Monte Carlo and Possibilistic Methods for Uncertainty Analysis

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‘The Uncertainty Factor’

Model parameters: $\alpha_1, \alpha_2, \ldots$

Uncertainty representation
Uncertainty propagation

OUTPUT

$u$

$y_1, y_2, \ldots, y_n$
Example 1: Uncertainty in Event Tree analysis

Safety system 1 (fire sprinkler system)  Safety system 2 (closing door)

\[
P(\text{Seq4}|\text{fire}) = p_1 p_2
\]
• How to characterize the uncertainty on the probability of occurrence of the events \((p_1, p_2)\)?
• How to propagate the uncertainties from the single events to the sequences \(\rho(seq_4|fire)\)?
Example 2: Reliability Assessment of a Degrading Structure

Monitored parameter

\[ h(t) \]

\[ h(t) = \text{actual crack depth} \]
\[ h(t) > h_{\text{max}} \rightarrow \text{structure fails} \]

MODEL:

\[
\frac{dh}{dt} = e^{\omega} C (\beta \sqrt{h})^\eta
\]

- \( h \) = actual crack depth
- \( \omega \) = random variable that describes the stochasticity of the degradation process
- \( C, \beta, \eta \) = constants related to the material properties
- \( t \) = time
Example 2: Reliability Assessment of a Degrading Structure

**STRUCTURE**

**MODEL**

- $\omega \sim N(0, \sigma^2_\omega)$
- $h_{\text{max}}$: “between 9 and 11, with a preference for 10”

**MONITORED PARAMETER**

$h(t) = \text{actual crack depth}$

$h(t) > h_{\text{max}} \rightarrow \text{structure fails}$

$$\frac{dh}{dt} = e^{\omega} C(\beta \sqrt{h})^\eta$$

- $h = \text{actual crack depth}$
- $\omega = \text{random variable that describes the stochasticity of the degradation process}$
- $C, \beta, \eta = \text{constants related to the material properties}$
- $t = \text{time}$
In This Lecture

Uncertainty representation
- Probability distributions
- Possibility distributions

Uncertainty propagation
- Level 1:
  - Purely probabilistic method
  - Purely possibilistic
  - Hybrid probabilistic and possibilistic method
- Level 2
  - Purely probabilistic method

Applications
- Event tree uncertainty analysis
- Reliability assessment of a degrading component
PART I: Uncertainty Representation
Uncertainty classification

Uncertain quantity: we do not known its value!

Aleatory uncertainty: randomness due to inherent variability in the system

Epistemic uncertainty: imprecision due to lack of knowledge and information on the system
• Aleatory uncertainty
  probability distributions
• Epistemic uncertainty
  Probability distributions
  Possibility distributions
Epistemic Uncertainty Representation
Sufficient informative data:
e.g. $N$ experiments $\rightarrow (y_1, y_2, \ldots, y_N)$

Probability distributions

$$P\{y \in [y_1, y_2]\} = \int_{y_1}^{y_2} f(y)\,dy$$

$$F(y_2) = P\{y < y_2\}$$
Scarce information and of qualitative nature on the parameter value.

Expert: “the value of $Y$ is between 9 and 11, with a preference for 10”

Information difficult to catch with a probability distribution

Possibility representation of the uncertainty

$\pi(y) = \text{possibility distribution} = \text{Degree of possibility that the uncertain variable } Y \text{ be equal to } y$
Possibility Measure:  \( \Pi(I) = \max_{y \in I} \pi(y) \)  

Necessity Measure  \( N(I) = 1 - \Pi(I) = 1 - \max_{y \notin I} \pi(y) \)  

« to what extent \( I \) is consistent with the knowledge \( \pi \)? »

«to what extent \( I \) is certainly implied by the knowledge \( \pi \)?»
Example 1

For any interval $I = [y_1, y_2]$

Possibility Measure: $\Pi(I) = ?$

Necessity Measure: $N(I) = ?$
Example 2

\[ I = [9.8, 10.5] \]

Possibility Measure: \[ \Phi(I) = ? \]

