[ L^{-1}\left[ \frac{1}{s + a} \right] = e^{-ax} \quad \text{and} \quad L^{-1}\left[ \frac{1}{s(s + a)} \right] = \frac{1}{a} (1 - e^{-ax}) \]

**STATE PROBABILITY VECTOR**

\[
P(t) = \begin{bmatrix} \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu)t} \\ \frac{\lambda}{\lambda + \mu} - \frac{\lambda}{\lambda + \mu} \cdot e^{-(\lambda + \mu)t} \end{bmatrix}
\]

- **System instantaneous availability** (probability of being in operational state 0 at time \( t \))
- **System instantaneous unavailability** (probability of being in failed state 1 at time \( t \))
Example 6: one component/one repairman – Solution to the fundamental equation (4)

- **TIME-DEPENDENT STATE PROBABILITIES**

\[ P_0(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \]

(system instantaneous availability)

\[ P_1(t) = \frac{\lambda}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\lambda + \mu)t} \]

(system instantaneous unavailability)

- **STEADY STATE PROBABILITIES**

\[ \Pi_0 = \lim_{t \to \infty} P_0(t) = \frac{\mu}{\mu + \lambda} = \frac{1/\lambda}{1/\mu + 1/\lambda} = \frac{MTBF}{MTTR + MTBF} \]

= average fraction of time the system is functioning

\[ \Pi_1 = \lim_{t \to \infty} P_1(t) = \frac{\lambda}{\mu + \lambda} = \frac{1/\mu}{1/\mu + 1/\lambda} = \frac{MTTR}{MTTR + MTBF} \]

= average fraction of time the system is down (i.e., under repair)
Quantity of Interest
Frequency of departure from a state

- **Unconditional** probability of arriving in state $j$ in the next $dt$ departing from state $i$ at time $t$:

$$P[X(t + dt) = j, X(t) = i]$$

$$P[X(t + dt) = j|X(t) = i]P[X(t) = i] = p_{ij}(dt)P_i(t)$$

- **Frequency of departure** from state $i$ to state $j$:

$$v_{ij}^{dep}(t) = \lim_{dt \to 0} \frac{p_{ij}(dt)P_i(t)}{dt} = \alpha_{ij}P_i(t) = v_{ij}^{dep} = \alpha_{ij} \cdot \Pi_i$$

  (at steady state)

- **Total frequency of departure** from state $i$ to any other state $j$:

$$v_i(t) = \sum_{j=0}^{N} \alpha_{ij} \cdot P_i(t) = -\alpha_{ii} \cdot P_i(t) = v_i = -\alpha_{ii} \cdot \Pi_i$$

  (at steady state)
Frequency of arrival to a state

- **In analogy**, considering the arrivals to state $i$ from any state $k$:

  \[ v_{i \text{arr}}^i(t) = \sum_{k=0}^{N} \alpha_{ki} \cdot P_k(t) \]

  \[ v_{i \text{arr}}^i = \sum_{k=0}^{N} \alpha_{ki} \cdot \Pi_k \] (at steady state)

- Since \( \Pi \cdot A = 0 = \sum_{k=0}^{N} \alpha_{ki} \cdot \Pi_k \) \( \Rightarrow -\alpha_{ii} \cdot \Pi_i = \sum_{k=0}^{N} \alpha_{ki} \cdot \Pi_k \)

**AT STEADY STATE:**
frequency of departures from state $i$ = frequency of arrivals to state $i$
• **SYSTEM FAILURE INTENSITY** \( W_f \):

  - **Rate** at which system failures occur
  - **Expected number** of system failures per unit of time
  - **Rate of exiting a success state** to go into one of fault

\[
W_f (t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F}
\]

\( S \) = set of success states of the system

\( F \) = set of failure states of the system

\( P_i(t) \) = probability of the system being in the functioning state \( i \) at time \( t \)

\( \lambda_{i \rightarrow F} \) = conditional (transition) probability of leaving success state \( i \) towards a failure state
• **SYSTEM REPAIR INTENSITY** \( W_r \):
  
  • **Rate** at which system repairs occur
  • **Expected number** of system repairs per unit of time
  • **Rate of exiting a failed state** to go into one of success

\[
W_r (t) = \sum_{j \in F} P_j (t) \cdot \mu_{j \rightarrow S}
\]