Necessity Measure: \[ N(I) = ? \]
Interpretation of the Possibility Distribution

For any interval $I = [y_1, y_2]$

Necessity Measure, $N(I)$

Possibility Measure, $\Pi(I)$

Lower limiting probability value $\leq P\{Y \in I\} \leq$ Upper limiting probability value

\[
1 - \max_{y \notin I}\{\pi(y)\} \leq P\{Y \in I\} \leq \max_{y \in I}\{\pi(y)\}
\]
Example 2: Interpretation

\[ \pi(y) \]

\[ I = [9.8, 10.5] \]

\[ 1 - \max_{y \notin I} \{ \pi(y) \} \leq P\{Y \in I\} \leq \max_{y \in I} \{ \pi(y) \} \]

\[ 1 - 0.8 \leq P\{Y \in I\} \leq 1 \]
Probability and Possibility: Comparison

<table>
<thead>
<tr>
<th>Probability</th>
<th>Possibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P{Y \in [y_1, y_2]}$</td>
<td>$\left[1 - \max_{y \notin I} {\pi(y)}, \max_{y \in I} {\pi(y)}\right]$</td>
</tr>
<tr>
<td>$\int_{y_1}^{y_2} f(y) , dy$</td>
<td>$\int_{-\infty}^{+\infty} f(y) , dy = 1$</td>
</tr>
<tr>
<td>Normalization</td>
<td>$\max {\pi(y)} = 1$</td>
</tr>
</tbody>
</table>

Where $f(y)$ is the probability density function and $\pi(y)$ is the possibility distribution function.
Possibility Distribution → Limiting Cumulative Distributions

\[ I = (-\infty, u] \]

\[ 1 - \max_{y \not\in I} \{ \pi(y) \} \leq P\{Y \leq u\} \leq \max_{y \in I} \{ \pi(y) \} \]

\[ 1 - \max_{y \not\in I} \{ \pi(y) \} \leq F(u) \leq \max_{y \in I} \{ \pi(y) \} \]

Upper cumulative distribution

Lower cumulative distribution

\[ 0 \leq F(9.5) \leq 0.5 \]
Possibility Distribution $\rightarrow$ Limiting Cumulative Distributions

\[ I = (-\infty, u] \]

\[ 1 - \max_{y \notin I} \{\pi(y)\} \leq P\{Y \leq u\} \leq \max_{y \in I} \{\pi(y)\} \]

\[ 1 - \max_{y \notin I} \{\pi(y)\} \leq F(u) \leq \max_{y \in I} \{\pi(y)\} \]

Family of probability distributions

Upper cumulative distribution

Lower cumulative distribution
Definition of an $\alpha$-cut

$$A_\alpha = I = \{y : \pi(y) \geq \alpha\}$$

Interpretation:

“A$_\alpha$ is certain to degree 1-\(\alpha\)”
Example

\[ I = A_\alpha = \{ y : \pi(y) \geq \alpha \} \]

\[ \pi(y) \]

\[ \alpha = 0.30 \]

\[ A_{0.30} \]

\[ 1 - \max_{y \not\in A_{0.30}} \{ \pi(y) \} < P\{ Y \in A_{0.30} \} \leq \max_{y \in A_{0.30}} \{ \pi(y) \} \]

\[ 0.7 < P( Y \in A_{0.30} ) \]
$0 \leq P(Y \in A_1) \leq 1$

$\pi$ possibility distribution

$A_1$ core
Possibility Distribution: Support

\[ P(Y \in A_0) = 1 \]

\( \pi \) possibility distribution

\( A_0 \) support
AVAILABLE INFORMATION

• physical limits \([a, b]\)
• most likely value \(c\)

(E.g., expert: “the true value of \(Y\) is between \(a\) and \(b\), with a preference for \(c\)”)

= family of all the probability distributions with support \([a, b]\) and mode \(c\)
Constructing a possibility distribution: Chebyshev inequality

**Available Information**

- mean value $\mu_x$
- standard deviation $\sigma_x$

Chebyshev inequality

\[
P\left(\mu_x - k\sigma_x \leq X \leq \mu_x + k\sigma_x\right) \geq 1 - \frac{1}{k^2},
\]

for $k \geq 1$

\[
A_{\frac{1}{k^2}} = [\mu_x - k\sigma_x, \mu_x + k\sigma_x] \rightarrow P\left\{X \in A_{\frac{1}{k^2}}\right\} \geq 1 - \frac{1}{k^2}
\]

for $k \geq 1$

= family of all the probability distributions with mean $\mu_x$ and standard deviation $\sigma_x$
LIMIT CASE: No available information on a model parameter $y$ except that its value is located somewhere between a value $y_{\text{min}}$ and $y_{\text{max}}$

*Laplace principle of insufficient reason:* “all that is equally plausible is equally probable”

$$f(y) = \frac{1}{y_{\text{max}} - y_{\text{min}}}$$

$$y_m = \frac{y_{\text{max}} - y_{\text{min}}}{2}$$

$$P(y \in [y_{\text{min}}, y_m]) = P(y \in [y_m, y_{\text{max}}])$$

This is a paradox!!!

No information available $\rightarrow$ No relation between $P(Y \in [y_{\text{min}}, y_m])$ and $P(Y \in [y_m, y_{\text{max}}])$
LIMIT CASE: No available information on a model parameter $y$ except that its value is located somewhere between a value $y_{\text{min}}$ and $y_{\text{max}}$

For any Interval $B \subseteq [y_{\text{min}}, y_{\text{max}}]$

$$1 - \max_{y \notin B} \{\pi(y)\} < P(Y \in B) \leq \max_{y \in B} \{\pi(y)\}$$

$$0 < P(Y \in B) \leq 1$$

the only information available is that: $0 \leq P(B) \leq 1$
Uncertainty representation: conclusions

Types:
- Randomness (aleatory)
- Lack of knowledge (epistemic)

Representation:
- Probability distributions
- Probability distributions (Sufficient statistical information, i.e., data)
- Possibility distributions (Scarce information and/or of qualitative nature, e.g., expert opinion)

Example:
- Gumbel pdf
- Gaussian pdf

"The value of $K_s$ is between 5 and 60, with a preference for 30"
PART II: Uncertainty propagation

\[ u = g(y_1, \ldots, y_n) \]
Uncertainty propagation: objective

MODEL
\[ g(y_1, \ldots, y_n) \]

\[ f(y_1) \]
\[ y_1 \]

\[ f(y_2) \]
\[ y_2 \]

\[ f(y_n) \]
\[ y_n \]

\[ f(u) \]
\[ u \]
Level 1 uncertainty propagation

\[ u = g(y_1, \ldots, y_n) \]

- **Level 1:**
  - \( y_1, \ldots, y_n \) are uncertain parameters described by the probability distributions \( f_{\theta_i}(y_i) \)
  - the parameters \( \theta_i \) of the probability distributions \( f_{\theta_i}(y_i) \) are known
Level 1 uncertainty propagation: Example

Model Input:

\[ y_1 = T_A = \text{failure time of A} \]

\[ y_2 = T_B = \text{failure time of B} \]

Model output:

\[ u = T_{sys} = \text{failure time of the system} = g(T_A, T_B) \]
Level 1 uncertainty propagation

\[ u = g(y_1, \ldots, y_n) \]

- **Level 1:**
  - \( y_1, \ldots, y_n \) are uncertain parameters described by the probability distributions \( f_{\theta_i}(y_i) \)
  - the parameters \( \theta_i \) of the probability distributions \( f_{\theta_i}(y_i) \) are known

\[ A \quad \quad B \]

**Model Input:**
- \( y_1 = t_A = \text{failure time of A} \)
- \( y_2 = t_B = \text{failure time of B} \)

**Model output:**
- \( u = t_{sys} = \text{failure time of the system} = g(t_A, t_B) \)

input uncertainty (aleatory)

\[ T_A \sim f_{\lambda_A}(t_A) = \lambda_A e^{-\lambda_A t_A} \]
\[ T_B \sim f_{\lambda_B}(t_B) = \lambda_B e^{-\lambda_B t_B} \]

**Parameter of \( f \)**

\[ \lambda_A = 0.001 \]
\[ \lambda_B = 0.0008 \]
Level 2 uncertainty propagation

\[ \mathbf{u} = g(\mathbf{y}_1, \ldots, \mathbf{y}_n) \]

- **Level 2:**
  - \( \mathbf{y}_1, \ldots, \mathbf{y}_n \) are uncertain parameters described by the probability distributions \( f_{\theta_i}(\mathbf{y}_1) \)
  - the parameters \( \theta_i \) of the probability distributions \( f_{\theta_i}(\mathbf{y}_1) \) are uncertain (epistemic uncertainty)

Model Input:
- \( \mathbf{y}_1 = t_A = \text{failure time of A} \)
- \( \mathbf{y}_2 = t_B = \text{failure time of B} \)

Model output:
- \( \mathbf{u} = t_{\text{sys}} = \text{failure time of the system} \)
  \[ \mathbf{u} = g(t_A, t_B) \]

Parameter of \( f \):
- \( \Lambda_A \sim N(\mu_A, \sigma_A) \)
- \( \Lambda_B \sim N(\mu_B, \sigma_B) \)

LEVEL 1 uncertainty

LEVEL 2 uncertainty

\[ T_A \sim f_{\lambda_A}(t_A) = \lambda_A e^{-\lambda_A t_A} \]
\[ T_B \sim f_{\lambda_B}(t_B) = \lambda_B e^{-\lambda_B t_B} \]
Uncertainty propagation – Level 1

\[ u = g(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) \]