\( S = \) set of success states of the system

\( F = \) set of failure states of the system

\( P_j (t) = \) probability of the system being in the failure state \( j \) at time \( t \)

\( \mu_{j \rightarrow S} = \) conditional (transition) probability of leaving failure state \( j \) towards a success state
Example 7: one component/one repairman – Failure and repair intensities

- $S = \text{set of success states of the system} = \{0\}$
- $F = \text{set of failure states of the system} = \{1\}$
- $\lambda_{i \rightarrow F} = \lambda$
- $\mu_{j \rightarrow S} = \mu$

The failure intensity rate is given by:

$$W_f(t) = \sum_{i \in S} P_i(t) \cdot \lambda_{i \rightarrow F}$$

The repair intensity rate is given by:

$$W_r(t) = \sum_{j \in F} P_j(t) \cdot \mu_{j \rightarrow S}$$

Where:

- $P_i(t)$ is the probability of being in state $i$ at time $t$.
- $\lambda_{i \rightarrow F}$ is the failure intensity rate from state $i$ to state $F$.
- $\mu_{j \rightarrow S}$ is the repair intensity rate from state $j$ to state $S$.

The functions $W_f(t)$ and $W_r(t)$ represent the rate at which failures and repairs occur, respectively, over time.
Sojourn Time in a state (1)

- **Sojourn time** $T_i$: time spent in a state $i$

- Markov property and time homogeneity imply that if at time $t$ the process is in state $i$, the time remaining in state $i$ is independent of time already spent in state $i$

\[
P(T_i > t + s | T_i > t) = P(X(t + u) = i, 0 \leq u \leq s | X(u) = i, 0 \leq u \leq t) =
\]

\[
= P(X(t + u) = i, 0 \leq u \leq s | X(t) = i) \text{ (by Markov property)}
\]

\[
= P(X(u) = i, 0 \leq u \leq s | X(0) = i) \text{ (by homogeneity)}
\]

\[
= P(T_i > s) \text{ Memoryless Property}
\]

- The only distribution satisfying the memoryless property is the **Exponential distribution** $T_i \sim \text{Exp}$
Sojourn Time in a state (2)

- **Sojourn time** $T_i$: time spent in a state $i$

- The system **remain** in state $i$ before leaving it with **constant** rate $-\alpha_{ii}$

\[ T_i \sim \text{Exp}(-\alpha_{ii}) \]

- **Expected sojourn time** $l_i$: average time of occupancy of state $i$

\[ l_i = \mathbb{E}\{T_i\} = \frac{1}{-\alpha_{ii}} \]
Sojourn Time in a state (3)

- Total frequency of departure at steady state:  \( \nu_i = -\alpha_{ii} \cdot \Pi_i \)
- Average time of occupancy of state:  \( l_i = \frac{1}{-\alpha_{ii}} \)

\[ \nu_i = -\alpha_{ii} \cdot \Pi_i = \frac{\Pi_i}{l_i} \]

\[ \Pi_i = \nu_i \cdot l_i \]

The **mean** proportion of time \( \Pi_i \) that the system spends in state \( i \) is equal to the visit frequency to state \( i \) multiplied by the mean duration of one visit in state \( i \).
• **System instantaneous availability** at time $t$

  $\text{sum}$ of the **probabilities** of being in a **success** state at time $t$

  $$p(t) = \sum_{i \in S} P_i(t) = 1 - q(t) = 1 - \sum_{j \in F} P_j(t)$$

  $S = \text{set of success states of the system}$

  $F = \text{set of failure states of the system}$

  In the Laplace domain

  $$\Phi(s) = \sum_{i \in S} \Phi_i(s) = \frac{1}{s} - \sum_{j \in F} \Phi_j(s)$$
• TWO CASES:

1) Non-Reparaible Systems  ➡ No repairs allowed

2) Reparaible Systems  ➡ Repairs allowed
System Reliability: Non-Reparaible Systems

- No repairs allowed ⇒ Reliability = Availability \( R(t) \equiv p(t) = 1 - q(t) \)

- In the Laplace Domain: \( \hat{R}(s) = \sum_{i \in S} \hat{P}_i(s) = \frac{1}{s} - \sum_{j \in F} \hat{P}_j(s) \)