1. Purely probabilistic method
   - \( y_1, \ldots, y_n \) = probability distributions

2. Purely possibilistic method
   - \( y_1, \ldots, y_n \) = possibility distributions

3. Hybrid Monte Carlo and Possibilistic method
   - \( y_1, \ldots, y_k \) = probability distribution
   - \( y_{k+1}, \ldots, y_n \) = possibility distribution
Uncertainty propagation – Level 1

\[ u = \text{g}(y_1, \ldots, y_n) \]

1. Purely probabilistic method
   - \( y_1, \ldots, y_n \) are uncertain parameters described by the probability distributions \( f(y_1), \ldots, f(y_n) \)
Purely probabilistic uncertainty propagation

\[ u = g(y_1, y_2) \]
\[ Y_1 \approx f_{Y_1}(y_1) \]
\[ Y_2 \approx f_{Y_1}(y_1) \]

\[ F_U(u) = P\{U \leq u\} = P\{g(Y_1, Y_2) \leq u\} = \int_{(y_1, y_2): g(y_1, y_2) \leq u} f_{Y_1}(y_1) f_{Y_2}(y_2) dy_1 dy_2 = \]
\[ = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} I_g(y_1, y_2) f_{Y_1}(y_1) f_{Y_2}(y_2) dy_1 dy_2 \]

with \[ I_g(y_1, y_2) = \begin{cases} 
1 & \text{if } f(y_1, y_2) \leq u \\
0 & \text{otherwise} 
\end{cases} \]

MC analog dart game:
- sample \((y_1^i, y_2^i)\) from \(f_{Y_1}(y_1), f_{Y_1}(y_1)\)
- corresponding award is: \(I_g(y_1^i, y_2^i)\)

Consider \(N\) trials \(\{(y_1^1, y_2^1), (y_1^2, y_2^2), \ldots, (y_1^N, y_2^N)\}\)

\[ F_U(u) = \frac{\sum_{i=1, \ldots, N} I_g(y_1^i, y_2^i)}{N} \]
Sample a realization $y^i_1, y^i_2, ..., y^i_n$ of the uncertain parameters from their probability distributions $f_{Y_1}(y_1), f_{Y_2}(y_2), ..., f_{Y_n}(y_n)$

Estimate the cumulative distribution $F(u)$ from: $u^1, ..., u^N$

Compute:

$$u^i = g(y^i_1, y^i_2, ..., y^i_n)$$

Purely probabilistic uncertainty propagation: the procedure
Uncertainty propagation – Level 1

\[ u = g(y_1, \ldots, y_n) \]

1. Purely possibilistic method
   - \( y_1, \ldots, y_n \) are uncertain parameters described by the possibility distributions \( \pi(y_1), \ldots, \pi(y_n) \)
Purely Possibilistic method: Uncertainty Propagation through $U=g(Y)$

Extension principle of fuzzy set theory

- Simple case:
  - 1 input quantity, $Y \in \mathbb{R}$, whose uncertainty is described by the possibility distribution $\pi(y)$
  - 1 output quantity, $U = g(Y), U \in \mathbb{R}$ whose uncertainty is described by the possibility distribution $\pi_U(u)$:

$$\pi_U(u) = \sup_{y,g(y)=u} \left( \pi_Y(y) \right)$$

Basic idea: extend the function $g$

- From a function from $\mathbb{R}$ to $\mathbb{R}$
- to a function from and to the class of all the possibility distributions defined on $\mathbb{R}$
For a given $u$, we proceed in three steps:

- Select the set of $y$ values such that $g(y) = u$
- Compute the corresponding set of $\pi_Y(y)$
- Choose for $\pi_U(u)$ the maximum among $\pi_Y(y)$

\[ \pi_U(u) = \sup_{y : g(y) = u}(\pi_Y(y)) \]
Extension Principle: Example

For a chemical reactor, we consider a physical model $g$ used for evaluating the consequence of a catastrophic failure event. We assume that the model specifies that the consequences $C$ (in arbitrary units) of a catastrophic failure are a quadratic function of the quantity of toxic gas release $R$ (in arbitrary units), i.e.

$$C = g(R) = R^2$$

The epistemic uncertainty related to $R$ is described by the triangular possibility distribution:

$$\pi_R(r) = \begin{cases} 
0 & \text{if } r < 0 \text{ or } r > 2 \\
r & \text{if } 0 \leq r \leq 1 \\
2 - r & \text{if } 1 < r \leq 2
\end{cases}$$
For a fixed value $c > 0$ of the consequences:

- find the quantity of toxic gas release $r$ such that $c = r^2$. In this particular case, the solutions are $r_1 = -\sqrt{c}$ and $r_2 = \sqrt{c}$.
- evaluate the possibility distribution of the quantity of toxic gas release, $\pi_R$, at the identified $r$ values, i.e. $\pi_R(r_1)$ and $\pi_R(r_2)$,
- assign to the possibility distribution of the consequence, $\pi_C$, the maximum of $\pi_R(r_1)$ and $\pi_R(r_2)$. Since, in this case, $\pi_R(r_1) = \pi_R(-\sqrt{c})$ is equal to 0, the maximum value is obtained as $\pi_R(r_1) = \pi_R(\sqrt{c})$ which, according to the definition of $\pi_R(r)$ is given by:

$$
\pi_C(c) = \pi_R(r_2) = \pi_R(\sqrt{c}) = \begin{cases} 
\sqrt{c} & \text{if } 0 < c \leq 1 \\
2 - \sqrt{c} & \text{if } 1 < c \leq 4 \\
0 & \text{if } c > 4.
\end{cases}
$$
Extension principle of fuzzy set theory

\[ \pi_C(c) \]
Multi-input case: $U=g(Y_1, Y_2, \ldots, Y_n)$

- input quantities: $Y_1, Y_2, \ldots, Y_n$, whose uncertainty is described by the possibility distribution $\pi(y_1), \pi(y_2), \ldots, \pi(y_n)$

- 1 output quantity, $U=g(Y_1, Y_2, \ldots, Y_n)$ whose uncertainty is described by the possibility distribution $\pi_U(u)$:

$$\pi_U(u) = \sup_{y_1, y_2, \ldots, y_n, g(y_1, y_2, \ldots, y_n) = u} \min\{\pi_{Y_1}(y_1), \pi_{Y_1}(y_1), \ldots, \pi_{Y_n}(y_n)\}$$

Basic idea: extend the function $g$

- The use of the minimum operator is justified by:

$$\pi_{Y_1, Y_1, \ldots, Y_n}(y_1, y_2, \ldots, y_n) = \min\{\pi_{Y_1}(y_1), \pi_{Y_1}(y_1), \ldots, \pi_{Y_n}(y_n)\}$$
Output possibility distribution is represented in the form of a nested set of intervals (alpha-cut)

\[ A_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha] = \{u : \pi_U(u) \geq \alpha\} \]

Indicating as \( Y_{1\alpha}, \ldots, Y_{n\alpha} \) the \( \alpha \)-cuts of the input quantities \( Y_1, \ldots, Y_N \), the extension principle, for a given value of \( \alpha \) in [0,1], becomes:

\[
\underline{u}_\alpha = \inf(g(y_1, \ldots, y_n), y_1 \in Y_{1\alpha}, \ldots, y_n \in Y_{n\alpha}),
\overline{u}_\alpha = \sup(g(y_1, \ldots, y_n), y_1 \in Y_{1\alpha}, \ldots, y_n \in Y_{n\alpha}).
\]
Uncertainty propagation – Level 1

\[ u = g(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) \]

2. Hybrid Monte Carlo and Possibilistic method
   - \( y_1, \ldots, y_k \) = probability distributions: \( p(y_1), \ldots, p(y_k) \)
   - \( y_{k+1}, \ldots, y_n \) = possibility distributions: \( \pi(y_{k+1}), \ldots, \pi(y_n) \)
Hybrid Monte Carlo and Possibilistic method

Repeat $N$ simulations

- Monte Carlo sampling of the variables described by probability distributions:
  \[ y_1^j \sim f_1(y_1), \ldots, y_k^j \sim f_k(y_k) \]

- Extension principle to process uncertainty described by possibility distributions:
  \[ \pi^j(u) \text{ of } u = g(y_1, \ldots, y_k, y_{k+1}, \ldots, y_n) \]

Possibility distribution \( \pi^j(u) \) of the input $u$ in each simulation.