- Mean Time to Failure (MTTF):

\[
\text{MTTF} = \int_0^\infty R(t) dt = \left[ \int_0^\infty R(t) e^{-st} dt \right]_{s=0} = \tilde{R}(0) = \sum_{i \in S} \tilde{P}_i(0) = \left[ \frac{1}{s} - \sum_{j \in F} \tilde{P}_j(s) \right]_{s=0}
\]
• TWO CASES:

1) Non-reparaible systems
   → No repairs allowed

2) Reparaible systems
   → Repairs allowed
System Reliability: Reparable Systems (1)

1. Exclude all the failed states \( j \in F \) from the transition rate matrix \( A \)

\[
A = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & \mu + \lambda & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix}
\]

The new matrix \( A' \) contains the transition rates for transitions only among the success states \( i \in S \)

\[
A' = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\mu + \lambda) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix}
\]
2. Solve the **reduced problem** of $A'$ for the probabilities $P_i^*(t), \ i \in S$ of being in these (transient) **safe states**

\[
\frac{d}{dt} P_i^*(t) = P_i^*(t) \cdot A'
\]

**Reliability**

\[
R(t) = \sum_{i \in S} P_i^*(t)
\]

**Mean Time To Failure (MTTF)**

\[
MTTF = \int_0^\infty R(t) \, dt = \sum_{i \in S} \Phi_i(0) = \Phi(0)
\]

**NOTICE:** in the reduced problem we have only transient states $\Rightarrow \Pi_i^* = P_i^*(\infty) = 0$
Example 8: system with two identical components and two repairmen available (1)

- **TWO CASES:**
  
  a) Parallel logic (1 out of 2)

b) Series logic (2 out of 2)

\[ A = \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & - (\mu + \lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \]
Example 8: system with two identical components and two repairmen available (2)

PARALLEL LOGIC

System discrete states:
- State 0: system is operating (both components functioning)
- State 1: system is operating (only one of the two components functioning)
- State 2: system is failed (both components failed)
Example 8: system with two identical components and two repairmen available (3)

1. Exclude all the failed states $j \in F$ from the transition rate matrix

- **System reliability** $R(t) :=$ probability of the system being in **safe states 0 or 1 continuously** from $t = 0$

\[
\begin{align*}
A &= \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\mu + \lambda) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix} \\
\Rightarrow \quad A' &= \begin{pmatrix}
-2\lambda & 2\lambda \\
\mu & -(\mu + \lambda)
\end{pmatrix}
\end{align*}
\]
Example 8: System with two identical components and two repairmen available (4)

2. Solve the **reduced problem** of \( \overline{A'} \) for the probabilities \( P_i^*(t), i \in S \) of being in these (transient) **safe states**

In the time domain:

\[
\begin{align*}
\frac{d P^*}{dt} &= P^*(t) \cdot A' \\
\implies \frac{d P^*}{dt} &= P^*(t) \begin{pmatrix} -2\lambda & 2\lambda \\
\mu & -(\lambda + \mu) \end{pmatrix} \\
P^*(0) &= (1 \ 0)
\end{align*}
\]

In the Laplace domain:

\[
\tilde{P}^*(s) = \tilde{P}^*(0) \cdot (sI - A')^{-1} \implies \tilde{P}^0(s) = (1 \ 0) \cdot (sI - A')^{-1}
\]
Example 8: system with two identical components and two repairmen available (5)

\[ f(s) = (1, 0) \cdot (sI - A')^{-1} \]

with \( A' = \begin{pmatrix} -2\lambda & 2\lambda \\ \mu & -(\lambda + \mu) \end{pmatrix} \)

\[ sI - A' = \begin{pmatrix} s + 2\lambda & -2\lambda \\ -\mu & s + \mu + \lambda \end{pmatrix} \]

\[
(sI - A')^{-1} = \frac{1}{(s + 2\lambda)(s + \mu + \lambda) - 2\lambda\mu} \begin{pmatrix} s + \mu + \lambda & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix} 
\cdot \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} s + \lambda + \mu & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix}
\]

where \( \omega_{0,1} = \frac{-3\lambda - \mu \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2} \)
Example 8: system with two identical components and two repairmen available (6)

\[ \tilde{P}^*(s) = C^* \cdot (sI - A')^{-1} = \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} 1 & 0 \\ s + \mu + \lambda & 2\lambda \\ \mu & s + 2\lambda \end{pmatrix} = \frac{1}{(s - \omega_0)(s - \omega_1)} \begin{pmatrix} s + \mu + \lambda & 2\lambda \end{pmatrix} \]