Simulation 1 and Simulation $N$.
Details of the Hybrid Monte Carlo and Possibilistic method

Outer loop: probabilistic parameters

\[ j=0 \]
\[ j=j+1 \]

Monte Carlo sampling of: \( y_j^1, \ldots, y_j^k \)
from \( f_1(y_1), \ldots, y_k \sim f_k(y_k) \)

Inner loop: possibilistic parameters

Assumption:

\[ y_1 = y_1^j \]
\[ \ldots \]
\[ y_k = y_k^j \]

Extension principle

\[ \underline{u}_\alpha = \inf_{y_{k+1} \in A^{Y_{k+1}}_\alpha, \ldots, y_n \in A^{Y_n}_\alpha} (g(y_1, \ldots, y_n)) \]
\[ \overline{u}_\alpha = \sup_{y_{k+1} \in A^{Y_{k+1}}_\alpha, \ldots, y_n \in A^{Y_n}_\alpha} (g(y_1, \ldots, y_n)) \]
Details of the Hybrid Monte Carlo and Possibilistic method

**Outer loop:**
- probabilistic parameters
- Montecarlo sampling of: \( y_1^j, \ldots, y_k^j \)
  from \( f_1(y_1), \ldots, y_k \sim f_k(y_k) \)
- \( \alpha = \alpha + \Delta \alpha \)

**Inner loop:**
- possibilistic parameters
- \( \alpha = 0 \)
- \( \alpha = 1? \)
- \( u_\alpha = \inf_{y_{k+1} \in A_{\alpha}^{Y_{k+1}}, \ldots, y_n \in A_{\alpha}^{Y_n}} g(y_1, \ldots, y_n) \)
Details of the Hybrid Monte Carlo and Possibilistic method

Outer loop: probabilistic parameters

\[ j = 0 \]
\[ j = j + 1 \]

Monte Carlo sampling of: \( y_1^j, \ldots, y_k^j \)
from \( f_1(y_1), \ldots, y_k \sim f_k(y_k) \)

Inner loop: possibilistic parameters

\[ \alpha = 0 \]
\[ \alpha = \alpha + \Delta \alpha \]

Find the \( \alpha \) - cuts \( A_{\alpha_1}^{Y_{k+1}}, \ldots, A_{\alpha}^{Y_n} \)

of \( \pi(Y_{k+1}), \ldots, \pi(Y_n) \)

\[ \bar{u}_\alpha = \sup_{y_{k+1} \in A_{\alpha}^{Y_{k+1}}, \ldots, y_n \in A_{\alpha}^{Y_n}} \left( g(y_1, \ldots, y_n) \right) \]
Details of the Hybrid Monte Carlo and Possibilistic method

Outer loop: probabilistic parameters

1. $j=0$
2. $j=j+1$

Monte Carlo sampling of: $y_1^j, \ldots, y_k^j$
from $f_1(y_1), \ldots, y_k \sim f_k(y_k)$

Inner loop: possibilistic parameters

1. $\alpha=0$
2. $\alpha=\alpha+\Delta\alpha$

$u_\alpha = \inf_{y_{k+1} \in A_{k+1}^Y, \ldots, y_n \in A_n^Y} (g(y_1, \ldots, y_n))$
$\bar{u}_\alpha = \sup_{y_{k+1} \in A_{k+1}^Y, \ldots, y_n \in A_n^Y} (g(y_1, \ldots, y_n))$
Details of the Hybrid Monte Carlo and Possibilistic method

Outer loop: probabilistic parameters

- \( j = 0 \)
- \( j = j + 1 \)

Monte Carlo sampling of: \( y_1^j, \ldots, y_k^j \)
from \( f_1(y_1), \ldots, y_k \sim f_k(y_k) \)

Inner loop: possibilistic parameters

- \( \alpha = 0 \)
- \( \alpha = \alpha + \Delta \alpha \)
- \( \ldots \)
- Find the \( \alpha \)-cuts \( A_{\alpha}^{Y_{k+1}}, \ldots, A_{\alpha}^{Y_n} \) of \( \pi(Y_{k+1}), \ldots, \pi(Y_n) \)

Exit

- \( j = N? \)
- \( \pi(u) \)
- \( \pi^j(u) \)

- \( u_{\alpha} = \inf_{y_{k+1} \in A_{\alpha}^{Y_{k+1}}, \ldots, y_n \in A_{\alpha}^{Y_n}} (g(y_1, \ldots, y_n)) \)
- \( \bar{u}_{\alpha} = \sup_{y_{k+1} \in A_{\alpha}^{Y_{k+1}}, \ldots, y_n \in A_{\alpha}^{Y_n}} (g(y_1, \ldots, y_n)) \)
Details of the Hybrid Monte Carlo and Possibilistic method