**SYSTEM RELIABILITY**

\[ \tilde{R}(s) = \tilde{P}_0(s) + \tilde{P}_1(s) \quad \text{In the Laplace domain} \]

- It is known that \( L^{-1} \left[ \frac{1}{s + a} \right] = e^{-ax} \) and \( L^{-1} \left[ \frac{1}{s(s + a)} \right] = \frac{1}{a} (1 - e^{-ax}) \)

**SYSTEM RELIABILITY**

\[ R(t) = \frac{\omega_0 \cdot e^{\omega_1 \cdot t} - \omega_1 \cdot e^{\omega_0 \cdot t}}{\omega_0 - \omega_1} \quad \text{In the time domain} \]
Example 8: system with two identical components and two repairmen available (7)

- **MEAN TIME TO FAILURE**
  
  \[ MTTF = \mathbb{E}(0) = \sum_{i} P_r^C(0) = \sum_{i=0}^{1} \tilde{P}_i^*(0) \]

- Starting from \( \mathcal{P}_i^C(s) = C^* \cdot \left(s \cdot I \div A' \right)^{-1} \)

\[
MTTF = C^* \cdot \left( -A' \right)^{-1} \cdot w^T \quad \text{with} \quad w = [1 \ 1 \ 1 \ \ldots \ \ 1]^T
\]

\[
MTTF = (1 \ 0) \cdot \left( \begin{array}{cc}
2\lambda & -2\lambda \\
-\mu & \mu + \lambda 
\end{array} \right)^{-1} \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \\
= (1 \ 0) \cdot \frac{1}{2\lambda(\lambda + \mu) - 2\lambda \mu} \cdot \left( \begin{array}{cc} \mu + \lambda & 2\lambda \\ \mu & 2\lambda \end{array} \right) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \\
= \frac{1}{2\lambda^2}(\mu + \lambda \ 2\lambda) \cdot \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \frac{3\lambda^2 + \mu}{2\lambda^2} = \\
= \frac{3}{2\lambda} + \frac{\mu}{2\lambda^2}
\]
Example 8: system with two identical components and two repairmen available (8)

\[ A \equiv \begin{pmatrix} -2\lambda & 2\lambda & 0 \\ \mu & -(\mu + \lambda) & \lambda \\ 0 & 2\mu & -2\mu \end{pmatrix} \]

- **TWO CASES:**
  
  a) Parallel logic (1 out of 2)
  
  b) Series logic (2 out of 2)
Example 8: system with two identical components and two repairmen available (9)

System discrete states:
State 0: system is operating (both components functioning)
State 1: system is failed (only one of the two components functioning)
State 2: system is failed (both components failed)

SERIES LOGIC

\[
A = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & - (\mu + \lambda) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix}
\]
Example 8: system with two identical components and two repairmen available (9)

1. Exclude all the failed states $j \in F$ from the transition rate matrix

- **System reliability** $R(t) :=$ probability of the system being in **safe states 0 or 1 continuously** from $t = 0$

\[
A = \begin{pmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\lambda + \mu) & \lambda \\
0 & 2\mu & -2\mu
\end{pmatrix} \Rightarrow A' = -2\lambda
\]
Example 8: system with two identical components and two repairmen available (10)

2. Solve the reduced problem of $A'$ for the probabilities $P_i^*(t), \ i \in S$ of being in these (transient) safe states

- Easy to solve in the time domain:

$$\begin{align*}
\frac{dP^*}{dt} &= P^* \cdot A' \\
\left[P^*(0) = C^*ight]
\end{align*}$$

which simplifies to

$$\begin{align*}
\frac{dP_0^*}{dt} &= -2\lambda \cdot P_0^* \\
\left[P_0^*(0) = 1\right]
\end{align*}$$

$$R(t) = P_0^*(t) = e^{-2\lambda t}$$