Outer loop: probabilistic parameters

- $j=0$
- $j = j + 1$

Monte Carlo sampling of: $y^j_1, ..., y^j_k$

from $f_1(y_1), ..., y_k \sim f_k(y_k)$

Inner loop: possibilistic parameters

- $\alpha = 0$
- $\alpha = \alpha + \Delta \alpha$

...$

\text{Find the } \alpha - \text{cuts } A^Y_{\alpha+1}, ..., A^Y_{\alpha} \text{ of } \pi(Y_{k+1}), ..., \pi(Y_n)$

\[ u_{\alpha} = \inf_{y_{k+1} \in A^Y_{\alpha+1}, ..., y_n \in A^Y_{\alpha}} (g(y_1, ..., y_n)) \]

\[ \bar{u}_{\alpha} = \sup_{y_{k+1} \in A^Y_{\alpha+1}, ..., y_n \in A^Y_{\alpha}} (g(y_1, ..., y_n)) \]
Details of the aggregation (1)

Find associated limiting Cumulative distributions

Upper and lower cumulative distributions of $u$: for any $u$ compute the average of the obtained cumulative distributions
Level 1 uncertainty propagation - applications:

1. Event tree analysis
2. Reliability Assessment of a Degrading Component
Application 1)

Event tree analysis

(only epistemic uncertainty is considered)
Case study

Anticipated Transient Without Scram (ATWS) in a PWR nuclear reactor

Objective: compute the probability of occurrence of a severe sequence
**Example of HED event**

<table>
<thead>
<tr>
<th>ADS inhibit</th>
<th>$X_I$</th>
<th>HED</th>
<th>$V_{12}$</th>
</tr>
</thead>
</table>

Automatic depressurization system (ADS) is designed to decrease the pressure of the reactor in order to start the low-pressure system. The low-pressure system will inject water into the reactor vessel to protect the fuel. When an ATWS event happens, the reactor power is controlled by the level of water in core. Since high-level water will cause high power, the operator should inhibit all ADS valves manually. If the operator fails to do so, event $X_I$ will occur.
Uncertainty representation:

- Probability of Hardware-Failure-Dominated (HFD) event $\rightarrow p.d.f$
- Probability of Human-Error-Dominated event (HED) $\rightarrow$ possibility distributions
Hardware-Failure-Dominated events
Human-Error-Dominated events

Event $X_1$

Event $X_{c1}$

Event $X_{c2}$

Event $X_v$
Uncertainty representation:

- Probability of Hardware-Failure-Dominated (HFD) event \(\rightarrow p.d.f\)
- Probability of Human-Error-Dominated event (HED) \(\rightarrow\) possibility distributions

Uncertainty propagation: Hybrid Monte Carlo and Possibilistic method

Event sequence probability \(\rightarrow\) limiting cumulative distributions
Severe consequence accident

Pure probabilistic approach
Upper cumulative distributions
Lower cumulative distributions

$p_{\text{sev}}$
Severe consequence accident

Pure probabilistic approach:
With probability 90% we can say that the probability of having a severe accident is lower than $7 \times 10^{-7}$
95% percentile for the probability of occurrence of a severe consequence accident

<table>
<thead>
<tr>
<th>Approach</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hybrid approach</td>
<td>$(3.78 \cdot 10^{-8}, 1.95 \cdot 10^{-7})$</td>
</tr>
<tr>
<td>Probabilistic approach</td>
<td>$1.09 \cdot 10^{-7}$</td>
</tr>
</tbody>
</table>
Application 2:

Reliability Assessment of a Degrading Structure
Example 3: Reliability Assessment of a Degrading Structure

$R(T_m) = P(X_s(T_m) = 0)$

$\omega \sim N(0, \sigma^2_\omega)$

$H_{\text{max}}$: “between 9 and 11, with a preference for 10”

$\frac{dh}{dt} = e^{\omega} C (\beta \sqrt{h})^\eta$

$h = \text{actual crack depth}$

$C, \beta, \eta = \text{constants related to the material properties}$

ASSUMPTIONS:

- $h$ monitored parameter
- $h > H_{\text{max}} \rightarrow \text{structure fails}$
Uncertainty in the case study

- aleatory \[ \frac{dh}{dL} = e^{\omega}C(\beta \sqrt{h})^n \] with \( \omega \sim N(0, \sigma_{\omega}^2) \)

- epistemic \( H_{\text{max}} \): between 9 and 11, with preference for 10
Uncertainty in the case study

- **aleatory** $\frac{dh}{dL} = e^{\omega} C(\beta \sqrt{h})^\eta$ with $\omega \sim N(0, \sigma_\omega^2)$
- **epistemic** $H_{\text{max}}$: between 9 and 11, with preference for 10

![Graph showing uncertainty in $H_{\text{max}}$]

$h(t_p) = 1$

Time

$TTF=?$

$h$

$TTF=?$

$t_p = 1$

9

10

11
Epistemic uncertainty representation in the case study

- aleatory $\frac{dh}{dL} = e^{\omega} C (\beta \sqrt{h})^{\eta}$ with $\omega \sim N(0, \sigma_\omega^2)$
- epistemic $H_{\text{max}}$: between 9 and 11, with preference for 10
Application of the Hybrid Monte Carlo and possibilistic method
Available information on $H_{\text{max}}$:

- $[9, 11]$
- Most likely value is 10
Uncertainty propagation

- \( X_{S}(T_{\text{miss}}) = f(h(T_{\text{miss}}), H_{\text{max}}) = \begin{cases} 0 & \text{if } h(T_{\text{miss}}) < H_{\text{max}} \\ 1 & \text{otherwise} \end{cases} \)
  - working
  - failed

\( \omega \)

Probabilistic distributions

Possibilistic distributions

- \( R(T_{\text{miss}}) = P(X_{S}(T_{\text{miss}}) = 0) \)
MC simulations of the degradation evolution: $h(T_{\text{miss}})$

\[ \omega_t \approx N(0, \sigma^2_\omega) \]

\[ h(t) = h(t - 1) + e^{\omega_t} C(\Delta K)^\eta \]

$T_{\text{miss}} = 500$
Possibility distributions of the component state: MC simulation 1

\[ X_s(T_{\text{miss}}) = f(h(T_{\text{miss}}), H_{\text{max}}) = \begin{cases} 0 & \text{if } h(T_{\text{miss}}) < H_{\text{max}} \\ 1 & \text{otherwise} \end{cases} \]

\[ h^1(T_{\text{Miss}}) = 13.7 \]

inputs

Extension principle

output
Possibility distributions of the component state: MC simulation 1

\[ X_S(T_{\text{miss}}) = f(h(T_{\text{miss}}), H_{\text{max}}) = \begin{cases} 
0 & \text{if } h(T_{\text{miss}}) < H_{\text{max}} \\
1 & \text{otherwise} 
\end{cases} \]

![Graph showing possibility distributions](image)

**Interpretation**

\[
1 - \max_{X_s \neq 0} \{ \pi(X_s) \} \leq P\{X_s = 0\} \leq \max_{X_s = 0} \{ \pi(X_s) \}
\]

\[
0 \leq P\{X_s(T_{\text{miss}}) = 0\} \leq 0
\]

\[
R(T_{\text{miss}}) = P\{X_s(T_{\text{miss}}) = 0\} = 0
\]

**Extension principle**
Possibility distributions of the component state: MC simulation

\[ X_S(T_{\text{miss}}) = f(h(T_{\text{miss}}), H_{\text{max}}) = \begin{cases} 
0 & \text{if } h(T_{\text{miss}}) < H_{\text{max}} \\
1 & \text{otherwise}
\end{cases} \]

\[ h^2(T_{\text{Miss}}) = 10.2 \]

**Interpretation**

\[ 1 - \max_{X_s \neq 0} \{\pi(X_s)\} \leq P\{X_s = 0\} \leq \max_{X_s = 0} \{\pi(X_s)\} \]

\[ 0 \leq P\{X_S(T_{\text{miss}}) = 0\} \leq 0.8 \]

\[ 0 \leq R(T_{\text{miss}}) \leq 0.8 \]
Example of realizations of Possibility distributions of the component state
• Hybrid Monte Carlo and possibilistic method: the component reliability is between [0.896,0.927]
Conclusions

How to collect the information and input it in the proper mathematical format?
Conclusions

How to collect the information and input it in the proper mathematical format?

- Sufficient information $\rightarrow$ probability distribution
- Scarce and qualitative information $\rightarrow$ possibility distribution

does not force information in the epistemic uncertainty representation
Conclusions

How to collect the information and input it in the proper mathematical format?

How to propagate the uncertainty through the model so as to obtain the proper representation of the uncertainty in the output of the analysis?

- Pure Probabilistic Approach
- Extension Principle
- Hybrid Monte Carlo and possibilistic method
Conclusions

How to collect the information and input it in the proper mathematical format?
How to propagate the uncertainty through the model so as to obtain the proper representation of the uncertainty in the output of the analysis?
How to interpret the uncertainty results in a manner that is understandable and useful to decision makers?
  - Interval of probabilities, e.g. the component reliability is between [0.896,0.927]
Uncertainty representation & propagation techniques:


Applications: